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Walrasian Equilibrium Problem with Memory Term

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Abstract The aim of this paper is to study the Walrasian equilibrium problem when the data are time-dependent. In order to have a more realistic model, the excess demand function depends on the current price and on previous events of the market. Hence, a memory term is introduced; it describes the precedent states of the equilibrium. This model is reformulated as an evolutionary variational inequality in the Lebesgue space $L^2([0,T],\mathbb{R})$, and, thanks to this characterization, existence and qualitative results on equilibrium solution are given.

Keywords Economic equilibrium problem · Evolutionary variational inequality · Memory term · Lagrangean theory

1 Introduction

This paper is concerned with a general economic equilibrium problem, and since we are interested in the evolution of the system with respect to time, we assume that all data are time-dependent. As a consequence, a dynamic Walrasian price equilibrium problem is studied. The advantage of the time-dependent approach lies in the fact that it allows us to examine a reliable model. Leon Walras, already in 1874, recognized the importance of dealing with models closer to reality and provided a sequence of models, each taking into account more aspects of a real economy. Then, in order to link this equilibrium model to the real world, he introduced a price-adjustment mechanism, called *tâtonement*. With this process Walras modeled the law of supply

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and demand: the price of a commodity will increase when demand for that commodity exceeds supply, and the price will decrease if supply exceeds demand.

This paper aims at exploring the dynamics of market adjustment processes in presence of a continuous lag response. Volterra [1–3] was the first who introduced some hereditary coefficients in the form of integral term in the constitutive equations for an elastic body with memory. Later, starting from the 1960s, the idea that a body is able to recollect only its recent past was suggested. As a result, the memory term represents only its recent history, and all the previous events can be neglected. Since then, several applications in different fields have been studied, ranging from economics to engineering problems, see, for instance, [4, 5] and [6]. In [7] authors generalize the Walrasian price equilibrium problem to the dynamic case and characterize the price equilibrium as a solution to a suitable evolutionary variational inequality. Furthermore, the same authors propose to consider the effects of the delay in a Walrasian price equilibrium problem. Following this idea, we introduce a memory term that is able to represent the history of the market. From an economic point of view, we are led to consider a memory term in the excess demand function. Consequently, this adjustment factor affects the equilibrium condition. Namely, in the *tâtonnement* process we take into account the contribution of the equilibrium price from the initial time of the observation time, and this contribution represents an adjustment factor. The mathematical framework chosen for the study of the model is that of evolutionary variational inequalities (see, e.g., [8–18] for both theory and applications of evolutionary variational inequalities). The paper is structured as follows. In Sect. 2, we present the Walrasian equilibrium problem with memory term and characterize the equilibrium as a solution to a suitable evolutionary variational inequality. In Sect. 3, by using the variational theory we give existence and Lipschitz continuity results. In Sect. 4, we characterize the dynamic equilibrium in terms of the Lagrange multipliers. Finally, in Sect. 5, an example concludes the paper.

2 Equilibrium Conditions

During a period of time $[0,T]$, a pure exchange economy with $l>1$ different commodities has been considered; at time t and at each commodity j, a nonnegative price $p^j(t)$ is associated, where

$$p^j:[0,T]\to\mathbb{R},\quad j=1,\dots,l,\ p^j\in L^2\big([0,T],\mathbb{R}\big).$$

Hence, the price vector $p=(p^1,p^2,\dots,p^l)\in L^2([0,T],\mathbb{R}^l)=L$. Let us denote by z^j the aggregate excess demand function relative to the commodity j:

$$z^j:[0,T]\times\mathbb{R}^l\to\mathbb{R},\quad j=1,\dots,l,\ \big(t,p(t)\big)\to z^j\big(t,p(t)\big),$$

and $z(t,p(t))=(z^1(t,p(t)),\dots,z^l(t,p(t)))$ represents the aggregate excess demand vector.

As usual in economy, we assume that z is homogeneous of degree zero in p, that is, for all p, $z(t,\alpha p(t))=z(t,p(t))$ with $\alpha>0$ a.e. in $[0,T]$. Because of homogeneity, the prices may be normalized, so that they take values in the set

$$S_0 := \left\{ p \in L : \ p^j(t) \geq 0, \, j = 1, \ldots, l, \sum_{j=1}^{l} p^j(t) = 1 \text{ a.e. in } [0, T] \right\}.$$

In order to avoid that in such market there is some “free” commodity, it is convenient to fix a minimum price for each commodity j. In this model, it is convenient to fix, for each commodity j, a minimal price $\underline{p}^j(t)$ at the time t. We suppose that $\underline{p} : [0, T] \to \mathbb{R}$ belongs to L and it is such that a.e. in $[0, T]$, $\underline{p}^j(t) > 0$ and for all $j = 1, \ldots, l$, $\underline{p}^j(t) < \frac{1}{l}$. Then the feasible set becomes:

$$S := \left\{ p \in L : \ p^j(t) \geq \underline{p}^j(t), \, j = 1, \ldots, l, \ \sum_{j=1}^{l} p^j(t) = 1 \text{ a.e. in } [0, T] \right\}.$$

Since our aim is to provide a model closer to reality, for a Walrasian pure exchange equilibrium problem, we suppose that the price trend at time t be affected to the previous events of the market. So, we introduce the aggregate excess demand function with memory term:

$$Z : [0, T] \times \mathbb{R}^l \to \mathbb{R}^l,$$

$$Z\big(t, p(t)\big) = z\big(t, p(t)\big) + \int_0^t I(t - s) p(s) \, ds,$$

where I is a nonnegative definite $l \times l$ matrix with entries in $L^2([0, T], \mathbb{R})$. It is worth emphasizing the role of the matrix I. The entries of the matrix I represent the information of past trade of the market, and they act on equilibrium solutions on the current time. Then, the new aggregate excess demand function takes into account a memory expressed in an integral form, and it can also be interpreted as adjustment factor of prices. The meaning of the integral term is that it expresses the equilibrium distribution in which the commodity price incur at time t and, hence, the effect of the previous situation on the present one. Moreover, the memory term is strictly connected with the concept of delay: the integral term represents the delay of the equilibrium solution, due to the previous equilibrium state.

