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# A Class of Chance Constrained Multi-objective Portfolio Selection Model Under Fuzzy Random Environment

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**Abstract** This paper deals with a class of chance constrained portfolio selection problems in the fuzzy random decision making system. An integrated fuzzy random portfolio selection model with a chance constraint is proposed on the basis of the mean-variance model and the safety-first model. According to different definitions of chance, we consider two types of fuzzy random portfolio selection models: one is for the optimistic investors and the other is for the pessimistic investors. In order to deal with the fuzzy random models, we develop a few theorems on the variances of fuzzy random returns and the equivalent partitions of two types of chance constraints. We then transform the fuzzy random portfolio selection models into their equivalent crisp models. We further employ the $\varepsilon$-constraint method to obtain the efficient frontier. Finally, we apply the proposed models and approaches to the Chinese stock market as an illustration.

**Keywords** Portfolio selection · Fuzzy random variable · Chance constraint · $\varepsilon$-Constraint method · Chinese stock market

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## 1 Introduction

Portfolio selection theory is concerned with how to allocate investment funds among different assets to maximize the return of a portfolio and minimize its risk. There are many sources of uncertainty including randomness and fuzziness in the stock markets. So far, most portfolio selection models are set up by using the probability theory. Recently, the existence of fuzziness in the stock markets has been gradually recognized by some researchers. Several fuzzy portfolio selection models have been proposed. However, both stochastic portfolio selection models and fuzzy portfolio selection models have been considered with only one type of uncertainty. Therefore, research on portfolio selection under both types of uncertainties is yet to be undertaken.

By assuming the future return of a security to be a random variable, the portfolio selection problem has been extensively studied using probability theory. In 1952, Markowitz [1, 2] published his pioneering work, which served as the basis for the development of the modern portfolio theory over the past several decades. The principle of diversification is the core of this method, and it still has a wide application in the financial industry. However, controlling (minimizing) the variance not only leads to low deviation from the expected return on the downside, but also on the upside. It thus may limit the potential profit as well. In 1952, Roy proposed the safety-first portfolio model, which minimizes the probability of the downside risk only [3, 4]. In addition, several researchers also analyzed the portfolio selection problem from other angles; see e.g., [5–8].

Although Markowitz ignored the expert judgments in the derivation of the efficient frontier, he emphasized the merit of combining the statistical techniques and the judgment of experts in the portfolio selection process. Yet, Markowitz neither proposed a method to tackle that issue, nor analysed the efficient set of portfolios for the investors in the presence of fuzziness or any subjective information.

In the information age, a great deal of financial information on the economy, industries, and individual companies is available to investors. In fact, investors are often faced with too many useful data that they find difficult or impossible to process. Their opinions on the information are often fuzzy. This calls for the utilization of fuzzy set theory [9, 10] in portfolio selection. By using the fuzzy approach, the expert judgement and the investor subjective opinions can be better taken into account in a portfolio selection model. Ramaswamy [11] presented a bond portfolio selection model based on the fuzzy decision theory. A given target rate of return can be achieved for an assumed market scenario through this approach. A similar approach for portfolio selection, by using the fuzzy decision theory, was proposed by Leon et al. [12]. By using the fuzzy decision principle, Ostermark [13] proposed a dynamic portfolio management model. Tanaka et al. [14] gave a special formulation of the fuzzy decision problems based on the possibility distributions. Watada [15] presented another type of portfolio selection model based on the fuzzy decision principle. The model is directly related to the mean-variance model, where the expected return and the corresponding risk are described by the logistic membership functions.

It is well-known that there are both random uncertainty and fuzzy uncertainty in the security markets; the future return of a security can be both random and fuzzy.

Randomness is the uncertainty that whether the event will happen or not, which means it is hard to predict whether the event will be happening or not. However, the states of the event are clear. So randomness can be understood as external uncertainty. Fuzziness is the uncertainty of the states, that is, the problem does not rest with the event's happening but rests with the states of the event being unclear. It leads to different people having different feelings while observing the same event. So fuzziness can encapsulate subjective uncertainty. It is thus necessary to take both types of uncertainty into account in a portfolio selection problem. Katagiri and Ishii [16] were the first ones who supposed the security's future return was a fuzzy random variable and did some research on fuzzy random assets portfolio selection problem. Xu, Zhou and Dash [17] use the $\lambda$ mean and hybrid entropy to deal with the portfolio selection with fuzzy random returns.

In this paper, we adopt the fuzzy random variable to describe the future return, and propose a new multi-objective fuzzy random portfolio selection model with chance constraints. This model enjoys the property to make satisfied personal portfolio selection according to the different investors' attitudes, and experts' opinions also could be introduced, when we use this model.

The remainder of this paper is organized as follows. In Sect. 2, we briefly review some basic portfolio selection models. In Sect. 3, we propose our models with chance constraints and we give the equivalent crisp models. We present the solution method in Sect. 4. In Sect. 5, an application to the Chinese stock market is given to illustrate how to apply the proposed models. Conclusion is given in Sect. 6.

## 2 Basic Portfolio Selection Models

In this section, we briefly review the mean-variance model and the safety-first model which are the basis of this paper.

### 2.1 Mean-Variance Model

Markowitz [1, 2] developed the famous mean-variance model for the portfolio selection problem. In the Markowitz portfolio theory, it is assumed that the future return of a portfolio is a random variable, and the variance of the return is used to measure the risk of a portfolio.

The basic idea of Markowitz [1, 2] is that investors search for the portfolio that minimizes the risk under an expected return level $R_0$, or maximize the return under a risk level $V_0$. Thus he proposed the following two equivalent models,

$$\begin{array}{ll} \min\ \boldsymbol{x}^{\mathrm{T}} V \boldsymbol{x}, \\ \text{s.t.} \begin{cases} E(\bar{\boldsymbol{r}})^{\mathrm{T}} \boldsymbol{x} \geq R_0, \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0, \quad i = 1, 2, \ldots, n, \end{cases} \end{array} \quad \text{or} \quad \begin{array}{ll} \max\ E(\bar{\boldsymbol{r}})^{\mathrm{T}} \boldsymbol{x}, \\ \text{s.t.} \begin{cases} \boldsymbol{x}^{\mathrm{T}} V \boldsymbol{x} \leq V_0, \\ \sum_{i=1}^{n} x_i = 1, \\ x_i \geq 0, \quad i = 1, 2, \ldots, n, \end{cases} \end{array} \tag{1}$$

where $n$ is the number of risky securities, $x_i$ is the proportion invested in Security $i$, random variable $\bar{r}_i$ is the future return of Security $i$, $V = [\sigma_{ij}]_{n \times n}$ is the covariance of the future returns on Securities $i$ and $j$, $R_0$ and $V_0$ are the predetermined levels of expected return and risk, respectively.

### 2.2 Safety-First Models

In 1952, Roy [3] proposed the safety-first model for the portfolio selection problem, which requires to minimize the probability that the return of the portfolio is less than a predetermined "disaster level". Based on this principle, Kataoka and Telser [4] later proposed two other forms of the safety-first model.

Altogether, there are three forms of the safety-first model based on probability theory as follows.

(1) Minimize the probability that the future return of the portfolio falls below a given return level $R$, i.e.

$$\min\ Pr\left\{\bar{\boldsymbol{r}}^{\mathrm{T}}\boldsymbol{x} < R\right\}, \tag{2}$$

where $Pr$ denotes the probability of a random event.

(2) Maximize the return level $R$ with the probability that the future value of the portfolio falls below $R$ is not greater than $\alpha$, i.e.

$$\max\ R, \quad \text{s.t. } Pr\left\{\bar{\boldsymbol{r}}^{\mathrm{T}}\boldsymbol{x} \le R\right\} \le \alpha. \tag{3}$$

(3) Maximize the expected value of future return with the probability that the future value of the portfolio falls below $R$ is not greater than $\alpha$, i.e.

$$\max\ E(\bar{\boldsymbol{r}})^{\mathrm{T}}\boldsymbol{x}, \quad \text{s.t. } Pr\left\{\bar{\boldsymbol{r}}^{\mathrm{T}}\boldsymbol{x} \le R\right\} \le \alpha. \tag{4}$$

Note that the above three models (2)–(4) are different from the mean-variance model as the measure of risk is not the same. All the above safety-first models are considered with the probability of loss, i.e., the downside risk, while Model (1) uses variance to measure risk.

