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# Existence, Uniqueness and Stability of Mild Solutions for Time-Dependent Stochastic Evolution Equations with Poisson Jumps and Infinite Delay

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**Abstract** In this paper, we study a class of time-dependent stochastic evolution equations with Poisson jumps and infinite delay. We establish the existence, uniqueness and stability of mild solutions for these equations under non-Lipschitz condition with Lipschitz condition being considered as a special case. An application to the stochastic nonlinear wave equation, with Poisson jumps and infinite delay, is given to illustrate the obtained theory.

**Keywords** Stochastic evolution equation · Evolution operator · Poisson point process · Successive approximation

## 1 Introduction

The stochastic evolution differential equations have attracted much attention because of their practical applications in many areas such as physics, population dynamics,

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electrical engineering, medicine biology, ecology and other areas of science and engineering. For more details, one can see Da Prato and Zabczyk [1], Ren and Sun [2], Wang and Zhang [3] and the references therein. In many areas of science, there has been an increasing interest in the investigation of the systems incorporating memory or aftereffect, i.e., there is the effect of infinite delay on state equations. Therefore, there is a real need to discuss stochastic evolution systems with infinite delay.

In addition, stochastic differential equations with Poisson jumps have become very popular in modeling the phenomena arising in the fields, such as economics, where the jump processes are widely used to describe the asset and commodity price dynamics (see [4]). Very recently, stochastic evolution equations with Poisson jumps have attracted the interest of many researchers. One can see [5–8] and the references therein.

Motivated by the above works, this paper is concerned with the time-dependent stochastic evolution equations with Poisson jumps and infinite delay of the form:

$$\begin{aligned} \mathrm{d}x(t) &= [A(t)x(t) + f(t, x_t)]\,\mathrm{d}t + \sigma(t, x_t)\,\mathrm{d}w(t) \\ &\quad + \int_Z h\big(t, x(t-), z\big)\widetilde{N}(\mathrm{d}t, \mathrm{d}z), \quad t \in J := [0, T], \qquad (1) \\ x_0 &= \phi \in \mathcal{B}, \qquad (2) \end{aligned}$$

in a real separable Hilbert space $H$ with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, where $A(t) : D \subset H \to H, t \in J$ is a family of unbounded operators defined on a common domain $D$, which is dense in the space $H$ and generates a strong evolution operator $U(s,t), 0 \le t \le s \le T$, and the history $x_t : ]-\infty, 0] \to H, x_t(\theta) = x(t+\theta)$, for $t \ge 0$, belongs to the phase space $\mathcal{B}$, which will be defined axiomatically in Sect. 2. Let $M$ be another real separable Hilbert space. Suppose that $w$ is a $M$-valued Wiener process with increment covariance given by a non-negative trace class operator $Q$ and $\widetilde{N}(\mathrm{d}s, \mathrm{d}z)$ is a compensating martingale measure induced by a Poisson point process $k(\cdot)$, which is dependent on the Wiener process $w$ and takes values in a measurable space $(Z, \mathcal{B}(Z))$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Let $L(M,H)$ be the space of all bounded, continuous and linear operators from $M$ into $H$. Assume that $f : J \times \mathcal{B} \to H$, $\sigma : J \times \mathcal{B} \to L_Q(M,H)$ and $h : J \times H \times (Z - \{0\}) \to H$ be appropriate mappings specified later. Here, $L_Q(M,H)$ denotes the space of all $Q$-Hilbert-Schmidt operators from $M$ into $H$, which will be defined in next section. The initial data $\phi = \{\phi(t) : -\infty < t \le 0\}$ is an $\mathcal{F}_0$-measurable, $\mathcal{B}$-valued stochastic process independent of the Wiener process $w$ and the Poisson point process $k(\cdot)$. Further, $\phi$ has finite second moment.

The present paper is devoted to prove the existence, uniqueness and stability of mild solutions for (1)–(2) with the functions $f$, $\sigma$ and $h$ satisfying some non-Lipschitz condition, which includes Lipschitz condition as a special case. Here, we adopt the methods, which have been used in our previous works Ren et al. [9] and Ren and Xia [10]. An application to the stochastic nonlinear wave equation with Poisson jumps and infinite delay is given to illustrate the obtained theory.

The paper is organized as follows. In Sect. 2, we introduce some preliminaries. Section 3 is devoted to the proof of existence and uniqueness of mild solutions. In Sect. 4, we study the continuous dependence of solutions on the initial values. An example is provided in Sect. 5 to illustrate the theory. Section 6 includes some concluding remarks.

## 2 Preliminaries

In this section, we recall some preliminaries needed to establish our results. For more details, one can see Da Prato and Zabczyk [1], Hale and Kato [11] and the references therein.

Throughout the paper, we assume that $(\Omega, \mathcal{F}, P; \mathbf{F})$ $(\mathbf{F} = \{\mathcal{F}_t\}_{0\le t\le T})$ be a complete filtered probability space such that $\mathcal{F}_0$ contains all $P$-null sets of $\mathcal{F}$, $\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$, $t \ge 0$ and $\mathcal{F} = \mathcal{F}_T$. Suppose that $\{k(t) : t \in J\}$ be a Poisson point process, taking its value in a measurable space $(Z, \mathcal{B}(Z))$ with a $\sigma$-finite intensity measure $\lambda(\mathrm{d}z)$. We denote by $N(\mathrm{d}s, \mathrm{d}z)$ the Poisson counting measure, which is induced by $k(\cdot)$, and the compensating martingale measure by

$$\widetilde{N}(\mathrm{d}s, \mathrm{d}z) := N(\mathrm{d}s, \mathrm{d}z) - \lambda(\mathrm{d}z)\mathrm{d}s.$$

Let $w$ be a $Q$-Wiener process, which is independent of the Poisson point process $\{k(t) : t \in J\}$, on $(\Omega, \mathcal{F}, P)$ with the non-negative, linear and bounded covariance operator $Q$ such that $\mathrm{Tr}(Q) < \infty$. We assume that there exist a complete orthonormal system $\{e_n\}$ in $M$, a bounded sequence of non-negative real numbers $\{\lambda_n\}$ such that $Qe_n = \lambda_n e_n$, $n = 1, 2, \ldots$, and a sequence of independent Wiener processes $\{\beta_n\}$ such that

$$\big\langle w(t), e\big\rangle = \sum_{k=1}^{\infty} \sqrt{\lambda_n}\langle e_n, e\rangle \beta_n(t), \quad e \in M.$$