We suppose that Z satisfies the Walras' law:

$$\big\langle Z\big(t, p(t)\big), p(t) - \underline{p}(t) \big\rangle = 0 \quad \text{a.e. in } [0, T] \ \forall p \in S. \tag{1}$$

We require that the following growth condition holds: there exist $B \in L^2([0, T])$ and $A \in L^\infty([0, T])$ such that for $t \in [0, T]$:

$$\big\| z\big(t, p(t)\big) \big\| \leq A(t) \big\| p(t) \big\| + B(t) \quad \forall p(t) \in S(t),$$

where

$$S(t) := \left\{ p(t) \in \mathbb{R}^l_+ : \ p(t) \geq \underline{p}^j(t), \ \sum_{j=1}^{l} p^j(t) = 1 \right\}.$$

Taking into account that matrix entries are in L^2, it is easy to prove that $\int_0^t I(t-s)p(s)\,ds$ is in L^2. The definition of a Walrasian equilibrium with memory term is now stated:

Definition 2.1 A price vector $\widehat{p}\in S$ is a dynamic Walrasian equilibrium vector for a pure exchange model with memory term if and only if

$$Z\big(t,\widehat{p}(t)\big)\le 0 \quad \text{a.e. in } [0,T].$$

We observe that, since Z satisfies the Walras' law, the equilibrium condition can be rewritten in the following way:

Definition 2.2 A price vector $\widehat{p}\in S$ is a dynamic Walrasian equilibrium vector for a pure exchange model with memory term if and only if a.e. in $[0,T]$,

$$Z^j\big(t,\widehat{p}(t)\big)\begin{cases}\le 0 & \text{if } \widehat{p}^j(t)=\underline{p}^j(t),\\ =0 & \text{if } \widehat{p}^j(t)>\underline{p}^j(t).\end{cases}$$

Now, we can characterize the equilibrium as a solution to an evolutionary variational inequality:

Theorem 2.1 *A price vector $\widehat{p}\in S$ is a dynamic Walrasian equilibrium with memory term if and only if $\widehat{p}$ is a solution to the following evolutionary variational inequality*:

$$\big\langle Z(\widehat{p}),p-\widehat{p}\big\rangle_L\le 0 \quad \forall p\in S. \tag{2}$$

Proof We observe that, by Walras' law,

$$\big\langle Z(\widehat{p}),p-\widehat{p}\big\rangle_L=-\big\langle Z(\widehat{p}),\widehat{p}-\underline{p}\big\rangle_L+\big\langle Z(\widehat{p}),p-\underline{p}\big\rangle_L=\big\langle Z(\widehat{p}),p-\underline{p}\big\rangle_L;$$

then the evolutionary variational inequality (2) is equivalent to

$$\big\langle Z(\widehat{p}),p-\underline{p}\big\rangle_L\le 0. \tag{3}$$

If $\widehat{p}$ is a dynamic Walrasian equilibrium with memory term, i.e., $Z^j(t,p(t))\le 0$ and $\widehat{p}^j(t)>\underline{p}^j(t)$ a.e. in $[0,T]$, then $\widehat{p}$ is a solution to evolutionary variational inequality (3) (or (2)).

Vice-versa, let $\widehat{p}$ be a solution to the evolutionary variational inequality (2); we prove that $\widehat{p}$ is a dynamic Walrasian equilibrium. We suppose ad absurdum that there exist $E\subset[0,T]$ and $j\in J$ such that $\mu(E)>0$ and

$$Z^j\big(t,\widehat{p}(t)\big)>0 \quad \text{for all } t\in E. \tag{4}$$

We set:

$$J_-=\big\{j\in J:\ Z^j\big(t,\widehat{p}(t)\big)\le 0 \text{ in } E\big\},\qquad J_+=\big\{j\in J:\ Z^j\big(t,\widehat{p}(t)\big)>0 \text{ in } E\big\}.$$

Since J_-, $J_+ \neq \emptyset$, it results $1 \leq |J_-| \leq 1-l$ and $1 \leq |J_+| \leq 1-l$. We consider:

$$\widetilde{p}^j(t) = \begin{cases} \widehat{p}^j(t) & \text{in } [0,T] \setminus E \ \forall j \in J, \\ \varepsilon & \text{in } E \ \forall j \in J_-, \\ (1 - |J_-|\varepsilon)\frac{1}{|J_+|} & \text{in } E \ \forall j \in J_+, \end{cases}$$

with $0 < \varepsilon < \min\{\frac{1}{|J_-|}, -\frac{A}{B}\}$, where

$$A = \int_E \sum_{j \in J_+} Z^j\big(t, \widehat{p}(t)\big)\left(\frac{1}{|J_+|} - \underline{p}^j(t)\right) dt,$$

$$B = \int_E \left(\sum_{j \in J_-} Z^j\big(t, \widehat{p}(t)\big) - \sum_{j \in J_+} \frac{Z^j(t, \widehat{p}(t))|J_-|}{|J_+|}\right) dt.$$