Moreover, when the future returns $\tilde{r}_j$ are considered as fuzzy variables, we have another form of the safety-first model. Similar to Model (3), Inuiguchi and Tanino [18] proposed the Fractile model:

$$\max\ R, \quad \text{s.t. } Nec\left\{\tilde{\boldsymbol{r}}^{\mathrm{T}}\boldsymbol{x} \ge R\right\} \ge h_0, \tag{5}$$

where $h_0$ is the necessary confidence level, and $Nec$ denotes the necessity of a fuzzy event [19].

## 3 Model Formulation

It is well-known that both randomness and fuzziness exist in the security markets. Thus, it is natural for us to assume that the future returns of risky securities are triangular left-right type (LR) fuzzy random variables as shown in Fig. 1. The fuzzy random returns are denoted by

$$\tilde{\bar{r}}_j(\omega) = \left(r_j(\omega), \alpha_j, \beta_j\right)_{LR} = \left(r_j(\omega) - \alpha_j, r_j(\omega), r_j(\omega) + \beta_j\right), \quad \omega \in \Omega,$$

where $r_j$ is a normally distributed random variable, i.e., $r_j \sim N(\mu_j, \sigma_j^2)$; $\mu_j$ is the expected value of $r_j$; $\sigma_j^2$ is the variance of $r_j$; $\alpha_j$ and $\beta_j$ are the levels of tolerance

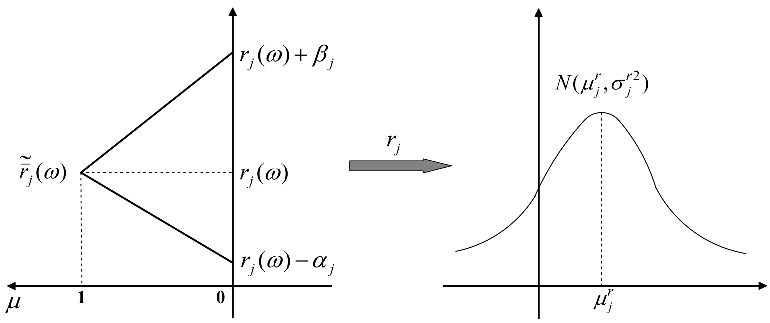



**Fig. 1** Triangular LR fuzzy random return $\tilde{\tilde{r}}_j$

of $\tilde{\tilde{r}}_j(\omega)$ according to the expert judgments; $r_j(\omega) - \alpha_j$ and $r_j(\omega) + \beta_j$ represent left-hand return and right-hand return, respectively.

### 3.1 Basic Knowledge of Fuzzy Random Variable

The fuzzy random variable used in this paper is defined on the real number set as proposed by Puri and Ralescu [20]. Let $\mathbb{R}$ denote the set of all real numbers, $\mathcal{F}_c(\mathbb{R})$ denote the set of all fuzzy variables, and $\mathcal{K}_c(\mathbb{R})$ denote all non-empty bounded close intervals.

**Definition 3.1** [20] For a probability space $(\Omega, \mathcal{F}, P)$, a mapping $\xi : \Omega \to \mathcal{F}_c(\mathbb{R})$ is a fuzzy random variable in $(\Omega, \mathcal{F}, P)$, iff for $\forall\ \alpha\ \in ]0, 1]$, the set value function $\xi_\alpha : \Omega \to \mathcal{K}_c(\mathbb{R})$

$$\xi_\alpha(\omega) := \big(\xi(\omega)\big)_\alpha := \big\{x | x \in \mathbb{R}, \mu_{\xi(\omega)}(x) \geq \alpha\big\}, \quad \forall \omega \in \Omega$$

is $\mathcal{F}$ measurable.

It has been proved [21] that if $\xi$ is an fuzzy random variable, then the left and right points of the $\alpha$-level sets of $\xi$, denoted by $(\xi)_\alpha^-$ and $(\xi)_\alpha^+$ are real-valued random variables for all $\alpha \in ]0, 1]$.

*Example 3.1* Suppose $\xi = (m, \alpha, \beta)$ is a triangular LR fuzzy variable, then $\xi$ is a fuzzy random variable if any one of $m, \alpha, \beta$ is a random variable.

#### *3.1.1 Chance of Fuzzy Random Variables*

We now give two definitions for the chance of a fuzzy random variable.

**Definition 3.2** [22] Let $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_n)$ be a fuzzy random vector in a probability space $(\Omega, \mathcal{F}, P)$, and $f_i : \mathbb{R}^{\mathrm{n}} \to \mathbb{R}, i = 1, 2, \ldots, m$ be a real-valued continuous function, then the chance measure $Ch$ of the fuzzy random event $f_i(\boldsymbol{\xi}) \leq 0, i = 1, 2, \ldots, m$ is defined as a function from ]0, 1] to ]0, 1],

$$Ch\big\{f_i(\boldsymbol{\xi}) \leq 0, i = 1, 2, \ldots, m\big\}(\alpha)$$

$$= \sup\big\{\beta|Pr\big\{\omega \in \Omega|Pos\big\{f_i\big(\boldsymbol{\xi}(\omega)\big) \leq 0, i = 1, 2, \ldots, m\big\} \geq \beta\big\} \geq \alpha\big\}, \tag{6}$$

where $\alpha, \beta \in ]0, 1]$ are predetermined confidence levels, $Pr$ is the probability of a random event, $Pos$ is the possibility of a fuzzy event [19].

From Definition 3.2, we get

$$Ch\big\{f(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\big\}(\alpha) \geq \beta \quad \Leftrightarrow \quad Pr\big\{Pos\big\{f(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\big\} \geq \beta\big\} \geq \alpha. \tag{7}$$

*Remark 3.1* $Ch\{f_i(\boldsymbol{\xi}) \leq 0, i = 1, 2, \ldots, m\}(\alpha)$ stands for the possibility that the fuzzy random event in $\{\cdot\}$ when the probability level is $\alpha$. $Ch\{f_i(\boldsymbol{\xi}) \leq 0, i = 1, 2, \ldots, m\}(\alpha) \geq \beta$ stands for the possibility that the fuzzy random event in $\{\cdot\}$ is no less than $\beta$ when the probability level is $\alpha$.

*Remark 3.2* If the fuzzy random vector $\boldsymbol{\xi}$ degenerates to a random vector, then the value of chance measure $Ch\{f_i(\boldsymbol{\xi}) \leq 0, i = 1, 2, \ldots, m\}(\alpha)$ is 0 or 1, that is,

$$Ch\big\{f_i(\boldsymbol{\xi}) \leq 0, i = 1, 2, \ldots, m\big\}(\alpha) = \begin{cases} 1, & Pr\{f_i(\boldsymbol{\xi}) \leq 0, i = 1, 2, \ldots, m\} \geq \alpha, \\ 0, & \text{otherwise}. \end{cases}$$

*Remark 3.3* If the fuzzy random vector $\boldsymbol{\xi}$ degenerates to a fuzzy vector, then the chance measure $Ch\{f_i(\boldsymbol{\xi}) \leq 0, i = 1, 2, \ldots, m\}(\alpha)$ $(\alpha > 0)$ is actually the possibility measure, that is,

$$Ch\big\{f_i(\boldsymbol{\xi}) \leq 0, i = 1, 2, \ldots, m\big\}(\alpha) = Pos\big\{f_i(\boldsymbol{\xi}) \leq 0, i = 1, 2, \ldots, m\big\}.$$

Note that we can also use the necessity $Nec$ to substitute the possibility $Pos$ in Definition 3.2 and get another definition of chance.