Let $\psi \in L(M, H)$ and define

$$|\psi|_Q^2 := \mathrm{Tr}(\psi Q \psi^*) = \sum_{n=1}^{\infty} \big|\sqrt{\lambda_n}\psi e_n\big|^2.$$

If $|\psi|_Q < \infty$, then $\psi$ is called a $Q$-Hilbert-Schmidt operator. Let $L_Q(M, H)$ denote the space of all $Q$-Hilbert-Schmidt operators $\psi : M \to H$. The completion $L_Q(M, H)$ of $L(M, H)$, with respect to the topology induced by the norm $|\cdot|_Q$, is a Hilbert space with the above norm topology, where $|\psi|_Q^2 = \langle \psi, \psi\rangle$. We assume that the filtration be generated by the $Q$-Wiener process $w(\cdot)$, the Poisson point process $k(\cdot)$ and be augmented, that is,

$$\mathcal{F}_t = \sigma\{w(s); s \le t\} \vee \sigma\{N(]0, s], A); s \le t, A \in \mathcal{B}(Z)\} \vee \mathcal{N}, \quad t \in J,$$

where $\mathcal{N}$ is the class of $P$-null sets. In the sequel, $L_2^0(\Omega, H)$ denotes the space of $\mathcal{F}_0$-measurable, $H$-valued and square integrable stochastic processes.

In this work, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato [11]. To establish the axiom of the phase space $\mathcal{B}$, we follow the terminology used in Hino et al. [12]. The axioms of the space $\mathcal{B}$ are established for $\mathcal{F}_0$-measurable functions from $]-\infty, 0]$ to $H$, endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$, which satisfies the following axiom.

**Axiom 2.1** (A1) If $x : ]-\infty, T] \to H, T > 0$ is such that $x_0 \in \mathcal{B}$, then, for every $t \in J$, the following properties hold

(1) $x_t \in \mathcal{B}$;
(2) $|x(t)| \le L\|x_t\|_{\mathcal{B}}$;
(3) $\|x_t\|_{\mathcal{B}} \le K(t)\sup_{0\le s\le t}|x(s)| + N(t)\|x_0\|_{\mathcal{B}}$,

where $L > 0$ is a constant; $K, N : [0, +\infty[\to [1, +\infty[$, $K$ is continuous, $N$ is locally bounded, and $L, K, N$ are independent of $x(\cdot)$.

(A2) The space $\mathcal{B}$ is complete.

The next result is a consequence of the phase space axiom.

**Lemma 2.1** *Let $x : ]-\infty, T] \to H$ be an $\mathcal{F}_t$-adapted measurable process such that the $\mathcal{F}_0$-adapted process $x_0 = \varphi \in L_2^0(\Omega, \mathcal{B})$; then*

$$E\|x_s\|_{\mathcal{B}} \le N_T E\|\phi\|_{\mathcal{B}} + K_T \sup_{0\le s\le T} E\|x(s)\|,$$

*where $N_T = \sup_{t\in J}\{N(t)\}$ and $K_T = \sup_{t\in J}\{K(t)\}$.*

*Remark 2.1* In functional differential equations with infinite delay, the axiom of the abstract phase space $\mathcal{B}$ includes the continuity of the function $t \to x_t$; see for instance [12]. However, this property is not satisfied in stochastic functional differential equations with Poisson jumps and infinite delay and for this reason, has been eliminated in our abstract description of $\mathcal{B}$.

**Definition 2.1** Denote by $\mathcal{M}^2(]-\infty, T], H)$ the space of all $H$-valued, right continuous with left limits processes and $\mathcal{F}_t$-adapted process $x = \{x(t)\}_{-\infty<t\le T}$ such that

(i) $x_0 = \phi \in \mathcal{B}$ and $x(t)$ is a right continuous with left limits process on $[0, T]$;
(ii) define the norm $\|\cdot\|_{\mathcal{M}}$ in $\mathcal{M}^2(]-\infty, T], H)$ by

$$\|x\|^2_{\mathcal{M}} = E\|\phi\|^2_{\mathcal{B}} + E\int_0^T |x(t)|^2\,\mathrm{d}t < \infty. \tag{3}$$

Then, $\mathcal{M}^2(]-\infty, T], H)$ with the norm (3) is a Banach space. In the sequel, when there will be no fear of confusion, we will use $\|\cdot\|$ for this norm.

In what follows, we need the following facts on evolution operators.

Let $\{A(t), t \in J\}$ be a family of linear operators satisfying that

(A1) the domain $D(A(t)) = D$ of $A(t)$ is dense in $H$ and independent of $t$, $A(t)$ is a closed linear operator for $t \in J$;
(A2) for each $t \in J$, the resolvent $R(\lambda, A(t))$ exists for all $\lambda$ with $\mathrm{Re}(\lambda) \le 0$ and there exists a constant $K > 0$ such that $\|R(\lambda, A(t))\| \le \frac{K}{|\lambda|+1}$;
(A3) there exist constants $L > 0$ and $\alpha \in ]0, 1[$ such that for $t, s, \tau \in J$

$$\|A(t) - A(s)A^{-1}(\tau)\| \le L|t-s|^{\alpha};$$

(A4) for each $t \in J$ and some $\lambda \in \rho(A(t))$, the resolvent set of $A(t)$, $R(\lambda, A(t))$ is a compact operator.

**Definition 2.2** (see [13]) A two parameter family of bounded linear operators $U(t,s), 0 \le s \le t \le T$, on $H$ is called an evolution system iff the following two conditions hold

(1) $U(s,s) = I,\ U(t,r)U(r,s) = U(t,s)$, for $0 \le s \le r \le t \le T$.
(2) $(t,s) \to U(t,s)$ is strongly continuous for $0 \le s \le t \le T$.