We observe that, since $\underline{p}^j(t) < \frac{1}{l}$ for all $j = 1, \ldots, l$ and $|J_+| \leq 1-l$, we have $A > 0$ and $B < 0$. It results $\widetilde{p} \in S$. We replace $\widetilde{p}$ in (3):

$$\begin{aligned}
\big\langle Z(\widehat{p}), \widetilde{p} - \underline{p}\big\rangle_L &= \int_{[0,T]\setminus E} \sum_{j \in J} Z^j\big(t, \widehat{p}(t)\big)\big(\widehat{p}^j(t) - \underline{p}^j(t)\big)\, dt \\
&\quad + \int_E \sum_{j \in J_-} Z^j\big(t, \widehat{p}(t)\big)\big(\varepsilon - \underline{p}^j(t)\big)\, dt \\
&\quad + \int_E \sum_{j \in J_+} Z^j\big(t, \widehat{p}(t)\big)\left(\big(1 - |J_-|\varepsilon\big)\frac{1}{|J_+|} - \underline{p}^j(t)\right) dt \\
&= \varepsilon \int_E \left(\sum_{j \in J_-} Z^j\big(t, \widehat{p}(t)\big) - \sum_{j \in J_+} Z^j\big(t, \widehat{p}(t)\big)\frac{|J_-|}{|J_+|}\right) dt \\
&\quad + \int_E \sum_{j \in J_+} Z^j\big(t, \widehat{p}(t)\big)\frac{1}{|J_+|}\, dt \\
&\quad - \int_E \sum_{j \in J_-} Z^j\big(t, \widehat{p}(t)\big)\underline{p}^j(t)\, dt - \int_E \sum_{j \in J_+} Z^j\big(t, \widehat{p}(t)\big)\underline{p}^j(t)\, dt \\
&= \varepsilon \int_E \left(\sum_{j \in J_-} Z^j\big(t, \widehat{p}(t)\big) - \sum_{j \in J_+} Z^j\big(t, \widehat{p}(t)\big)\frac{|J_-|}{|J_+|}\right) dt \\
&\quad - \int_E \sum_{j \in J_-} Z^j\big(t, \widehat{p}(t)\big)\underline{p}^j(t)\, dt \\
&\quad + \int_E \sum_{j \in J_+} Z^j\big(t, \widehat{p}(t)\big)\left(\frac{1}{|J_+|} - \underline{p}^j(t)\right) dt \leq 0.
\end{aligned}$$

Since

$$-\int_E \sum_{j\in J_-} Z^j\big(t,\widehat{p}(t)\big)\underline{p}^j(t)\,dt \geq 0,$$

replacing A and B, one has:

$$\varepsilon B + A \leq 0.$$

But, by choosing ε this results in that this estimate is false. Then (4) cannot occur, and we obtain $Z^j(t,\widehat{p}(t)) \leq 0$ a.e. in $[0,T]$ for all $j\in J$.

Then $\widehat{p}$, solution to the evolutionary variational inequality (2), is a dynamic Walrasian equilibrium whit memory term. □

3 Existence and Lipschitz Continuity Results

Thanks to the characterization of the dynamic Walrasian price equilibrium with memory term in terms of an evolutionary variational inequality, we can apply the variational inequality theory to give the existence and Lipschitz continuity of equilibrium solution. To this aim, it is worth noting that the evolutionary variational inequality problem (2) is equivalent to the following pointwise formulation:

$$\big\langle Z\big(t,\widehat{p}(t)\big), p(t)-\widehat{p}(t)\big\rangle \leq 0 \quad \forall p\in S(t),\ \forall t\in[0,T]. \tag{5}$$

Theorem 3.1 (Existence) *Let $Z: S\to\mathbb{R}^n$ be a continuous and strictly monotone function on S. Then there exists at least one dynamic Walrasian equilibrium with memory term.*

Proof We fix $t\in[0,T]$ and consider the pointwise problem (5). Z being continuous and strictly monotone and $S(t)$ compact, there exists a unique solution of variational inequality (5). Following [8] (see also [13] and [14]), the solution $\widehat{p}(t)$ is continuous in $[0,T]$, then $\widehat{p}(t)$ is also a solution to evolutionary variational inequality (2). □

Now we can adapt to our variational inequality problem a Lipschitz continuity result given in [6]. To this aim, we remark that in our case the following assumption, required in [6], holds:

$$\text{there exists } \kappa\geq 0 \text{ such that, for } t_1,t_2\in[0,T],$$
$$\big\| P_{S(t_2)}(q) - P_{S(t_1)}(q)\big\| \leq \kappa|t_2-t_1| \quad \forall q\in\mathbb{R}^n,$$

where $P_{S(t)}(q) = \arg\min_{x\in S(t)}|q-x|$, $t\in[0,T]$, denotes the projection onto the set $S(t)$.

Theorem 3.2 *Let the following assumptions be satisfied*:

(a) *z is strongly monotone, i.e., there exists $\alpha>0$ such that, for $t\in[0,T]$,*

$$\big\langle z(t,p_1)-z(t,p_2), p_1-p_2\big\rangle \geq \alpha\|p_1-p_2\|^2 \quad \forall p_1,p_2\in\mathbb{R}^n;$$

(b) *z is Lipschitz continuous with respect to p, i.e., there exists $\beta > 0$ such that, for $t \in [0, T]$,*

$$\left\| z(t, p_1) - z(t, p_2) \right\| \leq \beta \| p_1 - p_2 \| \quad \forall p_1, p_2 \in \mathbb{R}^n;$$

(c) *z is Lipschitz continuous with respect to t, i.e., there exists $M > 0$ such that, for $t_1, t_2 \in [0, T]$,*

$$\left\| z(t_2, p) - z(t_1, p) \right\| \leq M \| p \| |t_2 - t_1| \quad \forall p \in \mathbb{R}^n;$$