**Definition 3.3** [22] The chance measure $Ch$ of the fuzzy random event $f_i(\boldsymbol{\xi}) \leq 0, i = 1, 2, \ldots, m$ can also be defined as

$$\begin{aligned} &Ch\big\{f_i(\boldsymbol{\xi}) \leq 0, i = 1, 2, \ldots, m\big\}(\alpha) \\ &\quad = \sup\big\{\beta|Pr\big\{\omega \in \Omega|Nec\big\{f_i\big(\boldsymbol{\xi}(\omega)\big) \leq 0,\ i = 1, 2, \ldots, m\big\} \geq \beta\big\} \geq \alpha\big\}, \end{aligned} \tag{8}$$

where $\alpha, \beta \in ]0, 1]$ are predetermined confidence levels, $Pr$ is the probability of a random event, $Nec$ is the necessity of a fuzzy event [19].

The relationship between $Pos$ and $Nec$ satisfies the following condition [19]:

$$Pos\{A\} \geq Nec\{A\}. \tag{9}$$

Following (9), we know that if a decision maker is optimistic (pessimistic), it is better to use Definition 3.2(3.3) to measure the chance of an event under fuzzy random environment.

#### 3.1.2 Variance of Fuzzy Random Variables

Note that the variance of a fuzzy random variable reflects the spread between the fuzzy random variable and its mean, and the covariance of two fuzzy random variables reflects the degree of their linear correlation.

**Definition 3.4** [23] Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $\xi_1, \xi_2$ be fuzzy random variables that are quadratic defined in the space $(\Omega, \mathcal{F}, P)$, then the covariance of $\xi_1$ and $\xi_2$ is defined as

$$Cov(\xi_1, \xi_2) := \frac{1}{2}\int_0^1 \big[Cov\big((\xi_1)_\alpha^-, (\xi_2)_\alpha^-\big) + Cov\big((\xi_1)_\alpha^+, (\xi_1)_\alpha^+\big)\big]\, d\alpha \tag{10}$$

and variance of $\xi_1$ is defined as

$$Var(\xi_1) := Cov(\xi_1, \xi_1) = \frac{1}{2}\int_0^1 \big[Var\big((\xi_1)_\alpha^-\big) + Var\big((\xi_1)_\alpha^+\big)\big]\, d\alpha. \tag{11}$$

**Lemma 3.1** [23, 24] *Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and $\xi_1, \xi_2$ be quadratic fuzzy random variables defined in the space $(\Omega, \mathcal{F}, P)$, $\lambda, \gamma \in R$, then*

(i) $Var(\lambda\xi_1 + u) = \lambda^2 Var(\xi_1)$;
(ii) $Var(\xi_1 + \xi_2) = Var(\xi_1) + Var(\xi_2) + 2Cov(\xi_1, \xi_2)$;
(iii) $Cov(\lambda\xi_1 + u, \gamma\xi_2 + v) = \lambda\gamma Cov(\xi_1, \xi_2)$, *where $u, v$ are any fuzzy numbers*, $\lambda, \gamma \geq 0$.

### 3.2 Modelling

The mean-variance approach encourages risk diversification, while the safety-first approach discourages risk diversification sometimes. The mean-variance approach not only controls the risk on the downside, but also bounds the possible gain on the upside, while the mean safety-first approach only controls the risk on the downside. Another limitation of both approaches is that the underlying distribution of the return is not well understood, and there is no higher degree information available except for means and covariances (variances). Based on the above considerations, we propose a portfolio selection model combined by the mean-variance model and the safety-first model.

#### 3.2.1 Objectives

In this paper, we consider the following two objectives. The first one is about maximizing the expected return $R$ of a portfolio, i.e.

$$\max\ f_1 = R. \tag{12}$$

The second one is about minimizing the risk, i.e.

$$\min\ f_2 = Var\left(\sum_{j=1}^{n} \tilde{\bar{r}}_j x_j\right). \tag{13}$$

*3.2.2 Constraints*

For the expected return, it should satisfy the following chance constraint:

$$Ch\left\{\sum_{j=1}^{n} \tilde{\bar{r}}_j x_j \geq R\right\}(\gamma) \geq \delta, \tag{14}$$

where $\gamma, \delta \in ]0, 1]$ are predetermined confidence levels, *Ch* is the chance measure of the fuzzy random event.

By Definition 3.2, (14) can be written as

$$Pr\left\{\omega \,\middle|\, Pos\left\{\sum_{j=1}^{n} \tilde{\bar{r}}_j x_j \geq R\right\} \geq \delta\right\} \geq \gamma. \tag{15}$$

This *Pr–Pos* chance constraint can be used if the investor is relatively optimistic. It means that the event $\sum_{j=1}^{n} \tilde{\bar{r}}_j x_j \geq R$ holds at the confidence levels $\gamma$-*Pr*, $\delta$-*Pos*. That is, the possibility of the event $\sum_{j=1}^{n} \tilde{\bar{r}}_j x_j \geq R$ is more than $\delta$ with the probability $\gamma$.

*Remark 3.4* According to Remark 3.2 and Remark 3.3, if the fuzzy random variable $\tilde{\bar{r}}_j$ degenerates to a random variable $\bar{r}_j$, then the event $\sum_{j=1}^{n} \bar{r}_j x_j \geq R$ is a random event, for any $\omega \in \Omega$. As $Pos\{\sum_{j=1}^{n} \bar{r}_j(\omega) x_j \geq R\}$ implies $\sum_{j=1}^{n} \bar{r}_j(\omega) x_j \geq R$, the constraint

$$Pr\left\{\omega \,\middle|\, Pos\left\{\sum_{j=1}^{n} \bar{r}_j(\omega) x_j \geq R\right\} \geq \delta\right\} \geq \gamma$$

is equivalent to

$$Pr\left\{\omega \,\middle|\, \sum_{j=1}^{n} \bar{r}_j(\omega) x_j \geq R\right\} \geq \gamma.$$

*Remark 3.5* If the fuzzy random variable $\tilde{\bar{r}}_j$ degenerates to a fuzzy variable $\tilde{r}_j$, then

$$Pos\left\{\sum_{j=1}^{n} \tilde{r}_j x_j \geq R\right\} \geq \delta$$

is a crisp event. In order to satisfy $p := Pr\{\omega | Pos\{\sum_{j=1}^{n} \tilde{r}_j x_j \geq R\} \geq \delta\} \geq \gamma_i$, the probability $p$ should be 1. So, the constraint

$$Pr\left\{\omega \,\middle|\, Pos\left\{\sum_{j=1}^{n} \tilde{r}_j x_j \geq R\right\} \geq \delta\right\} = 1 \geq \gamma$$

is equivalent to

$$Pos\left\{\sum_{j=1}^{n} \tilde{r}_j x_j \geq R\right\} \geq \delta.$$

Similarly, by Definition 3.3, (14) can also be written as

$$Pr\left\{\omega\left|Nec\left\{\sum_{j=1}^{n}\tilde{\bar{r}}_j x_j \geq R\right\} \geq \delta\right.\right\} \geq \gamma. \tag{16}$$

This *Pr–Nec* chance constraint can be used if the investor is relatively pessimistic. It means that the event $\sum_{j=1}^{n}\tilde{\bar{r}}_j x_j \geq R$ holds at the confidence levels $\gamma$-*Pr*, $\delta$-*Nec*. That is, the necessity of the event $\sum_{j=1}^{n}\tilde{\bar{r}}_j x_j \geq R$ is more than $\delta$ with the probability $\gamma$.