**Lemma 2.2** (see [13]) *Assume that* (A1)–(A3) *hold. Then, there exist a unique evolution system* $U(t,s), 0 \le s \le t \le T$ *and a constant* $K > 0$ *such that*

(i) $\|U(t,s)\| \le K$ *for* $0 \le s \le t \le T$;
(ii) *for* $0 \le s \le t \le T, U(t,s): H \to D$ *and* $t \to U(t,s)$ *is strongly differentiable in* $H$. *The derivative* $\frac{\partial}{\partial t}U(t,s)$ *belongs to* $L(H)$ *and it is strongly continuous on* $0 \le s \le t \le T$. *Moreover, for all* $0 \le s \le t \le T$, *it holds*

$$\frac{\partial}{\partial t}U(t,s) + A(t)U(t,s) = 0,$$

$$\left\| \frac{\partial}{\partial t}U(t,s) \right\| = \|A(t)U(t,s)\| \le \frac{K}{t-s},$$

$$\|A(t)U(t,s)A(s)^{-1}\| \le K;$$

(iii) *for each* $v \in D$ *and* $t \in J$, $U(t,s)v$ *is differentiable with respect to* $s$ *on* $0 \le s \le t \le T$ *and* $\frac{\partial}{\partial t}U(t,s)v = U(t,s)A(s)v$.

**Lemma 2.3** (see [14]) *Let* $\{A(t), t \in J\}$ *be a family of linear operators satisfying* (A1)–(A4). *If* $\{U(t,s), 0 \le s \le t \le T\}$ *is the linear evolution system generated by* $\{A(t), t \in J\}$, *then* $\{U(t,s), 0 \le s \le t \le T\}$ *is a compact operator whenever* $t - s > 0$.

*Remark 2.2* If $A(t), t \ge 0$ is a second order differential operator $A$, i.e. $A(t) = A$ for each $t \ge 0$. Then, $A$ generates a $C_0$-semigroup $\{e^{At}, t \ge 0\}$.

Now, we present the definition of the mild solution of the system (1)–(2).

**Definition 2.3** A right continuous with left limits process $x: ]-\infty, T] \to H$ is called the mild solution of the system (1)–(2) iff

(i) $x(t)$ is $\mathcal{F}_t$-adapted and $\{x_t : t \in [0,T]\}$ is $\mathcal{B}$-valued;
(ii) $\int_0^T |x(s)|^2\, ds < \infty$, $P$-a.s.;
(iii) for each $t \in J$, $x(t)$ satisfies the following integral equation

$$x(t) = U(t,0)\phi(0) + \int_0^t U(t,s) f(s, x_s)\, ds + \int_0^t U(t,s)\sigma(s, x_s)\, dw(s)$$
$$+ \int_0^t U(t,s) \int_Z h\big(s, x(s-), z\big)\widetilde{N}(ds, dz);$$

(iv) $x_0 = \phi \in \mathcal{B}$.

In this work, we will work under the following assumptions:

(H1) $U(t,s)$ is a compact operator whenever $t-s>0$ and there exists a constant $K>0$ such that $\|U(t,s)\|^2 \le K$ for $0 \le s \le t \le T$.
(H2) The maps $f : J \times \mathcal{B} \to H$, $\sigma : J \times \mathcal{B} \to L_Q(M,H)$ and $h : J \times H \times (Z - \{0\}) \to H$ are all measurable functions satisfying for any $t \in J$, $\phi, \psi \in \mathcal{B}$, $x, y \in H$, $z \in Z - \{0\}$,

$$|f(t,\varphi) - f(t,\psi)|^2 \vee |\sigma(t,\varphi) - \sigma(t,\psi)|^2 \le \kappa\big(\|\varphi - \psi\|_{\mathcal{B}}^2\big),$$

$$\left(\int_Z |h(t,x,z) - h(t,x,z)|^2 \lambda(\mathrm{d}z)\mathrm{d}t\right) \vee \left(\int_Z |h(t,x,z) - h(t,y,z)|^4 \lambda(\mathrm{d}z)\mathrm{d}t\right)^{1/2} \le \kappa\big(|x-y|^2\big),$$

$$\left(\int_Z |h(t,x,z)|^4 \lambda(\mathrm{d}z)\mathrm{d}t\right)^{1/2} \le \kappa(|x|^2),$$

where $\kappa(\cdot)$ is a concave nondecreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa(0) = 0$, $\kappa(u) > 0$ for $u > 0$ and $\int_{0+} \frac{\mathrm{d}u}{\kappa(u)} = \infty$. Here, the symbol $\int_{0+}$ stands for $\lim_{\varepsilon \to 0+} \int_{\varepsilon}^{+\infty}$.
(H3) For all $t \in J$, there exists a positive constant $K$ such that

$$|f(t,0)|^2 \vee |\sigma(t,0)|^2 \vee \left(\int_Z |h(t,0,z)|^2 \lambda(\mathrm{d}z)\mathrm{d}t\right) \le K.$$

*Remark 2.3* Let us give some concrete functions $\kappa(\cdot)$. Let $K > 0$ and $\delta \in ]0,1[$ be sufficiently small. Define

$$\kappa_1(u) = Ku, u \ge 0.$$

$$\kappa_2(u) = \begin{cases} u \log(u^{-1}), & 0 \le u \le \delta, \\ \delta \log(\delta^{-1}) + \kappa_2'(\delta-)(u-\delta), & u > \delta. \end{cases}$$

$$\kappa_3(u) = \begin{cases} u \log(u^{-1}) \log\log(u^{-1}), & 0 \le u \le \delta, \\ \delta \log(\delta^{-1}) \log\log(\delta^{-1}) + \kappa_3'(\delta-)(u-\delta), & u > \delta. \end{cases}$$

They are all concave nondecreasing functions satisfying $\int_{0+} \frac{\mathrm{d}u}{\kappa_i(u)} = +\infty$ $(i = 1,2,3)$. In particular, we see that the Lipschitz condition is a special case of the proposed conditions.

In order to obtain the uniqueness of solutions, we give Bihari inequality which appeared in [15].