(d) *I is Lipschitz continuous on $[0, T]$, namely there exists $L > 0$ such that, for $t_1, t_2 \in [0, T]$,*

$$\left\| I(t_2) - I(t_1) \right\| \leq L |t_2 - t_1|.$$

Moreover, I is nonnegative definite for any $t \in [0, T]$. Then, the unique solution $\widehat{p}(t)$, $t \in [0, T]$, to (5) is Lipschitz continuous on $[0, T]$. Moreover, for any couple $t_1, t_2 \in [0, T]$, $t_1 \neq t_2$, the following estimate holds:

$$\frac{\|\widehat{p}(t_2) - \widehat{p}(t_1)\|^2}{|t_2 - t_1|^2} \leq \gamma \left(\|\widehat{p}\|^2_{C^0([0,T];\mathbb{R}^n)} + \sup_{\substack{t_1, t_2 \in [0,T] \\ t_1 \neq t_2}} \left\| \frac{P_{S(t_2)}(q) - P_{S(t_1)}(q)}{t_2 - t_1} \right\|^2 \right) \tag{6}$$

with $\gamma = \gamma(\alpha, \beta, M, T, L, \|I\|_{C^0([0,T];\mathbb{R}^{n \times n})})$.

Remark 3.1 The unique solution $\widehat{p}$ to problem (5) belongs to $W^{1,\infty}([0, T])$. In fact, from the Lipschitz continuity of the solution and, by Rademacher's theorem (see [19]), we immediately obtain the existence of bounded solution derivatives almost everywhere in $[0, T]$. In addition, by applying Sobolev embedding theorems, it follows that $W^{1,\infty}([0, T])$ can be compactly embedded in $C([0, T])$.

4 Lagrangean Theory

Now, our purpose is to characterize a dynamic Walrasian price equilibrium with memory term by means of the Lagrangean multipliers. In particular, we will prove the following result:

Theorem 4.1 *$\widehat{p} \in S$ is a solution to the variational problem (2) if and only if there exist $\widehat{\alpha} \in L^2([0, T], \mathbb{R}^l)$ and $\widehat{\beta} \in L^2([0, T], \mathbb{R})$ such that a.e. in $[0, T]$:*

(i) $\widehat{\alpha}^j(t) \geq 0, \ \forall j = 1, \ldots, l;$
(ii) $\widehat{\alpha}^j(t)(\widehat{p}^j(t) - \underline{p}^j(t)) = 0 \quad \forall j = 1, .., l;$
(iii) $\begin{cases} z(t, \widehat{p}(t)) + \int_0^t I(t-s)\widehat{p}(s)\, ds = -\widehat{\alpha}(t), \\ \widehat{\beta}(t) = 0. \end{cases}$

In order to prove Theorem 4.1, it is necessary to introduce some results (see, e.g., [20–26]). First of all, we recall the following definitions.

Definition 4.1 Let C be a nonempty subset of a real linear space X. Then, the set

$$cone(C) := \{\lambda x : x \in C, \lambda \in \mathbb{R}_+\}$$

is called the cone generated by C.

Definition 4.2 Let X denote a real normed space, and $C \subseteq X$. The set

$$T_C(x) := \Big\{h \in X : h = \lim_{n\to\infty} \lambda_n(x_n - x),\ \lambda_n \in \mathbb{R} \text{ and } \lambda_n > 0,\ \forall n \in \mathbb{N},$$
$$x_n \in C\ \forall n \in \mathbb{N} \text{ and } \lim_{n\to\infty} x_n = x\Big\}$$

is called the tangent cone to C at x.

Definition 4.3 Let C be a convex subset of a real linear space X. The quasi-relative interior of C, denoted by $qriC$, is the set of those $x \in C$ for which $T_C(x)$ is a linear subspace of X.

Now the following assumptions are made:

$$\begin{cases} \text{let } X \text{ be a linear topological space, and } Z \text{ be a real normed space;} \\ \text{let } Y \text{ be a real normed space ordered by a convex cone } C; \\ \text{let } S \text{ be a convex and nonempty subset of } X; \\ \text{let } f : S \to \mathbb{R},\ g : S \to Y, \text{ and } h : X \to Z \text{ be given;} \\ \text{let the constraint set be given as } \mathbb{K} = \{x \in S :\ g(x) \in -C,\ h(x) = \theta_Z\} \neq \emptyset. \end{cases} \tag{7}$$

Under these assumptions, we consider the constraint optimization problem

$$\min_{x\in\mathbb{K}} f(x), \tag{8}$$

and we associate it with the Lagrangean functional

$$\mathcal{L} : S \times C^* \times Z^* \to \mathbb{R},$$
$$\mathcal{L}(x, u, v) := f(x) + \langle u, g(x)\rangle + \langle v, h(x)\rangle,$$

where $C^* = \{u \in Y^* :\ \langle u, y\rangle \geq 0\ \forall y \in C\}$ is the dual cone of C, and Z^* is the dual space of Z. We consider the dual problem

$$\max_{(u,v)\in C^*\times Z^*} \inf_{x\in S} \mathcal{L}(x, u, v). \tag{9}$$

Definition 4.4 Given three functions f, g, h and a set $\mathbb{K}$ as in (7), we say that Assumption S is fulfilled at a point $x_0 \in \mathbb{K}$ if and only if

$$\text{(Assumption S)} \quad T_{\widetilde{M}}(0, \theta_Y, \theta_Z) \cap\,]-\infty, 0[\, \times \{\theta_Y\} \times \{\theta_Z\} = \emptyset, \tag{10}$$

where

$$\widetilde{M} = \big\{\big(f(x) - f(x_0) + \alpha, g(x) + y, h(x)\big) :\ x \in S \setminus \mathbb{K},\ \alpha \geq 0,\ y \in C\big\}.$$

Theorem 4.2 *Assume that the functions $f : S \to \mathbb{R}$, $g : S \to Y$ are convex and that $h : S \to Z$ is an affine-linear mapping. Assume that Assumption S is fulfilled at the optimization solution $x_0 \in \mathbb{K}$ to* (8). *Then also problem* (9) *is solvable, and if $\overline{u} \in C^*$, $\overline{v} \in Z^*$ are the extremal points of problem* (9), *then*

$$\langle \overline{u}, g(x_0) \rangle = 0,$$

and the extremes of the two problems are equal.

