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Convergence of Newton's Method for Sections on Riemannian Manifolds

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Abstract The present paper is concerned with the convergence problems of Newton's method and the uniqueness problems of singular points for sections on Riemannian manifolds. Suppose that the covariant derivative of the sections satisfies the generalized Lipschitz condition. The convergence balls of Newton's method and the uniqueness balls of singular points are estimated. Some applications to special cases, which include the Kantorovich's condition and the γ-condition, as well as the Smale's γ-theory for sections on Riemannian manifolds, are given. In particular, the estimates here are completely independent of the sectional curvature of the underlying Riemannian manifold and improve significantly the corresponding ones due to Dedieu, Priouret and Malajovich (IMA J. Numer. Anal. 23:395–419, 2003), as well as the ones in Li and Wang (Sci. China Ser. A. 48(11):1465–1478, 2005).

Keywords Newton's method · Riemannian manifold · Section · Generalized Lipschitz condition

1 Introduction

Many problems considered in optimization theory are formulated in linear spaces, where the linear structure plays an important role to develop numerical algorithms for solving the problems. At the same time, there are lots of optimization problems arising in various applications which can not be posed in linear spaces and requires

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a Riemannian manifold (in particular, a Hadamard manifold) structure for their formalization and study. For example, geometric models for human spine [3], eigenvalue optimization problems [4–6], nonconvex and nonsmooth problems of constrained optimization in $\mathbb{R}^n$ that can be reduced to convex and smooth unconstrained optimization problems on Riemannian manifolds as in [7–11], etc. We refer the reader to [6, 12–14] and the bibliographies therein for more examples and discussions.

In recent years, extensions of the concepts and techniques of optimization problems in linear spaces to the setting of Riemannian manifolds are done frequently, with theoretical objective and also to obtain effective algorithms as in [6, 8, 9, 14–21], etc. For example, in [8, 16, 17, 22], various derivative-like and subdifferential constructions for nondifferentiable functions on Riemannian manifolds have been developed, while, in [8, 23, 24], the extension of the maximal monotonicity to the setting of Riemannian manifolds makes it possible the development of a proximal-type method to approximate singularities of set-valued vector fields on a class of Riemannian manifolds with nonpositive sectional curvatures (i.e., on Hadamard manifolds). In [6, 14], several numerical algorithms such as steepest descent, conjugate gradient, trust-region method and so on, for solving optimization problems and/or equations are extended to the setting of Riemannian manifolds. As in the case of linear spaces, Newton's method is one of the most important methods to solve smooth optimization problems and/or to find singularity of vector fields on Riemannian manifolds. In particular, the extensions of the well-known Kantorovich theorem [25] and the famous Smale's α-theory and γ-theory [26] for Newton's method from linear spaces to Riemannian manifolds were done in [27] and [1, 28], respectively. We extended in [28] the notion of the γ-condition, which was presented by Wang in [29], to vector fields on Riemannian manifolds to establish the γ-theory and α-theory of Newton's method which improves the corresponding results in [1]. Alvarez et al. introduced in [30] a Lipschitz type radial function for the covariant derivative of vector fields and mappings on Riemannian manifolds to establish a unified curvature-free convergence criterion of Newton's method on Riemannian manifolds, which improves significantly the corresponding results in [1, 28]. Very recently, in [31], we extended the notion of the Lipschitz condition with L-average in [35] to Riemannian manifold settings and explored the semi-local behavior of Newton's method for sections on Riemannian manifolds. In particular, applications to analytic vector fields and mappings improve the corresponding results in [1, 28, 30].