**Lemma 2.4** (Bihari inequality) *Let $T > 0$, $u_0 \ge 0$, $u(t)$ and $v(t)$ be two continuous functions on* [0,*T*]. *Let $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ be a concave continuous and nondecreasing* *function such that* $\kappa(r) > 0$ *for* $r > 0$. *If*

$$u(t) \leq u_0 + \int_0^t v(s)\kappa(u(s))\,\mathrm{d}s \quad \text{for all } 0 \leq t \leq T,$$

*then*

$$u(t) \leq G^{-1}\left(G(u_0) + \int_0^t v(s)\,\mathrm{d}s\right)$$

*for all* $t \in [0, T]$ *such that*

$$G(u_0) + \int_0^t v(s)\,\mathrm{d}s \in \mathrm{Dom}(G^{-1}),$$

*where* $G(r) = \int_1^r \frac{\mathrm{d}s}{\kappa(s)}$, $r \geq 0$ *and* $G^{-1}$ *is the inverse function of* $G$. *In particular, if, moreover,* $u_0 = 0$ *and* $\int_{0+} \frac{\mathrm{d}s}{\kappa(s)} = \infty$, *then* $u(t) = 0$ *for all* $0 \leq t \leq T$.

In order to obtain the stability of solutions, we give the extended Bihari inequality, which appeared in [16] (Lemma 3.2) and its Corollary.

**Lemma 2.5** *Let the assumptions of Lemma 2.4 hold. If*

$$u(t) \leq u_0 + \int_t^T v(s)\kappa(u(s))\,\mathrm{d}s \quad \text{for all } 0 \leq t \leq T,$$

*then*

$$u(t) \leq G^{-1}\left(G(u_0) + \int_t^T v(s)\,\mathrm{d}s\right)$$

*for all* $t \in [0, T]$ *such that*

$$G(u_0) + \int_t^T v(s)\,\mathrm{d}s \in \mathrm{Dom}(G^{-1}),$$

*where* $G(r) = \int_1^r \frac{\mathrm{d}s}{\kappa(s)}$, $r \geq 0$ *and* $G^{-1}$ *is the inverse function of* $G$.

**Corollary 2.1** *Let the assumptions of Lemma 2.4 hold and* $v(t) \geq 0$ *for* $t \in [0, T]$. *If for all* $\varepsilon > 0$, *there exists* $t_1 \geq 0$ *such that for* $0 \leq u_0 < \varepsilon$, $\int_{t_1}^T v(s)\,\mathrm{d}s \leq \int_{u_0}^{\varepsilon} \frac{\mathrm{d}s}{\kappa(s)}$ *holds. Then for every* $t \in [t_1, T]$, *the estimate* $u(t) \leq \varepsilon$ *holds.*

## 3 Existence and Uniqueness of Mild Solutions

In this section, we establish the existence and uniqueness theorem of mild solutions for the system (1)–(2). The main result of this section is the following theorem.

**Theorem 3.1** *Let* (H1)–(H3) *be satisfied, and* $x_0 = \phi \in \mathcal{B}$ *satisfy* $x_0 \in L_2^0(\Omega, H)$. *Then, there exists a unique mild solution of* (1)–(2) *in space* $\mathcal{M}^2(]-\infty, T], H)$.

In order to prove the result, we consider the sequence of successive approximations defined as follows:

- $x^0(t) = U(t,0)\phi(0), \quad t \in J,$
- $$x^n(t) = U(t,0)\phi(0) + \int_0^t U(t,s)f(s,x_s^{n-1})\,\mathrm{d}s + \int_0^t U(t,s)\sigma(s,x_s^{n-1})\,\mathrm{d}w(s) + \int_0^t U(t,s)\int_Z h\big(s,x^{n-1}(s-),z\big)\widetilde{N}(\mathrm{d}s,\mathrm{d}z), \quad t \in J, n \geq 1, \tag{4}$$
- $x^n(t) = \phi(t), \quad -\infty < t \leq 0, n \geq 1.$

**Lemma 3.1** *Assume the assumptions of Theorem 3.1 hold. Then, for all* $t \in ]-\infty, T]$, $n \geq 0$, *it holds that* $x^n(t) \in \mathcal{M}^2(]-\infty,T],H)$. *That is, there exists a positive constant* $C_1$ *such that*

$$\|x^n(t)\|^2 \leq C_1.$$

*Proof* Obviously, $x^0(t) \in \mathcal{M}^2(]-\infty,T];H)$. By induction, we have $x^n(t) \in \mathcal{M}^2(]-\infty,T];H)$. In fact, from (4), using the Hölder inequality, the Doob martingale inequality and the Burkhölder-Davis-Gundy inequality for pure jump stochastic integral in Hilbert space ([17] Theorem 35), we have

$$\begin{aligned}
&E\Big(\sup_{0\leq s\leq t}|x^n(s)|^2\Big)\\
&\quad\leq 4E|U(t,0)\phi(0)|^2 + 4E\left|\int_0^t U(t,s)f\big(s,x_s^{n-1}\big)\,\mathrm{d}s\right|^2\\
&\qquad+4E\left|\int_0^t U(t,s)\sigma\big(s,x_s^{n-1}\big)\,\mathrm{d}w(s)\right|^2\\
&\qquad+4E\left|\int_0^t U(t,s)\int_Z h\big(s,x^{n-1}(s-),z\big)\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right|^2\\
&\quad\leq 4KE\|\phi\|_{\mathcal{B}}^2 + 4KTE\int_0^t |f(s,x_s^{n-1})|^2\,\mathrm{d}s\\
&\qquad+4KE\int_0^t|\sigma(s,x_s^{n-1})|^2\,\mathrm{d}s + 4C\int_0^t\int_Z |h(t,x^{n-1}(s-),z)|^2\lambda(\mathrm{d}z)\mathrm{d}t\\
&\qquad+4C\int_0^t\left(\int_Z |h(t,x^{n-1}(s-),z)|^4\lambda(\mathrm{d}z)\right)^{1/2}\mathrm{d}t\\
&\quad\leq 4KE\|\phi\|_{\mathcal{B}}^2 + 4KTE\int_0^t |f(s,x_s^{n-1}) - f(s,0) + f(s,0)|^2\,\mathrm{d}s\\
&\qquad+4KE\int_0^t|\sigma(s,x_s^{n-1}) - \sigma(s,0) + \sigma(s,0)|^2\,\mathrm{d}s
\end{aligned}$$