*Remark 3.6* If the fuzzy random variable $\tilde{\bar{r}}_j$ degenerates to a random variable $\bar{r}_j$, then

$$Pr\left\{\omega\left|Nec\left\{\sum_{j=1}^{n}\bar{r}_j(\omega)x_j \geq R\right\} \geq \delta\right.\right\} \geq \gamma \quad \Leftrightarrow \quad Pr\left\{\omega\left|\sum_{j=1}^{n}\bar{r}_j(\omega)x_j \geq R\right.\right\} \geq \gamma.$$

*Remark 3.7* If the fuzzy random variable $\tilde{\bar{r}}_j$ degenerates to a fuzzy variable $\tilde{r}_j$, then

$$Pr\left\{\omega\left|Nec\left\{\sum_{j=1}^{n}\tilde{r}_j x_j \geq R\right\} \geq \delta\right.\right\} = 1 \geq \gamma \quad \Leftrightarrow \quad Nec\left\{\sum_{j=1}^{n}\tilde{r}_j x_j \geq R\right\} \geq \delta.$$

In this situation, it is the same as the Fractile model (5) proposed by Inuiguchi and Tanino [18].

In addition to the chance constraint, we also impose the capital budget constraint as

$$\sum_{j=1}^{n} x_j = 1, \tag{17}$$

and the no short-selling constraint as

$$x_j \geq 0, \quad j = 1, 2, \ldots, n. \tag{18}$$

### 3.2.3 *Fuzzy Random Chance Constrained Multi-objective Model for Portfolio Selection*

Following the above discussion, we propose a model for the optimistic investors as follows:

$$\text{(OM)} \qquad \max\ f_1 = R, \qquad \min\ f_2 = Var\left(\sum_{j=1}^{n}\tilde{\bar{r}}_j x_j\right),$$
$$\text{s.t.} \begin{cases} Pr\{\omega | Pos\{\sum_{j=1}^{n}\tilde{\bar{r}}_j x_j \geq R\} \geq \delta\} \geq \gamma, \\ \sum_{j=1}^{n} x_j = 1, \\ x_j \geq 0, \quad j = 1, 2, \ldots, n, \end{cases}$$

and a model for the pessimistic investors as

$$\text{(PM)} \qquad \max\ f_1 = R,\ \min\ f_2 = Var\left(\sum_{j=1}^{n} \tilde{\bar{r}}_j x_j\right),$$
$$\text{s.t.} \begin{cases} Pr\{\omega | Nec\{\sum_{j=1}^{n} \tilde{\bar{r}}_j x_j \geq R\} \geq \delta\} \geq \gamma, \\ \sum_{j=1}^{n} x_j = 1, \\ x_j \geq 0, \quad j = 1, 2, \ldots, n. \end{cases}$$

Note that both (OM) and (PM) are fuzzy random multi-objective portfolio selection models.

## 4 Solution Method

In order to tackle the above (OM) and (PM) models, we first need to transform them into their equivalent crisp models which can be solved by the $\varepsilon$-constraint method.

### 4.1 Equivalent Crisp Models

In this section, we derive the equivalent crisp models for (OM) and (PM). Based on the above basic knowledge of fuzzy random variables, we show below how to transform the fuzzy random objective and the chance constraint. Let us consider the objective first.

**Lemma 4.1** [25] *Suppose the future return of security $j$ be $\tilde{\bar{r}}_j$, $x_j$ be the investment proportion in security $j$ $(j = 1, 2, \ldots, n)$, then we have*

$$Var\left(\sum_{j=1}^{n} \tilde{\bar{r}}_j x_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j.$$

It should be noted that the variance of the fuzzy random return of portfolio $\sum_{j=1}^{n} \tilde{\bar{r}}_j x_j$ coincides with the variance in the Markowitz mean-variance model.

We now derive the crisp equivalent constraints of the chance constraints in (OM) and (PM), respectively.

**Lemma 4.2** [10] *Suppose $\tilde{m}$ and $\tilde{n}$ be non-interactive fuzzy numbers with continuous membership function. For a deterministic confidence level $\alpha \in ]0, 1]$, we have the following conclusions*:

$$Pos\{\tilde{m} \geq \tilde{n}\} \geq \alpha \quad \Leftrightarrow \quad m_{\alpha}^{\mathrm{R}} \geq n_{\alpha}^{\mathrm{L}}, \qquad Nec\{\tilde{m} \geq \tilde{n}\} \geq \alpha \quad \Leftrightarrow \quad m_{1-\alpha}^{\mathrm{L}} \geq n_{\alpha}^{\mathrm{R}},$$

*where $m_{\alpha}^{\mathrm{L}}, m_{\alpha}^{\mathrm{R}}$ $(n_{\alpha}^{\mathrm{L}}, n_{\alpha}^{\mathrm{R}})$ denote the left and right endpoints of the $\alpha$-cut of $\tilde{m}$ $(\tilde{n})$. $Pos\{\tilde{m} \geq \tilde{n}\}$ denotes the possibility measure of fuzzy event $\tilde{m} \geq \tilde{n}$, and $Nec\{\tilde{m} \geq \tilde{n}\}$ denotes the necessity measure of fuzzy event $\tilde{m} \geq \tilde{n}$.*

**Theorem 4.1** *Assume $\tilde{\bar{r}}_j$ is a triangular LR fuzzy random variable, denoted by $\tilde{\bar{r}}_j(\omega) = (r_j(\omega) - \alpha_j, r_j(\omega), r_j(\omega) + \beta_j)$, $\omega \in \Omega$, the membership function is as follows*:

$$\mu_{\tilde{\bar{r}}_j(\omega)}(t) = \begin{cases} \frac{t - r_j(\omega) + \alpha_j}{\alpha_j}, & r_j(\omega) - \alpha_j < t \le r_j(\omega), \\ 1, & t = r_j(\omega), \\ \frac{r_j(\omega) + \beta_j - t}{\beta_j}, & r_j(\omega) < t \le r_j(\omega) + \beta_j, \end{cases} \tag{19}$$

*where $r_j$ is normally distributed, i.e., $r_j \sim N(\mu_j, \sigma_j^2)$, $\alpha_j$ and $\beta_j (> 0)$ are the left and right width of the $\tilde{\bar{r}}_j(\omega)$, respectively.*

*Then*

$$Pr\left\{\omega \middle| Pos\left\{\sum_{j=1}^{n} \tilde{\bar{r}}_j x_j \ge R\right\} \ge \delta\right\} \ge \gamma$$

*is equivalent to*

$$R \le \sum_{j=1}^{n} \mu_j x_j + (1-\delta)\sum_{j=1}^{n} \beta_j x_j + \Phi^{-1}(1-\gamma)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j},$$

*where $\Phi$ is standard normal distribution function, and $\delta, \gamma \in ]0, 1]$ are predetermined confidence levels.*

*Proof* For $\omega \in \Omega$, $\tilde{\bar{r}}_j(\omega)$ is a fuzzy number, the membership function is $\mu_{\tilde{\bar{r}}_j(\omega)}(t)$. By the extension principle [9], we can get the membership function of the fuzzy number $\sum_{j=1}^{n} \tilde{\bar{r}}_j(\omega) x_j$ as

$$\mu_{\sum_{j=1}^{n} \tilde{\bar{r}}_j(\omega) x_j}(t)$$
$$= \begin{cases} \frac{t - \sum_{j=1}^{n}(r_j(\omega)+\alpha_j)x_j}{\sum_{j=1}^{n}\alpha_j x_j}, & \sum_{j=1}^{n}(r_j(\omega) - \alpha_j)x_j < t \le \sum_{j=1}^{n} r_j(\omega) x_j, \\ 1, & t = \sum_{j=1}^{n} r_j(\omega) x_j, \\ \frac{\sum_{j=1}^{n}(r_j(\omega)+\beta_j)x_j - t}{\sum_{j=1}^{n}\beta_j x_j}, & \sum_{j=1}^{n} r_j(\omega) x_j < t \le \sum_{j=1}^{n}(r_j(\omega) + \beta_j)x_j. \end{cases} \tag{20}$$

So we obtain

$$\sum_{j=1}^{n} \tilde{\bar{r}}_j(\omega) = \left(\sum_{j=1}^{n}(r_j - \alpha_j)(\omega)x_j, \sum_{j=1}^{n} r_j(\omega)x_j, \sum_{j=1}^{n}(r_j + \beta_j)(\omega)x_j\right).$$