Proof See, e.g., [24]. □

Using Theorem 4.2, let us characterize a solution of constraint optimization problem (8) as a saddle point of Lagrangean functional.

Theorem 4.3 *Let the assumptions of Theorem* 4.2 *be satisfied. Then $x_0 \in \mathbb{K}$ is a minimal solution to problem* (8) *if and only if there exist $\overline{u} \in C^*$ and $\overline{v} \in Z^*$ such that $(x_0, \overline{u}, \overline{v})$ is a saddle point of the Lagrangean functional, namely*:

$$\mathcal{L}(x_0, u, v) \le \mathcal{L}(x_0, \overline{u}, \overline{v}) \le \mathcal{L}(x, \overline{u}, \overline{v}) \quad \forall x \in S,\ u \in C^*, v \in Z^*,$$

and, moreover,

$$\langle \overline{u}, g(x_0) \rangle = 0.$$

Proof See, e.g., [24]. □

Now, Theorem 4.1 can be proven.

Proof of Theorem 4.1 Let $\widehat{p} \in S$ be a solution to evolutionary variational inequality (2); then $\widehat{p}$ is a solution to the optimization problem

$$\min_{p \in S} \langle Z(\widehat{p}), \widehat{p} - p \rangle_L = 0. \tag{11}$$

The minimal problem (11) is a constrained optimization problem of the kind (8); in fact, we have:

$$\begin{gathered}
X = Y = L, \qquad Z = L^2([0,T], \mathbb{R}), \\
C = C^* = \{\alpha \in L : \alpha(t) \ge 0 \text{ a.e. in } [0,T]\}, \\
f : L \to \mathbb{R}, \qquad f(p) = \langle Z(\widehat{p}), \widehat{p} - p \rangle_L, \\
g : L \to L, \qquad g(p) = \underline{p} - p \in L, \quad \text{and} \\
h : L \to Z, \qquad h(p) = 1 - \sum_{j=1}^{l} p^j \in Z, \\
\mathbb{K} = \{p \in L : g(p) \in -C,\ h(p) = \theta_Z\}.
\end{gathered}$$

We associate to the problem (11) the Lagrangean functional

$$\mathcal{L} : L \times C^* \times Z^* \to \mathbb{R}$$

such that

$$\begin{aligned}
\mathcal{L}(p, \alpha, \beta) &= \big\langle Z(\widehat{p}), \widehat{p} - p \big\rangle_L - \langle \alpha, p - \underline{p} \rangle_L - \left\langle \beta, \sum_{j=1}^{l} p^j - 1 \right\rangle_Z \\
&= \int_0^T \big\langle z\big(t, \widehat{p}(t)\big), \widehat{p}(t) - p(t) \big\rangle \, dt \\
&\quad + \int_0^T \left\langle \int_0^t I(t-s)\widehat{p}(s)\, ds, \widehat{p}(t) - p(t) \right\rangle dt \\
&\quad - \int_0^T \big\langle \alpha(t), p(t) - \underline{p}(t) \big\rangle \, dt - \int_0^T \left\langle \beta(t), \sum_{j=1}^{l} p^j(t) - 1 \right\rangle dt.
\end{aligned}$$

We consider the dual problem

$$\max_{(\alpha,\beta) \in C^* \times Z^*} \inf_{p \in S} \mathcal{L}(p, \alpha, \beta).$$

We verify that the assumptions of Theorem 4.3 are satisfied. S is a convex set, f, g are convex, and h is an affine-linear mapping; we have to verify Assumption S. We have:

$$\widetilde{M} = \left\{ \left(f(p) - f(\widehat{p}) + k, \underline{p} - p + y, \sum_{j=1}^{l} p^j - 1 \right) : p \in L \setminus S,\ k \geq 0,\ \ y \in C \right\}$$

and

$$\begin{aligned}
&T_{\widetilde{M}}(0, \theta_L, \theta_Z) \\
&\quad = \Bigg\{ y : y = \lim_{n \to +\infty} \lambda_n \left(f(p_n) - f(\widehat{p}) + k_n, \underline{p} - p_n + y_n, \sum_{j=1}^{l} p_n^j(t) - 1 \right) \\
&\qquad \text{with } \lambda_n > 0,\ \lim_{n \to +\infty} \big(f(p_n) - f(\widehat{p}) + k_n \big) = 0, \\
&\qquad \lim_{n \to +\infty} (\underline{p} - p_n + y_n) = \theta_L,\ \lim_{n \to +\infty} \sum_{j=1}^{l} p_n^j - 1 = \theta_Z,\ p_n \in L \setminus S, \\
&\qquad k_n \geq 0,\ y_n \in C \Bigg\}.
\end{aligned}$$