$$+ 4C \int_0^t \int_Z |h(t, x^{n-1}(s-), z) - h(t, 0, z) + h(t, 0, z)|^2 \lambda(\mathrm{d}z)\mathrm{d}t$$
$$+ 4C \int_0^t \left( \int_Z |h(t, x^{n-1}(s-), z)|^4 \lambda(\mathrm{d}z) \right)^{1/2} \mathrm{d}t$$
$$\leq K_1 + 8K(T+1)E \int_0^t \kappa(\|x_s^{n-1}\|_{\mathcal{B}}^2)\,\mathrm{d}s + 12CE \int_0^t \kappa(|x^{n-1}(s)|^2)\,\mathrm{d}s,$$

where $C > 0$ is a constant and $K_1 = 4KE\|\phi\|_{\mathcal{B}}^2 + 8K^2T(T+1) + 8TCK$. Given that $\kappa(\cdot)$ is concave and $\kappa(0) = 0$, we can find a pair of positive constants $a$ and $b$ such that

$$\kappa(u) \leq a + bu, \quad \text{for all } u \geq 0.$$

So, by Lemma 2.1, we get

$$E\Big(\sup_{0\leq s\leq t} |x^n(s)|^2\Big) \tag{5}$$
$$\leq K_2 + 8M(T+1)bE \int_0^t \|x_s^{n-1}\|_{\mathcal{B}}^2\,\mathrm{d}s + 12CbE \int_0^t |x^{n-1}(s)|^2\,\mathrm{d}s$$
$$\leq K_2 + 8K(T+1)bE \int_0^t \Big(\sup_{0\leq r\leq s} |x^{n-1}(r)| + N\|x_0^n\|_{\mathcal{B}}\Big)^2 \mathrm{d}s$$
$$+ 12CbE \int_0^t \Big(\sup_{0\leq r\leq s} |x^{n-1}(r)|\Big)^2 \mathrm{d}s$$
$$\leq K_3 + \big[16K(T+1)b + 12Cb\big] \int_0^t E\Big(\sup_{0\leq r\leq s} |x^{n-1}(r)|\Big)^2 \mathrm{d}s, \tag{6}$$

where $K_2 = K_1 + 8TaK(T+1) + 12CTa$, $K_3 = K_2 + 8K(T+1)bTN^2E\|\phi\|_{\mathcal{B}}^2$.

For any $k \geq 1$, it follows from (5),

$$\max_{1\leq n\leq k} E\Big(\sup_{0\leq s\leq t} |x^n(s)|^2\Big)$$
$$\leq K_3 + \big[16K(T+1)b + 12Cb\big] \int_0^t \Big[E|x^0(s)|^2 + E \max_{1\leq n\leq k}\Big(\sup_{0\leq r\leq s} |x^n(r)|\Big)^2\Big] \mathrm{d}s$$
$$\leq K_3 + \big[16K(T+1)b + 12Cb\big]TE\|\phi\|_{\mathcal{B}}^2$$
$$+ \big[16K(T+1)b + 12Cb\big] \int_0^t E \max_{1\leq n\leq k}\Big(\sup_{0\leq r\leq s} |x^n(r)|\Big)^2 \mathrm{d}s$$
$$\leq K_4 + \big[16K(T+1)b + 12Cb\big] \int_0^t E \max_{1\leq n\leq k}\Big(\sup_{0\leq r\leq s} |x^n(r)|\Big)^2 \mathrm{d}s,$$

where $K_4 = K_3 + [16K(T+1)b + 12Cb]TE\|\phi\|_{\mathcal{B}}^2$.

From the Gronwall inequality, we derive that

$$\max_{1\le n\le k} E\Big(\sup_{0\le s\le t} |x^n(s)|^2\Big) \le K_4 \mathrm{e}^{16K(T+1)b+12Cb}.$$

Since $k$ is arbitrary, we have

$$E\Big(\sup_{0\le s\le t} |x^n(s)|^2\Big) \le K_4 \mathrm{e}^{16K(T+1)b+12Cb}, \quad \text{for all } 0\le t\le T,\ n\ge 1.$$

Thus, by the above result, we obtain

$$\begin{aligned}\|x^n\|^2 &= E\|x_0^n\|_{\mathcal{B}}^2 + E\int_0^T |x^n(s)|^2\,\mathrm{d}s \\ &\le E\|\phi\|_{\mathcal{B}}^2 + T K_4 \mathrm{e}^{16K(T+1)b+12Cb} < \infty,\end{aligned}$$

which shows that the desired result holds with $C_1 = E\|\phi\|_{\mathcal{B}}^2 + K_4\mathrm{e}^{16K(T+1)b+12Cb}$. □

**Lemma 3.2** *Under the assumptions of Theorem 3.1, there exists a positive constant $C_2$ such that*

$$E\Big(\sup_{0\le s\le t} |x^{n+m}(s)-x^n(s)|^2\Big) \le C_2\int_0^t \kappa\Big(E\Big(\sup_{0\le r\le s} |x^{n+m-1}(r)-x^{n-1}(r)|^2\Big)\Big)\,\mathrm{d}s$$

*for all* $0\le t\le T, n, m\ge 1$.

*Proof* From (4), for $n, m\ge 1$ and $0\le t\le T$, we derive that

$$\begin{aligned}&x^{n+m}(t)-x^n(t)\\ &\quad= \int_0^t U(t,s)\big[f\big(s,x_s^{n+m-1}\big)-f\big(s,x_s^{n-1}\big)\big]\,\mathrm{d}s\\ &\qquad+\int_0^t U(t,s)\big[\sigma\big(s,x_s^{n+m-1}\big)-\sigma\big(s,x_s^{n-1}\big)\big]\,\mathrm{d}w(s)\\ &\qquad+\int_0^t U(t,s)\int_Z\big[h\big(s,x^{n+m-1}(s-),z\big)-h\big(s,x^{n-1}(s-),z\big)\big]\widetilde{N}(\mathrm{d}s,\mathrm{d}z).\end{aligned}$$