Since $r_j \sim N(\mu_j, \sigma_j^2)$, we have

$$\sum_{j=1}^{n} r_j x_j \sim N\left(\sum_{j=1}^{n} \mu_j x_j, \sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j\right).$$

Using Lemma 4.2, we obtain

$$Pos\left\{\sum_{j=1}^{n}\tilde{\bar{r}}_j x_j \geq R\right\} \geq \delta \quad \Leftrightarrow \quad \sum_{j=1}^{n}\tilde{\bar{r}}_j x_j \geq R - (1-\delta)\sum_{j=1}^{n}\beta_j x_j.$$

So, for predetermined confidence levels $\delta, \gamma \in ]0, 1]$, there holds

$$\begin{aligned}
&Pr\left\{\omega \middle| Pos\left\{\sum_{j=1}^{n}\tilde{\bar{r}}_j x_j \geq R\right\} \geq \delta\right\} \geq \gamma \\
&\Leftrightarrow \quad Pr\left\{\omega \middle| \sum_{j=1}^{n}\tilde{\bar{r}}_j x_j \geq R - (1-\delta)\sum_{j=1}^{n}\beta_j x_j\right\} \geq \gamma \\
&\Leftrightarrow \quad Pr\left\{\omega \middle| \frac{\sum_{j=1}^{n}\tilde{\bar{r}}_j x_j - \sum_{j=1}^{n}\mu_j x_j}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j}} \right. \\
&\geq \left. \frac{R - \sum_{j=1}^{n}\mu_j x_j - (1-\delta)\sum_{j=1}^{n}\beta_j x_j}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j}}\right\} \geq \gamma \\
&\Leftrightarrow \quad \Phi\left(\frac{R - \sum_{j=1}^{n}\mu_j x_j - (1-\delta)\sum_{j=1}^{n}\beta_j x_j}{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j}}\right) \leq 1-\gamma \\
&\Leftrightarrow \quad R \leq \sum_{j=1}^{n}\mu_j x_j + (1-\delta)\sum_{j=1}^{n}\beta_j x_j + \Phi^{-1}(1-\gamma)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j}.
\end{aligned}$$

The proof is completed. □

**Theorem 4.2** *Assume $\tilde{\bar{r}}_j$ is the same triangular LR fuzzy random variable defined in Theorem* 4.1. *Then*

$$Pr\left\{\omega \middle| Nec\left\{\sum_{j=1}^{n}\tilde{\bar{r}}_j x_j \geq R\right\} \geq \delta\right\} \geq \gamma$$

*is equivalent to*

$$R \leq \sum_{j=1}^{n}\mu_j x_j - \delta\sum_{j=1}^{n}\alpha_j x_j + \Phi^{-1}(1-\gamma)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j},$$

*where $\Phi$ is standard normal distributed function*, $\delta, \gamma \in ]0, 1]$ *are predetermined confidence levels.*

*Proof* The proof is similar to that of Theorem 4.1, and thus omitted. □

Based on the Lemma 4.1 and Theorems 4.1, we can transform (OM) into the following equivalent model:

$$\max\ f_1 = R, \qquad \min\ f_2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j,$$
$$\text{s.t.} \begin{cases} R \le \sum_{j=1}^{n}\mu_j x_j + (1-\delta)\sum_{j=1}^{n}\beta_j x_j + \Phi^{-1}(1-\gamma)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j}, \\ \sum_{j=1}^{n} x_j = 1, \\ x_j \ge 0, \quad j = 1, 2, \ldots, n, \end{cases} \tag{21}$$

which is also equivalent to

$$\begin{aligned} &\max \sum_{j=1}^{n}\mu_j x_j + (1-\delta)\sum_{j=1}^{n}\beta_j x_j + \Phi^{-1}(1-\gamma)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j}, \\ &\min \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j, \\ &\text{s.t.} \begin{cases} \sum_{j=1}^{n} x_j = 1, \\ x_j \ge 0, \quad j = 1, 2, \ldots, n. \end{cases} \end{aligned} \tag{22}$$

Similarly, based on Lemma 4.1 and Theorem 4.2, we can transform (PM) into the following equivalent model:

$$\begin{aligned} &\max \sum_{j=1}^{n}\mu_j x_j - \delta\sum_{j=1}^{n}\alpha_j x_j + \Phi^{-1}(1-\gamma)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j}, \\ &\min \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j, \\ &\text{s.t.} \begin{cases} \sum_{j=1}^{n} x_j = 1, \\ x_j \ge 0, \quad j = 1, 2, \ldots, n. \end{cases} \end{aligned} \tag{23}$$

Note that Models (22) and (23) are crisp models, which can be tackled as described below.

### 4.2 $\varepsilon$-Constraint Method

In this section, we illustrate how to use the $\varepsilon$-constraint method to solve the equivalent crisp models (22) and (23).

For a crisp multi-objective decision making model, we can use the $\varepsilon$-constraint method to solve it. The $\varepsilon$-constraint method was proposed by Haimes [26, 27] in 1971. The essence of this method is that we choose a main referenced objective $f_{i0}$, and put the other objective functions into the constraints. Here we give a general description of this method.

Let us consider the following general multi-objective model:

$$\min\ f_i(\boldsymbol{x}), \quad i = 1, 2, \ldots, m, \quad \text{s.t. } \boldsymbol{x} \in X. \tag{24}$$

Using the $\varepsilon$-constraint method, we can get the single objective model:

$$\min\ f_{i_0}(\boldsymbol{x}), \quad \text{s.t.} \begin{cases} f_i(\boldsymbol{x}) \le \varepsilon_i, & i = 1, 2, \ldots, m, i \ne i_0, \\ \boldsymbol{x} \in X, \end{cases} \tag{25}$$

where the parameter $\varepsilon_i$ is predetermined by the decision maker and it represents the threshold value that the decision maker can accept.

Let us denote the feasible domain of Model (25) as $X_1$.

**Theorem 4.3** *If $\bar{\boldsymbol{x}}$ is the optimal solution of Model* (25), *then $\bar{\boldsymbol{x}}$ is a weak efficient solution of Model* (24).

*Proof* Here we prove this theorem by contradiction. Let $\bar{\boldsymbol{x}}$ be the optimal solution of Model (25). If $\bar{\boldsymbol{x}}$ is not a weak efficient solution of Model (24), then there exists $\boldsymbol{x}' \in X_1$ such that for $\forall i \in \{1, 2, \ldots, m\}$, $f_i(\boldsymbol{x}') < f_i(\bar{\boldsymbol{x}})$ holds. And we obtain

$$f_{i_0}(\boldsymbol{x}') < f_{i_0}(\bar{\boldsymbol{x}}), \tag{26}$$

and

$$f_i(\boldsymbol{x}') < f_i(\bar{\boldsymbol{x}}) \le \varepsilon_i, \quad i = 1, 2, \ldots, m, i \ne i_0. \tag{27}$$

It follows from (26) and (27) that $\boldsymbol{x}'$ should be the optimal solution of Model (25), which conflicts with the fact that $\bar{\boldsymbol{x}}$ is the optimal solution. So above all, we can conclude that if $\bar{\boldsymbol{x}}$ is the optimal solution of Model (25), then there does not exist any $\boldsymbol{x}' \in X_1$ such that $f_i(\boldsymbol{x}') < f_i(\bar{\boldsymbol{x}})$, $\forall\, i \in \{1, 2, \ldots, m\}$. That is, if $\bar{\boldsymbol{x}}$ is the optimal solution of Model (25), then $\bar{\boldsymbol{x}}$ is a weak efficient solution of Model (24). Thus the theorem is proved. □

**Theorem 4.4** *Let $\bar{\boldsymbol{x}}$ be an efficient solution of Model* (24), *then there exists a parameter $\varepsilon_i$ $(i = 1, 2, \ldots, m, i \ne i_0)$, such that $\bar{\boldsymbol{x}}$ is the optimal solution of Model* (25).