In order to achieve Assumption S (10), we prove that if (l,θ_L,θ_Z) belongs to $T_{\widetilde{M}}(0,\theta_L,\theta_Z)$, then $l\geq 0$. We have:

$$\begin{aligned}\lambda_n\big(f(p_n)-f(\widehat{p})+k_n\big)&=\lambda_n\big(\langle Z(\widehat{p}),\widehat{p}-p_n\rangle_L+k_n\big)\\&=\lambda_n\langle Z(\widehat{p}),\widehat{p}-\underline{p}\rangle_L+\langle Z(\widehat{p}),\lambda_n(\underline{p}-p_n+y_n)\rangle_L\\&\quad+\langle Z(\widehat{p}),-\lambda_n y_n\rangle_L+\lambda_n k_n\\&\geq\langle Z(\widehat{p}),\lambda_n(\underline{p}-p_n+y_n)\rangle_L,\end{aligned}$$

because

$$\langle Z(\widehat{p}),\widehat{p}-\underline{p}\rangle_L=0,\qquad Z(\widehat{p})\leq 0,\qquad \lambda_n y_n\geq 0;$$

passing to the limit, one has

$$l=\lim_{n\to+\infty}\lambda_n\big(f(p_n)-f(\widehat{p})+k_n\big)\geq\lim_{n\to+\infty}\langle Z(\widehat{p}),\lambda_n(\underline{p}-p_n+y_n)\rangle_L=0.$$

Hence assumption (10) holds.

By Theorem 4.3 there exists $(\widehat{\alpha},\widehat{\beta})\in C^*\times Z^*$ such that $(\widehat{p},\widehat{\alpha},\widehat{\beta})$ is a saddle point of the Lagrangean functional $\mathcal{L}$:

$$\mathcal{L}(\widehat{p},\alpha,\beta)\leq\mathcal{L}(\widehat{p},\widehat{\alpha},\widehat{\beta})\leq\mathcal{L}(p,\widehat{\alpha},\widehat{\beta})\quad\forall(\alpha,\beta)\in C^*\times Z^*,\ p\in L,\tag{12}$$

and furthermore:

$$\langle\widehat{\alpha},\widehat{p}-\underline{p}\rangle_L=0.\tag{13}$$

From (13), since $\widehat{\alpha}\in C^*$ and $\widehat{p}\in S$, we derive

$$\widehat{\alpha}^j(t)\big(\widehat{p}^j(t)-\underline{p}^j(t)\big)=0\quad\text{a.e. in }[0,T]\ \forall j=1,\dots,l\tag{14}$$

and $\mathcal{L}(\widehat{p},\widehat{\alpha},\widehat{\beta})=0$. From right-hand side of (12) we have:

$$\mathcal{L}(p,\widehat{\alpha},\widehat{\beta})\geq 0\quad\forall p\in L.\tag{15}$$

Assuming in (15) $p_1=\widehat{p}+\varepsilon$ and $p_2=\widehat{p}-\varepsilon$ $\forall\varepsilon\in\mathcal{D}([0,T])$, we have:

$$\begin{aligned}\mathcal{L}(p_1,\widehat{\alpha},\widehat{\beta})&=\langle Z(\widehat{p}),\widehat{p}-p_1\rangle_L-\langle\widehat{\alpha},p_1-\underline{p}\rangle_L-\left\langle\widehat{\beta},\sum_{j=1}^l p_1^j-1\right\rangle\\&=\langle Z(\widehat{p}),-\varepsilon\rangle_L-\langle\widehat{\alpha},\widehat{p}-\underline{p}\rangle_L-\langle\widehat{\alpha},\varepsilon\rangle_L\\&\quad-\left\langle\widehat{\beta},\sum_{j=1}^l\widehat{p}^j-1\right\rangle-\left\langle\widehat{\beta},\sum_{j=1}^l\varepsilon^j\right\rangle\\&=-\langle Z(\widehat{p})+\widehat{\alpha},\varepsilon\rangle_L-\left\langle\widehat{\beta},\sum_{j=1}^l\varepsilon^j\right\rangle\geq 0,\end{aligned}\tag{16}$$

$$\mathcal{L}\big(p^2, \widehat{\alpha}, \widehat{\beta}\big) = \langle Z(\widehat{p}) + \widehat{\alpha}, \varepsilon\rangle_L + \left\langle \widehat{\beta}, \sum_{j=1}^{l} \varepsilon^j \right\rangle \geq 0. \tag{17}$$

Hence, from (16) and (17) we have that, for all $\varepsilon = \{\varepsilon^j\}_{j=1,\dots,l} \in \mathcal{D}([0,T], \mathbb{R}^l)$,

$$\langle Z(\widehat{p}) + \widehat{\alpha}, \varepsilon\rangle_L + \left\langle \widehat{\beta}, \sum_{j=1}^{l} \varepsilon^j \right\rangle_Z = \int_0^T \sum_{j=1}^{l} \big(Z^j\big(\widehat{p}(t)\big) + \widehat{\alpha}^j(t) + \widehat{\beta}(t)\big)\varepsilon^j(t)\,dt = 0.$$

Taking

$$\varepsilon^j \begin{cases} = 0 & \text{for } j = 1, \dots, l,\ j \neq h, \\ \neq 0 & \text{for } j = h, \end{cases}$$

we get:

$$\text{for all } h = 1, \dots, l, \quad \widehat{\alpha}^h(t) = -Z^h\big(\widehat{p}(t)\big) - \widehat{\beta}(t) \quad \text{a.e. in } [0,T]. \tag{18}$$

By (14), (18), and Walras' law (1), we have a.e. in $[0,T]$:

$$\begin{aligned} 0 &= \sum_{j=1}^{l} \widehat{\alpha}^j(t)\big(\widehat{p}^j(t) - \underline{p}^j(t)\big) = \sum_{j=1}^{l} \big(-Z^j\big(t, \widehat{p}(t)\big) - \widehat{\beta}(t)\big)\big(\widehat{p}^j(t) - \underline{p}^j(t)\big) \\ &= \big\langle -Z\big(t, \widehat{p}(t)\big), \widehat{p}(t) - \underline{p}(t)\big\rangle - \sum_{j=1}^{l} \widehat{\beta}(t)\big(\widehat{p}^j(t) - \underline{p}^j(t)\big) \\ &= -\widehat{\beta}(t) \sum_{j=1}^{l} \big(\widehat{p}^j(t) - \underline{p}^j(t)\big). \end{aligned}$$