Our interests in the present paper are focused on the local behavior of Newton's method for sections on Riemannian manifolds. Recall that the first results on the estimates of the radii of convergence balls of Newton's method on Riemannian manifolds are due to [1] for analytic vector fields and mappings. Then, the authors of [2] extended one kind of Lipschitz condition with L-average to vector fields on Riemannian manifolds and provided the estimates of convergence balls and of uniqueness balls of singular points of vector fields, which extend the corresponding estimate results in [1]. All the estimates are in terms of one geometric number depending on the sectional curvature of the underlying Riemannian manifold. In the present paper, under the assumption that the covariant derivatives of the sections satisfy one kind of Lipschitz condition with L-average, we give new estimates of the radii of convergence balls of Newton's method and the radii of uniqueness balls of singular points of sections on Riemannian manifolds. In particular, our estimates are completely independent of the sectional curvature and hence improve the corresponding results due to [2]. Applications to special cases, which include the Kantorovich's condition and the γ-condition, as well as the Smale's γ-theory for sections on Riemannian manifolds, are provided, which consequently improve the corresponding result in [1].

The paper is organized as follows. In Sect. 2, some basic concepts, results and notations on Riemannian manifolds are introduced. Results on estimates of the radii of uniqueness balls of singular points of sections and the radii of convergence balls are provided, respectively, in Sects. 3 and 4. Applications to special cases are presented in Sects. 5 and 6. In the last section, we give a criterion to determine a point being an approximate singular point of an analytic section.

2 Notions and Preliminaries

Let $\kappa \in \mathbb{N} \cup \{\infty, \omega\}$, and let M be a complete m-dimensional C^κ-Riemannian manifold with countable bases, where C^κ means smooth or analytic in the case when $\kappa = \infty$ or ω. Let $p \in M$ and T_pM denote the tangent space at p to M. We denote by $\langle \cdot, \cdot \rangle_p$ the scalar product on T_pM with the associated norm $\|\cdot\|_p$, where the subscript p is sometimes omitted. The tangent bundle TM of M is defined by

$$TM := \bigcup_{p \in M} T_pM.$$

Thus, a vector field X on M is a mapping from M to TM satisfying $X(p) \in T_pM$ for each $p \in M$. For $p, q \in M$, let $c : [0, 1] \to M$ be a piecewise smooth curve connecting p and q. Then the arc-length of c is defined by $l(c) := \int_0^1 \|c'(t)\| \mathrm{d}t$, while the Riemannian distance from p to q is defined by $d(p, q) := \inf_c l(c)$, where the infimum is taken over all piecewise smooth curves $c : [0, 1] \to M$ connecting p and q. Thus (M, d) is a complete metric space by the Hopf-Rinow Theorem (cf. [32]). Noting that M is complete, the exponential map $\exp_p : T_pM \to M$ at p is well-defined on T_pM. Recall that a geodesic c in M connecting p and q is called a minimizing geodesic iff its arc-length is equal to its Riemannian distance between p and q. Clearly, a curve $c : [0, 1] \to M$ is a minimizing geodesic connecting p and q if and only if there exists a vector $v \in T_pM$ such that $\|v\| = d(p, q)$ and $c(t) = \exp_p(tv)$ for each $t \in [0, 1]$.

Let ∇ denote the Levi-Civita connection on M and let $c : \mathbb{R} \to M$ be a C^κ-curve. Then we use $P_{c,\cdot,\cdot}$ to denote the parallel transport on tangent bundle TM along c with respect to ∇. In particular, we write $P_{\cdot,\cdot}$ for $P_{c,\cdot,\cdot}$ in the case where c is a minimizing geodesic.

In the remainder of this section, we describe simply the notion of sections, connections and parallel transports as well as some relative facts. For the details, the readers are refereed to [31] and some text books, for example, [32–34]. Recall that $\kappa \in \mathbb{N} \cup \{\infty, \omega\}$. Throughout the whole paper, we assume that E and M be C^κ-manifolds. Let $\pi : E \to M$ be a C^κ-vector bundle and ξ a C^κ-section of this vector bundle. Let D be a connection of this vector bundle. Since D is tensorial in X, the value of $\mathrm{D}_X\xi$ at $p\in M$ depends only on the tangent vector $v=X(p)\in T_pM$. Hence, the map $\mathrm{D}\xi(p): T_pM\to\pi^{-1}(p)$, given by

$$\mathrm{D}\xi(p)v := \mathrm{D}_X\xi(p)\quad \text{for each } v\in T_pM, \tag{1}$$

is well-defined and is a linear map from T_pM to $\pi^{-1}(p)$.