*Proof* Set $\varepsilon_i = f_i(\bar{\boldsymbol{x}})$ $(i = 1, 2, \ldots, m, i \ne i_0)$. By the definition of efficient solution, we can obtain that $\bar{\boldsymbol{x}}$ is an optimal solution of Model (25). □

The advantages of the $\varepsilon$-constraint method are as follows:

(1) Every efficient solution of Model (24) can be obtained by properly choosing parameter $\varepsilon_i$ $(i = 1, 2, \ldots, m, i \ne i_0)$.
(2) The $i_0$th objective must be satisfied, gives consideration to other objectives as well.

Note that the above discussion is for a general multi-objective model. The specific form of the $\varepsilon$-constraint method for Model (22) and (23) will be discussed in detail in the next section.

## 5 Application in Chinese Stock Market

In this section, we illustrate the effectiveness of our approach by applying the proposed models to the Chinese stock market. To this end, we choose the top

**Table 1** The expected values and the left-right spreads of the return of the 20 sample stocks (%)

| No. | Name | Code | $\mu_j$ | $\alpha_j$ | $\beta_j$ |
|---|---|---|---|---|---|
| S1 | China Petroleum | 600028 | 0.1557 | 0.2000 | 0.2500 |
| S2 | Air China | 601111 | 0.2372 | 0.2000 | 0.4000 |
| S3 | China Coal Energy | 601898 | 0.1334 | 0.2500 | 0.5000 |
| S4 | China Life Insurance | 601628 | 0.2292 | 0.4000 | 0.4000 |
| S5 | Bank of China | 601988 | 0.1034 | 0.5000 | 0.5000 |
| S6 | Kweichow Moutai | 600519 | 0.3030 | 0.2500 | 0.2500 |
| S7 | Shandong Gold Mine | 600547 | 0.2632 | 0.2500 | 0.3000 |
| S8 | Heilongjiang Agricultural | 600598 | 0.0284 | 0.0200 | 0.0600 |
| S9 | Daqin Railway | 601006 | 0.0592 | 0.0100 | 0.0900 |
| S10 | Baoshan Iron | 600019 | 0.0933 | 0.0300 | 0.0800 |
| S11 | Citic Securities | 600030 | 0.0437 | 0.0350 | 0.0300 |
| S12 | Poly Real Estate | 600048 | 0.0889 | 0.0200 | 0.0200 |
| S13 | Aluminum Corp China | 601600 | 0.2535 | 0.1500 | 0.4000 |
| S14 | China South Locomotive | 601766 | 0.0125 | 0.0200 | 0.0900 |
| S15 | Jinduicheng Molybdenum | 601958 | 0.4576 | 0.0800 | 0.0500 |
| S16 | China United Network | 600050 | 0.0157 | 0.0900 | 0.5000 |
| S17 | Tebian Electric | 600089 | 0.1015 | 0.0500 | 0.0900 |
| S18 | Ping An insurance | 601318 | 0.3943 | 0.2500 | 0.3500 |
| S19 | China Merchants Bank | 600036 | 0.0518 | 0.0500 | 0.0900 |
| S20 | Saic Motor | 600104 | 0.4867 | 0.0500 | 0.9000 |

20 stocks from the Shanghai Stock Exchange (SSE) Composite Index. One hundred historical returns before November 1, 2009 are collected from Yahoo Finance (http://finance.yahoo.com) for each of the 20 stocks. We use Excel to compute the mean value of the return $\mu_j$ and the covariance $\sigma_{ij}$. The values of $\alpha_j$, $\beta_j$ are chosen according to some expert views. Note that the return $\mu_j$ can be positive, 0 or negative, which means that if an investor gets positive (negative) return, then he or she wins (loses) in a certain considered period; and if an investor gets 0 return, then he breaks even in the period.

Table 1 shows the names, the transaction codes, the mean values and the left–right spreads of the future returns of the sample stocks. Table 2 presents the covariance matrix.

### 5.1 Portfolio Selection for Optimistic Investors

For an optimistic investor, we use *Pr–Pos* chance based (OM) to get efficient portfolios. According to the $\varepsilon$-constraint method, we can solve the bi-objective crisp equivalent optimistic portfolio selection model (22) by splitting it into two single objective

**Table 2** The variance-covariance matrix for the 20 sample stocks (%)

| | S1 | S2 | S3 | S4 | S5 | S6 | S7 | S8 | S9 | S10 | S11 | S12 | S13 | S14 | S15 | S16 | S17 | S18 | S19 | S20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| S1 | 0.0691 | | | | | | | | | | | | | | | | | | | |
| S2 | 0.0255 | 0.1181 | | | | | | | | | | | | | | | | | | |
| S3 | 0.0161 | 0.0466 | 0.1176 | | | | | | | | | | | | | | | | | |
| S4 | −0.0055 | 0.0201 | 0.0583 | 0.0630 | | | | | | | | | | | | | | | | |
| S5 | 0.0051 | 0.0205 | 0.0375 | 0.0296 | 0.0427 | | | | | | | | | | | | | | | |
| S6 | 0.0054 | 0.0186 | 0.0243 | 0.0182 | 0.0111 | 0.0480 | | | | | | | | | | | | | | |
| S7 | 0.0127 | 0.0257 | 0.0761 | 0.0479 | 0.0361 | 0.0320 | 0.1728 | | | | | | | | | | | | | |
| S8 | 0.0327 | 0.0345 | 0.0115 | −0.0003 | 0.0019 | 0.0083 | 0.0144 | 0.0735 | | | | | | | | | | | | |
| S9 | 0.0051 | 0.0317 | 0.0506 | 0.0295 | 0.0242 | 0.0229 | 0.0396 | 0.0110 | 0.0682 | | | | | | | | | | | |
| S10 | 0.0013 | 0.0411 | 0.0676 | 0.0435 | 0.0409 | 0.0192 | 0.0555 | 0.0041 | 0.0390 | 0.0906 | | | | | | | | | | |
| S11 | 0.0108 | 0.0429 | 0.0638 | 0.0412 | 0.0350 | 0.0157 | 0.0474 | 0.0039 | 0.0337 | 0.0645 | 0.0773 | | | | | | | | | |
| S12 | 0.0104 | 0.0336 | 0.0637 | 0.0451 | 0.0298 | 0.0216 | 0.0623 | −0.0028 | 0.0317 | 0.0459 | 0.0463 | 0.0995 | | | | | | | | |
| S13 | 0.0412 | 0.0696 | 0.0470 | 0.0317 | 0.0170 | 0.0296 | 0.0474 | 0.0433 | 0.0209 | 0.0412 | 0.0472 | 0.0353 | 0.1711 | | | | | | | |
| S14 | 0.0235 | 0.0050 | 0.0032 | −0.0058 | 0.0047 | 0.0039 | 0.0055 | 0.0173 | −0.0047 | 0.0040 | −0.0002 | −0.0094 | 0.0216 | 0.0591 | | | | | | |
| S15 | 0.0279 | 0.0500 | 0.1180 | 0.0606 | 0.0485 | 0.0316 | 0.0988 | 0.0125 | 0.0715 | 0.0744 | 0.0656 | 0.0682 | 0.0582 | 0.0049 | 0.1965 | | | | | |
| S16 | 0.0000 | 0.0286 | 0.0504 | 0.0397 | 0.0330 | 0.0202 | 0.0365 | −0.0004 | 0.0299 | 0.0523 | 0.0452 | 0.0300 | 0.0067 | 0.0034 | 0.0582 | 0.0628 | | | | |
| S17 | 0.0208 | 0.0265 | 0.0121 | 0.0076 | 0.0026 | 0.0085 | 0.0215 | 0.0379 | 0.0047 | −0.0010 | 0.0084 | 0.0069 | 0.0356 | 0.0169 | 0.0133 | 0.0009 | 0.0770 | | | |
| S18 | 0.0118 | 0.0420 | 0.0543 | 0.0378 | 0.0310 | 0.0108 | 0.0317 | 0.0129 | 0.0300 | 0.0434 | 0.0428 | 0.0356 | 0.0431 | −0.0053 | 0.0602 | 0.0258 | 0.0190 | 0.0884 | | |
| S19 | 0.0294 | 0.0169 | 0.0187 | 0.0046 | −0.0005 | 0.0105 | 0.0203 | 0.0151 | −0.0019 | −0.0075 | 0.0049 | 0.0219 | 0.0456 | 0.0181 | 0.0252 | −0.0047 | 0.0069 | 0.0131 | 0.1031 | |
| S20 | 0.0063 | 0.0377 | 0.0591 | 0.0393 | 0.0278 | 0.0199 | 0.0489 | 0.0037 | 0.0340 | 0.0501 | 0.0432 | 0.0448 | 0.0186 | 0.0000 | 0.0686 | 0.0383 | 0.0206 | 0.0364 | 0.0008 | 0.1044 |