From the last equality it follows that $\widehat{\beta}(t) = 0$ a.e. in $[0,T]$. In fact, if there exists $E \subset [0,T]$ with $\mu(E) > 0$ such that $\widehat{\beta}(t) \neq 0$ for all $t \in E$, we have $\sum_{j=1}^{l}(\widehat{p}^j(t) - \underline{p}^j(t)) = 0$ in E; then, since $\widehat{p} \in S$, by assumption on $\underline{p}$ we get:

$$1 = \sum_{j=1}^{l} \widehat{p}^j(t) = \sum_{j=1}^{l} \underline{p}^j(t) < 1 \quad \text{in } E.$$

Then $\widehat{\beta}(t) = 0$ and $\widehat{\alpha}(t) = -Z(\widehat{p}(t))$ a.e. in $[0,T]$.

Conversely, if there exist $\widehat{p} \in S$, $\widehat{\alpha} \in L^2([0,T], \mathbb{R}^l)$, and $\widehat{\beta} \in L^2([0,T], \mathbb{R})$ that satisfy conditions (i), (ii), and (iii), then $(\widehat{p}, \widehat{\alpha}, \widehat{\beta})$ is a saddle point of the Lagrangean functional $\mathcal{L}$. Then, from Theorem 4.3 it follows that $\widehat{p}$ is a solution to the evolutionary variational inequality (2). □

Remark 4.1 The dynamic Walrasian price equilibrium problem with memory term can be expressed in the following way:

$$\begin{cases} \mathcal{M}(\widehat{p})\mathcal{B}(\widehat{p}) = 0, \\ \mathcal{M}(\widehat{p}) \geq 0, \quad \mathcal{B}(\widehat{p}) \geq 0, \quad \widehat{p} \in S, \end{cases} \tag{19}$$

where $S = L$, and $\mathcal{M}$, $\mathcal{B}$ are the operators

$$\mathcal{M}(p) = -z\big(t, p(t)\big) - \int_0^t I(t-s)p(s)\,ds \quad \text{and} \quad \mathcal{B}(p) = p - \underline{p}.$$

Many equilibrium problems arising from various fields of science may be expressed in a unified way under general conditions (19) (see, e.g., [23]).

5 Example

During the trading session represented by the time interval $[0, T]$, we consider an economy with two commodities, with typical commodity denoted by j, and two agents, with typical agent denoted by a. The typical agent has a demand function

$$x_a^j(t) = \frac{\gamma(t)\sum_{j=1}^2 p_j(t)e_a^j(t)}{p_j(t)}, \quad j = 1, 2,$$

with $e_a^j(t)$, $j = 1, 2$, is the initial endowment, and $\gamma(t) \in L^2([0, T])$, $\gamma(t) \geq 0$ a.e. in $[0, T]$. The excess aggregate demand is then given by

$$z^j\big(p(t)\big) = \sum_{a=1}^2 x_a^j(t) - \sum_{a=1}^2 e_a^j(t), \quad j = 1, 2.$$

Now, we want to focus on the price formation for informed agents, where information is meant as memory of past trade and is represented as an integral term that leads to the price adjustment. Thus, in order to study the effective behavior of the excess aggregate demand function, we introduce a memory term of the form

$$\int_0^t I(t-s)p(s)\,ds = \beta(t)\int_0^t \mathrm{e}^{-\alpha(t-s)}p(s)\,ds,$$

namely an exponentially weighted average of past prices, given by an exponentially distributed adjustment. Here $\alpha(t), \beta(t) \in L^2([0, T])$ represent the duration and intensity of memory, respectively.

Exponential decay, the decrease at a rate proportional to its value, is a feature that appears in many fields to describe the decay of a perturbation. In such a way it is quite natural to consider a decay of memory of an exponential form. Of course, other functions could be considered.

The effective excess aggregate demand is

$$Z^j\big(t, p(t)\big) = z^j\big(t, p(t)\big) + \beta(t) \int_0^t \mathrm{e}^{-\alpha(t-s)} p^j(s)\, ds, \quad j = 1, 2,$$

where $\alpha(t), \beta(t) \in L^2([0, T])$ and $\alpha(t), \beta(t) > 0$ a.e. in $[0, T]$.
We set $A = \sum_{a=1}^2 e_a^1(t)$, $B = \sum_{a=1}^2 e_a^2(t)$, and $D = \frac{1}{\alpha}(\mathrm{e}^{-\alpha t} - 1)$, where we suppose that $A - B - \beta D > 0$ and $2B + \beta D > 0$, and we fix the minimum prices

$$\underline{p}^1(t) = \frac{-2B - \beta D + \sqrt{(\beta D)^2 + 4AB}}{2(A - B - 3\beta D)},$$
$$\underline{p}^2(t) = \frac{2A - \beta D + \sqrt{(\beta D)^2 + 4AB}}{2(A - B - 3\beta D)}.$$

Thus, we are led to consider the following variational inequality:

$$\int_0^T \big[Z^1\big(t, \widehat{p}(t)\big)\big(p^1(t) - \widehat{p}^1(t)\big) + Z^2\big(t, \widehat{p}(t)\big)\big(p^2(t) - \widehat{p}^2(t)\big)\big]\, dt \le 0 \quad \forall p \in S,$$

where

$$S = \big\{p \in L^2\big([0, T], \mathbb{R}^2\big) : \underline{p}(t) \le p(t), p^1(t) + p^2(t) = 1 \text{ a.e. in } [0, T]\big\}.$$