Let $c:\mathbb{R}\to M$ be a C^κ-curve. For any $a,b\in\mathbb{R}$, define the mapping

$$\mathcal{P}_{c,c(b),c(a)}:\pi^{-1}(c(a))\to\pi^{-1}(c(b))$$

by $\mathcal{P}_{c,c(b),c(a)}(v)=\eta_v(c(b))$ for each $v\in\pi^{-1}(c(a))$, where η_v is the unique C^κ-section such that $\mathrm{D}_{c'(t)}\eta_v=0$ and $\eta_v(c(a))=v$. Then $\mathcal{P}_{c,\cdot,\cdot}$ is called the parallel transport on vector bundle E along c. In particular, we write $\mathcal{P}_{c(b),c(a)}$ for $\mathcal{P}_{c,c(b),c(a)}$ in the case when c is a minimizing geodesic.

Let k be a positive integer and let ξ be a C^κ-section. Following [31], we now define inductively the covariant derivative of order k for ξ. Recall that D is a connection on the vector bundle $\pi:E\to M$ and ∇ is the Levi-Civita connection on M. Define the map $\mathcal{D}^1\xi=\mathcal{D}\xi:(C^\kappa(TM))^1\to C^{\kappa-1}(M,E)$ by

$$\mathcal{D}\xi(X)=\mathrm{D}_X\xi\quad\text{for each } X\in C^\kappa(TM) \tag{2}$$

and define the map $\mathcal{D}^k\xi:(C^\kappa(TM))^k\to C^{\kappa-k}(M,E)$ by

$$\begin{aligned}\mathcal{D}^k\xi(X_1,\ldots,X_{k-1},X) &= \mathrm{D}_X(\mathcal{D}^{k-1}\xi(X_1,\ldots,X_{k-1}))\\ &\quad-\sum_{i=1}^{k-1}\mathcal{D}^{k-1}\xi(X_1,\ldots,\nabla_X X_i,\ldots,X_{k-1})\end{aligned} \tag{3}$$

for each $X_1,\ldots,X_{k-1},X\in C^\kappa(TM)$. Then, one can use mathematical induction to prove easily that $\mathcal{D}^k\xi(X_1,\ldots,X_k)$ is tensorial with respect to each component X_i, that is, k multi-linear map from $(C^\kappa(TM))^k$ to $C^{\kappa-k}(M,E)$, where the linearity refers to the structure of $C^k(M)$-module. This implies that the value of $\mathcal{D}^k\xi(X_1,\ldots,X_k)$ at $p\in M$ only depends on the k-tuple of tangent vectors

$$(v_1,\ldots,v_k)=(X_1(p),\ldots,X_k(p))\in(T_pM)^k.$$

Consequently, for a given $p\in M$, the map $\mathcal{D}^k\xi(p):(T_pM)^k\to E_p$, defined by

$$\mathcal{D}^k\xi(p)v_1\cdots v_k:=\mathcal{D}^k\xi(X_1,\ldots,X_k)(p)\quad\text{for any } (v_1,\ldots,v_k)\in(T_pM)^k, \tag{4}$$

is well-defined, where $X_i\in C^\kappa(TM)$ satisfy $X_i(p)=v_i$ for each $i=1,\ldots,k$. Let $p_0\in M$ be such that $\mathrm{D}\xi(p_0)^{-1}$ exists. Thus, for any piece-geodesic curve c connecting p_0 and p, $\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathcal{D}^k\xi(p)$ is a k-multilinear map from $(T_pM)^k$ to $T_{p_0}M$. We define the norm of $\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathcal{D}^k\xi(p)$ by

$$\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathcal{D}^k\xi(p)\|=\sup\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p}\mathcal{D}^k\xi(p)v_1v_2\cdots v_k\|_{p_0},$$

where the supremum is taken over all k-tuple of vectors $(v_1, \ldots, v_k) \in (T_pM)^k$ with each $\|v_j\|_p = 1$. Furthermore, for any geodesic $c : \mathbb{R} \to M$ on M, since $\nabla_{c'(s)}c'(s) = 0$, it follows from (3) that

$$\mathcal{D}^k\xi(c(s))(c'(s))^k = \mathrm{D}_{c'(s)}(\mathcal{D}^{k-1}\xi(c(s))(c'(s))^{k-1}) \quad \text{for each } s \in \mathbb{R}. \tag{5}$$

The following lemma is taken from [31, Lemma 2.2], which plays an important role in this paper.