Thus, we have

$$\begin{aligned}&E\Big(\sup_{0\le s\le t} |x^{n+m}(s)-x^n(s)|^2\Big)\\ &\quad\le 3KTE\int_0^t \big|f\big(s,x_s^{n+m-1}\big)-f\big(s,x_s^{n-1}\big)\big|^2\,\mathrm{d}s\\ &\qquad+3KE\int_0^t \big|\sigma\big(s,x_s^{n+m-1}\big)-\sigma\big(s,x_s^{n-1}\big)\big|^2\,\mathrm{d}s\end{aligned}$$

$$+ 3C \int_0^t \int_Z |h(t, x^{n+m-1}(s-), z) - h(t, x^{n-1}(s-), z)|^2 \lambda(\mathrm{d}z)\mathrm{d}t$$

$$+ 3C \int_0^t \left( \int_Z |h(t, x^{n+m-1}(s-), z) - h(t, x^{n-1}(s-), z)|^4 \lambda(\mathrm{d}z) \right)^{1/2} \mathrm{d}t$$

$$\leq 3K(T+1)E \int_0^t \kappa(\|x_s^{n+m-1} - x_s^{n-1}\|_{\mathcal{B}}^2)\,\mathrm{d}s$$

$$+ 6CE \int_0^t \kappa(|x^{n+m-1}(s) - x^{n-1}(s)|^2)\,\mathrm{d}s$$

$$\leq 3[K(T+1) + 2C]E \int_0^t \kappa\Big(\sup_{0\leq r\leq s} |x^{n+m-1}(r) - x^{n-1}(r)|^2\Big)\,\mathrm{d}s.$$

From the Jensen inequality, we get

$$E\Big(\sup_{0\leq s\leq t} |x^{n+m}(s) - x^n(s)|^2\Big)$$
$$\leq 3[K(T+1) + 2C] \int_0^t \kappa\Big(E\Big(\sup_{0\leq r\leq s} |x^{n+m-1}(r) - x^{n-1}(r)|^2\Big)\Big)\,\mathrm{d}s.$$

If we choose $C_2 = 3[K(T+1) + 2C]$, we obtain the desired result. □

**Lemma 3.3** *Under the assumptions of Theorem 3.1, there exists a positive constant $C_3$ such that*

$$E\Big(\sup_{0\leq s\leq t} |x^{n+m}(s) - x^n(s)|^2\Big) \leq C_3 t \tag{7}$$

*for all* $0 \leq t \leq T, n, m \geq 1$.

*Proof* From Lemma 3.1 and 3.2, we have

$$E\Big(\sup_{0\leq s\leq t} |x^{n+m}(s) - x^n(s)|^2\Big) \leq C_2 \int_0^t \kappa\Big(E\Big(\sup_{0\leq r\leq s} |x^{n+m-1}(r) - x^{n-1}(r)|^2\Big)\Big)\,\mathrm{d}s$$
$$\leq C_2 \int_0^t \kappa(2C_1)\,\mathrm{d}s$$
$$\leq C_2 \kappa(2C_2)t = C_3 t.$$

The proof is complete.

Define

$$\varphi_1(t) = C_3 t,$$
$$\varphi_{n+1}(t) = C_2 \int_0^t \kappa(\varphi_n(s))\,\mathrm{d}s, \quad n \geq 1,$$
$$\varphi_{n,m}(t) = E\Big(\sup_{0\leq r\leq t} |x^{n+m}(r) - x^n(r)|^2\Big), \quad n, m \geq 1.$$

Choose $T_1 \in [0, T[$ such that

$$C_2\kappa(C_3t) \le C_3, \quad \text{for all } 0 \le t \le T_1. \qquad \square$$

**Lemma 3.4** *There exists a positive* $0 \le T_1 < T$ *such that for all* $n, m \ge 1$,

$$0 \le \varphi_{n,m}(t) \le \varphi_n(t) \le \varphi_{n-1}(t) \le \cdots \le \varphi_1(t) \tag{8}$$

*for all* $0 \le t \le T_1$.

*Proof* We prove this Lemma by induction with respect to $n$. By Lemma 3.3, we have

$$\varphi_{1,m}(t) = E\Big(\sup_{0\le r\le t} |x^{1+m}(r) - x^1(r)|^2\Big) \le C_3t = \varphi_1(t).$$

By Lemma 3.3,

$$\begin{aligned}\varphi_{2,m}(t) &= E\Big(\sup_{0\le r\le t} |x^{2+m}(r) - x^2(r)|^2\Big)\\ &\le C_2\int_0^t \kappa\Big(E\Big(\sup_{t_0\le r\le s} |x^{1+m}(r) - x^1(r)|^2\Big)\Big)\,\mathrm{d}s\\ &\le C_2\int_0^t \kappa\big(\varphi_{1,m}(s)\big)\,\mathrm{d}s\\ &\le C_2\int_0^t \kappa\big(\varphi_1(s)\big)\,\mathrm{d}s = \varphi_1(t).\end{aligned}$$

So, we also have

$$\begin{aligned}\varphi_2(t) &= C_2\int_0^t \kappa\big(\varphi_1(s)\big)\,\mathrm{d}s \le C_2\int_0^t \kappa(C_3s)\,\mathrm{d}s\\ &\le C_2\int_0^t C_3ds = \varphi_1(t).\end{aligned}$$

We have already showed that

$$\varphi_{2,m}(t) \le \varphi_2(t) \le \varphi_1(t), \quad \text{for all } 0 \le t \le T_1.$$

Now, we assume that (8) holds for some $n \ge 1$. Then, using the same inequalities as above yields that

$$\begin{aligned}\varphi_{n+1,m}(t) &= C_2\int_0^t \kappa\Big(E\Big(\sup_{0\le r\le s} |x^{n+m}(r) - x^n(r)|^2\Big)\Big)\,\mathrm{d}s\\ &\le C_2\int_0^t \kappa\big(\varphi_{n,m}(s)\big)\,\mathrm{d}s\\ &\le C_2\int_0^t \kappa\big(\varphi_n(s)\big)\,\mathrm{d}s = \varphi_{n+1}(t)\end{aligned}$$

for all $0 \le t \le T_1$. On the other hand, we have

$$\varphi_{n+1}(t) = C_2 \int_0^t \kappa(\varphi_n(s))\,\mathrm{d}s \le C_2 \int_0^t \kappa(\varphi_{n-1}(s))\,\mathrm{d}s = \varphi_n(t),$$

for all $0 \le t \le T_1$. This completes the proof. □

*Proof of Theorem 3.1* *Uniqueness.* Let $x(t)$ and $\bar{x}(t)$ be two solutions of (1)–(2). Then, the uniqueness is obvious on the interval $]-\infty, 0]$ and for $0 \le t \le T$, by the same calculations as Lemma 3.2, we have

$$E\Big(\sup_{0\le s\le t} |x(s) - \bar{x}(s)|^2\Big) \le 3\big[K(T+1) + 2C\big] \int_0^t \kappa\Big(E\Big(\sup_{0\le r\le s} |x(r) - \bar{x}(r)|^2\Big)\Big)\,\mathrm{d}s.$$