models (28) and (29):

$$\begin{aligned}&\max \sum_{j=1}^{n}\mu_j x_j+(1-\delta)\sum_{j=1}^{n}\beta_j x_j+\Phi^{-1}(1-\gamma)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j},\\ &\text{s.t.}\begin{cases}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j\le V_0,\\ \sum_{j=1}^{n}x_j=1,\\ x_j\ge 0,\quad j=1,2,\dots,n,\end{cases}\end{aligned}\tag{28}$$

and

$$\begin{aligned}&\min \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j,\\ &\text{s.t.}\begin{cases}\sum_{j=1}^{n}\mu_j x_j+(1-\delta)\sum_{j=1}^{n}\beta_j x_j+\Phi^{-1}(1-\gamma)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j}\ge R_0,\\ \sum_{j=1}^{n}x_j=1,\\ x_j\ge 0,\quad j=1,2,\dots,n,\end{cases}\end{aligned}\tag{29}$$

where $V_0$ and $R_0$ are given risk level and return level, respectively.

Following Theorem 4.4, we know the optimal solutions of (28) and (29) are the efficient solutions of Model (22). Thus we can get the efficient frontier using the $\varepsilon$-constraint method.

Next, we discuss the value ranges of $R_0$ and $V_0$. Consider the following two single objective models:

$$\begin{aligned}&\max\ R=\sum_{j=1}^{n}\mu_j x_j+(1-\delta)\sum_{j=1}^{n}\beta_j x_j+\Phi^{-1}(1-\gamma)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j},\\ &\text{s.t.}\begin{cases}\sum_{j=1}^{n}x_j=1,\\ x_j\ge 0,\quad j=1,2,\dots,n,\end{cases}\end{aligned}\tag{30}$$

and

$$\min\ V=\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j,\quad \text{s.t.}\begin{cases}\sum_{j=1}^{n}x_j=1,\\ x_j\ge 0,\quad j=1,2,\dots,n.\end{cases}\tag{31}$$

The optimal objective value of Model (30) is $R_{\max}$, the solution is $x^R$. We use $x^R$ to compute the corresponding $V_{\max}$. Similarly, The optimal objective value of Model (31) is $V_{\min}$, the solution is $x^V$, and we use $x^V$ to get the corresponding $R_{\min}$.

Here we suppose that an optimistic investor sets the probability confidence level $\gamma$ to be 0.6, and the possibility $\delta$ of that the portfolio's return is more than $R$ to be no less than 0.8. In the following, we show the detailed process to obtain efficient portfolios for the investor under the confidence level (0.6-*Pr*, 0.8-*Pos*).

First, we solve Models (30) and (31) to get the return level range and the risk level range, respectively:

$$[R_{\min}, R_{\max}]=[0.29\%, 0.819\%],\qquad [V_{\min}, V_{\max}]=[0.0166\%, 0.0422\%].$$

**Table 3** Optimal portfolios

| | Maximizing return portfolio | Minimizing risk portfolio | Portfolio under risk level $V_1$ |
|---|---|---|---|
| Proportion of S1 | 47.61% | 6.68% | 41.80% |
| Proportion of S2 | 0 | 0 | 0.96% |
| Proportion of S3 | 0 | 0 | 0 |
| Proportion of S4 | 0 | 8.71% | 2.46% |
| Proportion of S5 | 0 | 20.35% | 6.58% |
| Proportion of S6 | 6.43% | 16.40% | 19.95% |
| Proportion of S7 | 0 | 0 | 0 |
| Proportion of S8 | 0 | 6.25% | 0 |
| Proportion of S9 | 0 | 6.53% | 0 |
| Proportion of S10 | 0 | 0 | 0 |
| Proportion of S11 | 0 | 0 | 0 |
| Proportion of S12 | 0 | 0 | 0 |
| Proportion of S13 | 0 | 0 | 0 |
| Proportion of S14 | 0 | 17.37% | 0 |
| Proportion of S15 | 0 | 0 | 0 |
| Proportion of S16 | 0 | 1.43% | 0 |
| Proportion of S17 | 0 | 8.37% | 0 |
| Proportion of S18 | 0 | 0 | 0 |
| Proportion of S19 | 0 | 7.91% | 0 |
| Proportion of S20 | 45.96% | 0 | 28.25% |
| Total | 1 | 1 | 1 |
| Return | 0.819% | 0.29% | 0.77% |
| Risk | 0.0422% | 0.0166% | 0.03% |

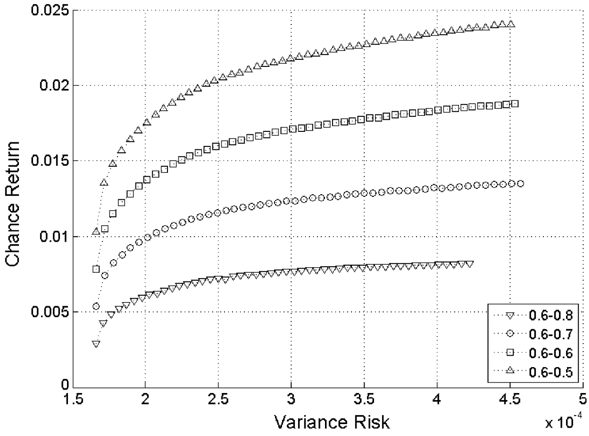
When we consider Model (30), we just maximize the return without considering the risk, and we can obtain the optimal portfolio as reported in the second column of Table 3; When we consider Model (31), we just minimize the risk without considering the return, and we can obtain the optimal portfolio given in the third column of Table 3.

Suppose the investor sets a risk level $V_1 = 0.03\%$, the optimal portfolio for this investor is reported in the fourth column of Table 3, and the chance return of this portfolio is 0.77% per day.

When $R_0$ gets all of the values in the range $[R_{\min}, R_{\max}]$, we can get all of the efficient matches of Model (22); Similarly, when $V_0$ gets all of the values in the range $[V_{\min}, V_{\max}]$, we can obtain all of the efficient matches of Model (22). Then by using the $\varepsilon$-constraint method we can obtain the efficient frontier (EF) under the confidence level (0.6-*Pr*, 0.8-*Pos*) for this optimistic investor as shown in Fig. 1.

We set the same *Pr* confidence level as $\gamma = 0.6$, and change the *Pos* confidence level from $\delta = 0.8$ to $\delta = 0.5$ with a step of 0.1. Using the $\varepsilon$-constraint method, we

**Fig. 2** EFs for optimistic investors under different *Pos* confidence levels



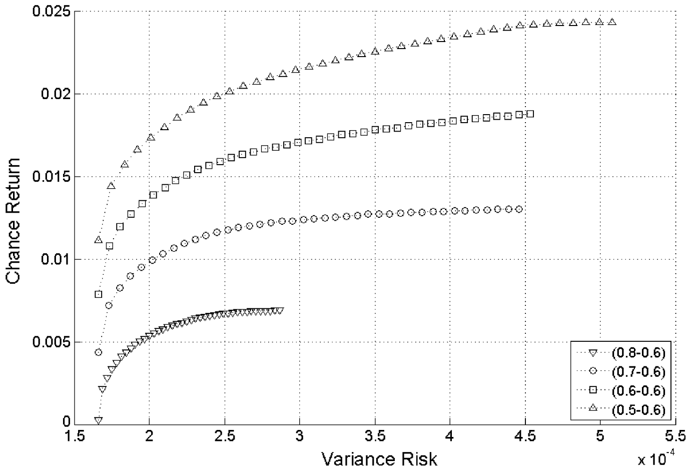




**Fig. 3** EFs for optimistic investors under different *Pr* confidence levels

can obtain the EFs under the confidence levels (0.6-*Pr*, 0.8-*Pos*), (0.6-*Pr*, 0.7-*Pos*), (0.6-*Pr*, 0.6-*Pos*) and (0.6-*Pr*, 0.5-*Pos*), respectively. They are shown in Fig. 2.