Lemma 2.1 *Let $c : \mathbb{R} \to M$ be a geodesic and let ζ be a C^κ-section. Then, for each $t \in \mathbb{R}$,*

$$\mathcal{P}_{c,c(0),c(t)}\zeta(c(t)) = \zeta(c(0)) + \int_0^t \mathcal{P}_{c,c(0),c(s)}(\mathrm{D}\zeta(c(s))c'(s))\mathrm{d}s. \tag{6}$$

In the remainder of this paper, we assume that ξ be a C^1-section and $p_0 \in M$. We end this section by giving Newton's method to sections on M (*cf.* [31]). Newton's method with initial point p_0 for ξ is defined as follows.

$$p_{n+1} = \exp_{p_n}(-\mathrm{D}\xi(p_n)^{-1}\xi(p_n)) \quad \text{for each } n = 0, 1, 2, \ldots. \tag{7}$$

3 Uniqueness Ball of Singular Points of Sections

For a Banach space or a Riemannian manifold Z, we use $\mathbf{B}_Z(p, r)$ and $\overline{\mathbf{B}_Z(p, r)}$ to denote respectively the open metric ball and the closed metric ball at p with radius r, that is,

$$\mathbf{B}_Z(p, r) = \{q \in Z : d(p, q) < r\} \quad \text{and} \quad \overline{\mathbf{B}_Z(p, r)} = \{q \in Z : d(p, q) \le r\}.$$

We often omit the subscript Z if no confusion caused.

Let $L(\cdot)$ be a positive integrable function on $[0, R]$, where R is a positive number large enough such that $\int_0^R (R - u)L(u)\mathrm{d}u \ge R$. The notion of center Lipschitz condition with the L average for operators from Banach spaces to Banach spaces was first introduced in [35] by Wang for the study of Smale's point estimate theory. The following definition extends this notion to sections on Riemannian manifold M. Recall that $\pi : E \to M$ is a C^κ-vector bundle with a connection D and ξ is a C^κ-section of this vector bundle.

Definition 3.1 *Let $0 < r \le R$. Let $p^* \in M$ be such that $\mathrm{D}\xi(p^*)^{-1}$ exists. Then $\mathrm{D}\xi(p^*)^{-1}\mathrm{D}\xi$ is said to satisfy the center Lipschitz condition with the L-average on $\mathbf{B}(p^*, r)$ iff for each $p \in \mathbf{B}(p^*, r)$, we have*

$$\|\mathrm{D}\xi(p^*)^{-1}(\mathcal{P}_{p^*,p}\mathrm{D}\xi(p)P_{p,p^*} - \mathrm{D}\xi(p^*))\| \le \int_0^{d(p^*,p)} L(u)\mathrm{d}u. \tag{8}$$

The main theorem of this section is as follows which shows that the uniqueness ball of singular points of sections is independent of the sectional curvature of the underlying manifold and hence improves the corresponding result in [2].

Theorem 3.1 *Let $r_u > 0$ be such that*

$$\frac{1}{r_u}\int_0^{r_u} L(u)(r_u - u)\mathrm{d}u \le 1. \tag{9}$$

Suppose that $\xi(p^*) = 0$ and $\mathrm{D}\xi(p^*)^{-1}\mathrm{D}\xi$ satisfies the center Lipschitz condition with the L-average on $\mathbf{B}(p^*, r_u)$. Then p^* is the unique singular pint of ξ on $\mathbf{B}(p^*, r_u)$.