The Bihari inequality yields that

$$E\Big(\sup_{0\le s\le t} |x(s) - \bar{x}(s)|^2\Big) = 0, \quad 0 \le t \le T. \tag{9}$$

The above expression means that $x(t) = \bar{x}(t)$ for all $0 \le t \le T$. Therefore, for all $-\infty < t \le T$, $x(t) = \bar{x}(t)$ a.s. This establish the uniqueness.

*Existence.* We claim that

$$E\Big(\sup_{0\le s\le t} |x^{n+m}(s) - x^n(s)|^2\Big) \to 0 \tag{10}$$

for all $0 \le t \le T_1$, as $n, m \to \infty$. Note that $\varphi_n$ is continuous on $[0, T_1]$. Note also that for each $n \ge 1$, $\varphi_n(\cdot)$ is decreasing on $[0, T_1]$, and for each $t$, $\varphi_n(t)$ is a decreasing sequence. Therefore, we can define the function $\varphi(t)$ as

$$\varphi(t) = \lim_{n\to\infty} \varphi_n(t) = \lim_{n\to\infty} C_2 \int_{t_0}^t \kappa\big(\varphi_{n-1}(s)\big)\,\mathrm{d}s = C_2 \int_{t_0}^t \kappa\big(\varphi(s)\big)\,\mathrm{d}s \tag{11}$$

for all $0 \le t \le T_1$. The Bihari inequality implies that $\varphi(t) = 0$ for all $0 \le t \le T_1$. Now, from Lemma 3.4, we have

$$\varphi_{n,n}(t) \le \sup_{t_0\le t\le T_1} \varphi_n(t) \le \varphi_n(T_1) \to 0 \tag{12}$$

as $n \to \infty$. That is $x^n(t)$ is a Cauchy sequence in $L^2$ on $]-\infty, T_1]$. From Lemma 3.1, we can easily derive that

$$\|x(t)\|^2 \le C,$$

where $C$ is a positive constant.

Using the property of the function to $\kappa(\cdot)$, we can obtain that for all $0 \le t \le T_1$,

$$E\left|\int_0^t U(t,s)[f(s, x_s^n) - f(s, x_s)]\,\mathrm{d}s\right|^2 \to 0, \quad \text{as } n \to \infty,$$

$$E\left|\int_0^t U(t,s)[\sigma(s,x_s^n)-\sigma(s,x_s)]\,\mathrm{d}w(s)\right|^2 \to 0, \quad \text{as } n\to\infty,$$

and

$$E\left|\int_0^t U(t,s)\int_Z [h(s,x^{n-1}(s-),z)-h(s,x(s-),z)]\widetilde{N}(\mathrm{d}s,\mathrm{d}z)\right|^2 \to 0, \quad \text{as } n\to\infty.$$

For all $0\le t\le T_1$, taking limits on both sides of (4), we obtain

$$x(t)=U(t,0)\phi(0)+\int_0^t U(t,s)f(s,x_s)\,\mathrm{d}s+\int_0^t U(t,s)\sigma(s,x_s)\,\mathrm{d}w(s) + \int_0^t U(t,s)\int_Z h\big(s,x(s-),z\big)\widetilde{N}(\mathrm{d}s,\mathrm{d}z).$$

The above expression demonstrates that $x(t)$ is one solution of (1)–(2) on $[0,T_1]$. By iteration, the existence of solutions to (1)–(2) on $[0,T]$ can be obtained. □

## 4 Stability of Solutions

In this section, we will give the continuous dependence of solutions on the initial value by means of the Corollary of Bihari inequality.

**Definition 4.1** A mild solution $x^\xi(t)$ of the system (1)–(2) with initial value $\xi$ is said to be stable in mean square iff for all $\varepsilon>0$ there exists $\delta>0$ such that

$$E\Big(\sup_{0\le s\le t}|x^\xi(s)-y^\eta(s)|^2\Big)\le\varepsilon, \quad \text{when } E\|\xi-\eta\|_{\mathcal{B}}^2<\delta, \tag{13}$$

where $x^\eta(t)$ is another solution of the system (1)–(2) with initial value $\eta$.

**Theorem 4.1** *Let $x^\xi(t)$ and $y^\eta(t)$ be solutions of the system* (1)–(2) *with initial value $\xi$ and $\eta$, respectively. Assume the assumptions of Theorem* 3.1 *be satisfied*; *then the solution of the system* (1)–(2) *is stable in mean square.*

*Proof* By assumptions, let $x(t)$ and $y(t)$ be two solutions of the system (1)–(2) with initial value $\xi$ and $\eta$, respectively. Using the same arguments as Lemma 3.2, we get

$$\begin{aligned}E\Big(\sup_{0\le s\le t}|x(s)-y(s)|^2\Big) &\le 4KE\|\xi-\eta\|_{\mathcal{B}}^2+4K(1+T)\int_0^t\kappa\big(E\|x_s-y_s\|_{\mathcal{B}}^2\big)\,\mathrm{d}s \\ &\quad+4C\int_0^t\kappa\big(E|x(s)-y(s)|^2\big)\,\mathrm{d}s \\ &\le 4KE\|\xi-\eta\|_{\mathcal{B}}^2+4\big[K(1+T)+C\big] \\ &\quad\times\int_0^t\kappa\Big(E\Big(\sup_{0\le r\le s}|x(r)-y(r)|^2\Big)\Big)\,\mathrm{d}s.\end{aligned}$$