In virtue of the continuity of solutions, we are entitled to solve

$$Z^1\big(t, \widehat{p}(t)\big)\big(p^1(t) - \widehat{p}^1(t)\big) + Z^2\big(t, \widehat{p}(t)\big)\big(p^2(t) - \widehat{p}^2(t)\big) \le 0$$
$$\forall p \in S(t),\ \forall t \in [0, T].$$

By applying the direct method we have to equate the aggregate excess demands of consumers

$$Z^1\big(t, \widehat{p}(t)\big) - Z^2\big(t, \widehat{p}(t)\big) = 0$$

with $\widehat{p}^2(t) = 1 - \widehat{p}^1(t)$ and $\underline{p}^1(t) < \widehat{p}^1(t) < 1$, namely

$$\begin{aligned} 0 &= \sum_{a=1}^2 \Big[\big(x_a^1(t) - e_a^1(t)\big) - \big(x_a^2(t) - e_a^2(t)\big) \\ &\quad + \beta(t) \int_0^t \mathrm{e}^{-\alpha(t-s)} \widehat{p}^1(s)\, ds - \beta(t) \int_0^t \mathrm{e}^{-\alpha(t-s)} \widehat{p}^2(s)\, ds\Big] \\ &= \sum_{a=1}^2 \Big[\Big(\frac{\gamma(t) \sum_{j=1}^2 \widehat{p}^j(t) e_a^j(t)}{\widehat{p}^1(t)} - e_a^1(t)\Big) - \Big(\frac{\gamma(t) \sum_{j=1}^2 \widehat{p}^j(t) e_a^j(t)}{1 - \widehat{p}^1(t)} - e_a^2(t)\Big) \\ &\quad + \beta(t) \int_0^t \mathrm{e}^{-\alpha(t-s)} \widehat{p}^1(s)\, ds - \beta(t) \int_0^t \mathrm{e}^{-\alpha(t-s)} \big(1 - \widehat{p}^1(s)\big)\, ds\Big]. \end{aligned}$$

After some steps, we find

$$\sum_{a=1}^{2}\big(2\gamma(t)-1\big)\big(e_a^1(t)-e_a^2(t)\big)=\gamma(t)\sum_{a=1}^{2}\frac{e_a^1(t)}{1-\widehat{p}^1(t)}-\gamma(t)\sum_{a=1}^{2}\frac{e_a^2(t)}{\widehat{p}^1(t)} \\ -2\beta\int_0^t \mathrm{e}^{-\alpha(t-s)}\widehat{p}^1(s)\,ds+\frac{\beta}{\alpha}\big(\mathrm{e}^{-\alpha t}-1\big).$$

Setting $\gamma(t)=1$, we are led to solve the equation

$$\bigg(A-B+2\beta\int_0^t \mathrm{e}^{-\alpha(t-s)}\widehat{p}^1(s)\,ds-\frac{\beta}{\alpha}\big(\mathrm{e}^{-\alpha t}-1\big)\bigg)\big(\widehat{p}^1(t)\big)^2 \\ +\bigg(2B-2\beta\int_0^t \mathrm{e}^{-\alpha(t-s)}\widehat{p}^1(s)\,ds+\frac{\beta}{\alpha}\big(\mathrm{e}^{-\alpha t}-1\big)\bigg)\widehat{p}^1(t)-B=0,$$

whose solution is

$$\widehat{p}^1(t)=\frac{-2B+2\beta C-\beta D+\sqrt{(2\beta C-\beta D)^2+4AB}}{2(A-B+2\beta C-\beta D)},$$

where $C=\int_0^t \mathrm{e}^{-\alpha(t-s)}\widehat{p}^1(s)\,ds$ is in turn a solution to the equation

$$16\beta^2C^4+\big(16\beta A-32\beta^2D-16\beta B\big)C^3 \\ +\big(20\beta^2D^2-16\beta AD-8AB+4B^2+32\beta BD+4A^2\big)C^2 \\ +\big(-20\beta BD^2-8B^2D+8ABD-4\beta^2D^3+4\beta AD^2\big)C \\ +4\beta BD^3+4B^2D^2-4ABD^2=0.$$

Finally, it is easy to verify that if $A-B+2\beta C-\beta D>0$, then $\underline{p}^1(t)<\widehat{p}^1(t)<1$. Then we find the following equilibrium price for a pure exchange economy with memory term:

$$\widehat{p}^1(t)=\frac{-2B+2\beta C-\beta D+\sqrt{(2\beta C-\beta D)^2+4AB}}{2(A-B+2\beta C-\beta D)},$$

$$\widehat{p}^2(t)=\frac{2A+2\beta C-\beta D+\sqrt{(2\beta C-\beta D)^2+4AB}}{2(A-B+2\beta C-\beta D)}.$$

6 Concluding Remarks

In [7] the Walrasian price equilibrium problem was generalized to the dynamic case, and the price equilibrium was characterized as a solution to a suitable evolutionary variational inequality. In this paper, in order to have a model closer to reality, it was supposed that the excess demand function depends on the current price and on previous events of the market. Hence, a memory term was introduced in the excess demand function; it is able to represent the history of the market. Namely, we took into account of the contribution of the equilibrium price from the initial time of the observation time, and this contribution represents an adjustment factor. It has been shown that a time-dependent Walrasian price equilibrium problem with memory term can be characterized by a suitable evolutionary variational inequality. By means of this mathematical formulation the existence of the dynamic Walrasian price equilibrium solution was given, and the Lipschitz continuity of price solution was studied. Furthermore the dynamic equilibrium with memory term was characterized in terms of the Lagrange multipliers.

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