From Fig. 2 we make the following remarks:

(a) With the same *Pr* confidence level, the minimal risks of all EFs are the same;
(b) With the same *Pr* confidence level, the maximal chance return increases when *Pos* decreases;
(c) With the same *Pr* confidence level and under a certain risk level, the chance return increases when *Pos* decreases.

We set the same *Pos* confidence level as $\delta = 0.6$, and adjust the *Pr* confidence level from $\gamma = 0.8$ to $\gamma = 0.5$ with a step of 0.1. Then we can obtain the EFs under the confidence levels (0.8-*Pr*, 0.6-*Pos*), (0.7-*Pr*, 0.6-*Pos*), (0.6-*Pr*, 0.6-*Pos*) and (0.5-*Pr*, 0.6-*Pos*), respectively. The results are depicted in Fig. 3.

Figure 3 shows the following:

(a) With the same *Pr* confidence level, the minimal risks of all EFs are the same;
(b) With the same *Pos* confidence level, the maximal chance return increases when the when *Pr* decreases;
(c) With the same *Pos* confidence level and under a certain risk level, the chance return is bigger when *Pr* decreases;
(d) With the same *Pos* confidence level, when *Pr* decreases, the investor can obtain more choices of portfolio, that is, the investor can get more return-risk matches by selecting different portfolios.

Note that the conclusions are coincident with our insights. The EFs obtained by solving (OM) have the same characteristics as the traditional EF, i.e., all of the EFs are the curves sloping upward to the right, which reflects as the high return-high risk principle. Furthermore, all of the EFs are convex, and there is no cupped segment on the EFs.

## 5.2 Portfolio Selection for Pessimistic Investors

For a pessimistic investor, we use *Pr–Nec* based portfolio selection model (PM) to get efficient portfolios. By the $\varepsilon$-constraint method, we get the following two models:

$$\begin{aligned} &\max \sum_{j=1}^{n}\mu_j x_j - \delta\sum_{j=1}^{n}\alpha_j x_j + \Phi^{-1}(1-\gamma)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j} \\ &\text{s.t.}\begin{cases}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j \le V_0\\ \sum_{j=1}^{n}x_j = 1\\ x_j \ge 0, \quad j=1,2,\dots,n\end{cases}\end{aligned} \tag{32}$$

and

$$\begin{aligned} &\min \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j \\ &\text{s.t.}\begin{cases}\sum_{j=1}^{n}\mu_j x_j - \delta\sum_{j=1}^{n}\alpha_j x_j + \Phi^{-1}(1-\gamma)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j} \ge R_0\\ \sum_{j=1}^{n}x_j = 1\\ x_j \ge 0, \quad j=1,2,\dots,n.\end{cases}\end{aligned} \tag{33}$$

Similar to Sect. 5.1, we present the EFs for the pessimistic investors under different confidence levels in Figs. 4 and 5.

From Fig. 4, we can draw the following observations:

(a) With the same *Pr* confidence level, the minimal risks of all EFs are the same;
(b) With the same *Pr* confidence level, the maximal chance return increases when *Nec* decreases;
(c) With the same *Pr* confidence level and under a certain risk level, the chance return increases when *Nec* decreases;

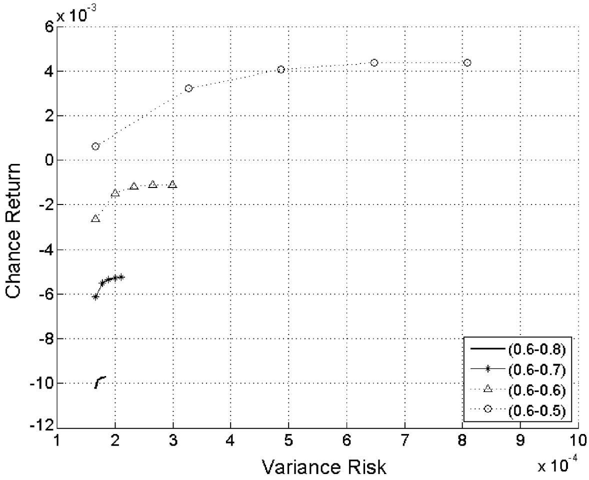



**Fig. 4** EFs for pessimistic investors under different *Nec* confidence levels

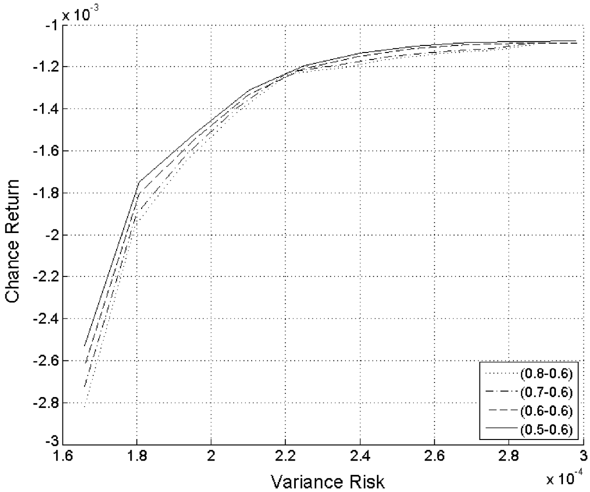



**Fig. 5** EFs for pessimistic investors under different *Pr* confidence levels

(d) With the same *Pr* confidence level, when *Nec* decreases, the investor will obtain more choices of portfolio. We note that the differences are obvious in this situation.

It is worth noting that, if *Nec* is too big, then there are very few portfolios for a pessimistic investor with such high confidence levels, that is, the efficient frontier under the high confidence levels is rather short.

From Fig. 5, we have the following remarks:

(a) With the same *Pr* confidence level, the minimal risks of all EFs are the same;
(b) With the same *Nec* confidence level, the maximal chance return increases when the when *Pr* decreases;
(c) With the same *Nec* confidence level and under a certain risk level, the chance return increases when *Pr* decreases;

Above all, we can draw the following conclusions which are consistent with our expectations in practice:

(1) Different investors have different EFs, and the anticipant chance returns are different when the investors hold different levels of optimistic-pessimistic attitude: the EF under the lower confidence levels is longer and higher than the EF under the higher confidence levels;
(2) For the optimistic investors, the difference caused by the external probability is greater than that caused by the internal possibility. It shows that, to a larger extent, the optional portfolios for the optimistic investors are influenced by the objective factor;
(3) For the pessimistic investors, the difference caused by the internal necessity is greater than that caused by the external probability. It indicates that, to a larger extent, the optional portfolios for the pessimistic investors are determined by the subjective factor.

## 6 Conclusion

In this paper, a portfolio selection problem with fuzzy random parameters is studied. On the basis of the mean-variance model and the safety-first model, we provide a new type of fuzzy random multi-objective model with a chance constraint. We introduce a few theorems about the variance of fuzzy random portfolio, and develop the equivalent partition of two kinds of chance constraint: one is for the optimistic investor and the other is for the pessimistic investor. We transform the proposed fuzzy random portfolio selection models into their equivalent crisp models. We further use the $\varepsilon$-constraint method to obtain the efficient frontier of the portfolio selection problem. We also apply the proposed models to the Chinese stock market to illustrate the effectiveness of our approach. Our research reveals that the decision for portfolio selection problem is dependent on the expert advice, the risk attitude of the investor, and the confidence level of the investor.

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