Let $\kappa_1(u) = 4[K(1+T)+C]\kappa(u)$, where $\kappa$ is a concave increasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa(0)=0, \kappa(u)>0$ for $u>0$ and $\int_{0^+}\frac{\mathrm{d}u}{\kappa(u)} = +\infty$. Thus, $\kappa_1(u)$ is obvious a concave function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $\kappa_1(0)=0, \kappa(u)\geq \kappa(1)u$, for any $0\leq u\leq 1$ and $\int_{0^+}\frac{\mathrm{d}u}{\kappa_1(u)} = \infty$. So, for any $\varepsilon>0, \varepsilon_1 := \frac{1}{2}\varepsilon$, we have

$$\lim_{s\to 0}\int_s^{\varepsilon_1}\frac{\mathrm{d}u}{\kappa_1(u)} = +\infty.$$

Thus, there is a positive constant $\delta < \varepsilon_1$ such that $\int_\delta^{\varepsilon_1}\frac{\mathrm{d}u}{\kappa_1(u)} \geq T$. From Corollary 2.1, let $u_0 = 4KE\|\xi-\eta\|^2_{\mathcal{B}}, u(t) = E(\sup_{0\leq s\leq t}|x(s)-y(s)|^2), v(t)=1$, when $u_0\leq\delta\leq\varepsilon_1$, we have $\int_{u_0}^{\varepsilon_1}\frac{\mathrm{d}u}{\kappa_1(u)} \geq \int_\delta^{\varepsilon_1}\frac{\mathrm{d}u}{\kappa_1(u)} \geq T = \int_0^T v(s)\,\mathrm{d}s$. So, for any $t\in[0,T]$, the estimate $u(t)\leq\varepsilon_1$ holds. This completes the proof of the theorem. □

## 5 An Example

In this section, an example is provided to illustrate the obtained theory.

*Example 5.1* We consider the following stochastic partial differential equations with Poisson jumps and infinite delay of the form:

$$\begin{cases} \mathrm{d}v(t,\xi) = \left[\frac{\partial^2}{\partial x^2}v(t,\xi) + b(t,\xi)v(t,\xi)\right]\mathrm{d}t + F(t, v(t-h,\xi))\,\mathrm{d}t \\ \qquad\qquad + \Theta(t, v(t-h,\xi))\,\mathrm{d}\beta(t) + \int_Z h(t, v(t-,\xi), z)\widetilde{N}(\mathrm{d}t,\mathrm{d}z), \\ \quad 0\leq\xi\leq\pi, h>0, t\in J := [0,T], \\ v(t,0) = v(t,\pi) = 0, \quad t\in J, \\ v(t,\xi) = \varphi(t,\xi), \quad t\in\,]-\infty,0], 0\leq x\leq\pi, \end{cases} \tag{14}$$

where $\beta(t)$ denotes a standard cylindrical Wiener process and $\{k(t): t\in J\}$ is a Poisson point process (independent of $\beta(t)$) taking its value in the space $M = [0,\infty[$ with a $\sigma$-finite intensity measure $\lambda(\mathrm{d}z)$ defined on a stochastic space $(\Omega, \mathcal{F}, P)$. We denote by $N(\mathrm{d}s,\mathrm{d}z)$ the Poisson counting measure, which is induced by $k(\cdot)$, and the compensating martingale measure by

$$\widetilde{N}(\mathrm{d}s,\mathrm{d}z) := N(\mathrm{d}s,\mathrm{d}z) - \lambda(\mathrm{d}z)\mathrm{d}s.$$

To rewrite (14) into the abstract form of (1)–(2), we consider the space $H = L^2([0,\pi])$ and define the operator $A: D(A)\subset H\to H$ by $Az = z''$ with domain

$$D(A) := \{z\in H, z, z' \text{ are absolutely continuous } z''\in H, z(0)=z(\pi)=0\}.$$

Then, $A$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ on $H$. $A$ has discrete spectrum with eigenvalues $-n^2, n\in\mathbb{N}$, and the corresponding normalized eigenfunctions is given by $z_n(\xi) = \sum_{n=1}^\infty \mathrm{e}^{-n^2t}\sin(n\xi)$. Moreover, $\{z_n, n\in\mathbb{N}\}$ is an orthonormal basis of $H$ and for $x\in H, t\geq 0$, it holds $S(t)x = \sum_{n=1}^\infty \mathrm{e}^{-n^2t}(x,z_n)z_n$.

In addition, it follows that $S(t)$ is compact for every $t > 0$ and $\|S(t)\| \leq \mathrm{e}^{-t}$ for every $t \geq 0$.

Now, we define an operator $A(t) : D(A) \subset H \to H$ by

$$A(t)x(\xi) = Ax(\xi) + b(t,\xi)x(\xi).$$

Let $b(\cdot)$ be continuous and $b(t,\xi) \leq -\gamma$ $(\gamma > 0)$ for every $t \in \mathbb{R}$. Then, the system

$$\begin{cases} v'(t) = A(t)v(t), & t \geq s, \\ v(s) = x \in H, \end{cases}$$

has an associated evolution family, given by

$$U(t,s)x(\xi) = [S(t-s)\mathrm{e}^{\int_s^t b(\tau,\xi)\,\mathrm{d}\tau} x](\xi).$$

From the above expression, it follows that $U(t,s)$ is a compact operator and for every $t, s \in J$ with $t > s$

$$\|U(t,x)\| \leq \mathrm{e}^{-(1+\gamma)(t-s)}.$$

Let $v(t)(\cdot) = v(t,\cdot)$, for $t \in J$.We define respectively,

$$f(t,v)(\cdot) = F(t,v(\cdot)),\ \sigma(t,v)(\cdot) = \Theta(t,v(\cdot)).$$

Then, we can rewrite (14) in the form of (1)–(2). Furthermore, if we impose suitable conditions on $G, \Theta$ and $h$ to verify assumptions on Theorem 3.1, then we can conclude that the system (14) has a unique mild solution on $J$.

## 6 Concluding Remarks

In this paper, we prove the existence, uniqueness and stability of mild solutions for a class of time-dependent stochastic evolution equations, with Poisson jumps and infinite delay, under non-Lipschitz condition with Lipschitz condition being considered as a special case. An application to the stochastic nonlinear wave equation, with Poisson jumps and infinite delay, is given to illustrate the obtained theory.

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