Proof Let $q^* \in \mathbf{B}(p^*, r_u)$ be another singular pint of ξ in $\mathbf{B}(p^*, r_u)$. As $L(u) > 0$, it follows from [35] that the function ψ defined by

$$\psi(t) = \frac{1}{t}\int_0^t L(u)(t-u)\mathrm{d}u, \quad \forall t \in [0, r)$$

is strictly monotonically increasing. Set

$$\lambda = \frac{1}{d(q^*, p^*)}\int_0^{d(q^*,p^*)} L(u)(d(q^*, p^*) - u)\mathrm{d}u.$$

Then, by (9), we get

$$\lambda < \frac{1}{r_u}\int_0^{r_u} L(u)(r_u - u)\mathrm{d}u \le 1.$$

To complete the proof, it suffices to show that

$$d(q^*, p^*) \le \lambda d(q^*, p^*). \tag{10}$$

Granting this, one has that $q^* = p^*$. In fact, there exists $v \in T_{p^*}M$ such that $q^* = \exp_{p^*} v$ and $\|v\| = d(p^*, q^*)$. Define the curve $c(t) := \exp_{p^*} tv$ for each $t \in [0, 1]$. Thus,

$$d(p^*, q^*) = \|v\| = \|-\mathrm{D}\xi(p^*)^{-1}(\mathcal{P}_{c,p^*,q^*}\xi(q^*) - \xi(p^*)) + v\|. \tag{11}$$

Using Lemma 2.1 and (8), we obtain that

$$\begin{aligned}
&\|-\mathrm{D}\xi(p^*)^{-1}(\mathcal{P}_{c,p^*,q^*}\xi(q^*) - \xi(p^*)) + v\| \\
&\quad \le \|-\mathrm{D}\xi(p^*)^{-1}\int_0^1 \mathcal{P}_{c,p^*,c(\tau)}\mathrm{D}\xi(c(\tau))P_{c,c(\tau),p^*}v\mathrm{d}\tau + v\| \\
&\quad \le \int_0^1 \|-\mathrm{D}\xi(p^*)^{-1}(\mathcal{P}_{c,p^*,c(\tau)}\mathrm{D}\xi(c(\tau))P_{c,c(\tau),p^*} - \mathrm{D}\xi(p^*))\|\|v\|\mathrm{d}\tau \\
&\quad \le \int_0^1\int_0^{\tau d(q^*,p^*)} L(u)\mathrm{d}u \cdot d(q^*, p^*)\mathrm{d}\tau \\
&\quad = \int_0^{d(q^*,p^*)} L(u)(d(q^*, p^*) - u)\mathrm{d}u \\
&\quad = \lambda d(q^*, p^*).
\end{aligned}$$

This, together with (11), gives (10). This completes the proof of the theorem. □

4 Convergence Ball of Newton's Method

This section provides estimates of the radii of the convergence balls of Newton's method, which are independent of the sectional curvature of the underlying manifold.

Let $L(\cdot)$ be a positive nondecreasing integrable function on $[0, R]$, where R is a positive number large enough such that $\int_0^R (R-u)L(u)\mathrm{d}u \geq R$. The notion of the 2-piece L-average Lipschitz condition has been presented in [31] for the study of Newton's method on Riemannian manifolds. Let k be a positive integer. The following definition extends the k-piece L-average Lipschitz condition to sections on Riemannian manifolds.

Definition 4.1 *Let* $0 < r \leq R$. *Let* $p_0 \in M$ *be such that* $\mathrm{D}\xi(p_0)^{-1}$ *exists. Then* $\mathrm{D}\xi(p_0)^{-1}\mathrm{D}\xi$ *is said to satisfy the* k*-piece* L*-average Lipschitz condition on* $\mathbf{B}(p_0, r)$ *iff for any* k *points* $p_1, \ldots, p_k \in \mathbf{B}(p_0, r)$ *and for any geodesics* c_i *connecting* p_i, p_{i+1} *with* $i = 0, 1, \ldots, k-1$, c_0 *a minimizing geodesic connecting* p_0, p_1, *and* $\sum_{i=0}^{k-1} l(c_i) < r$, *we have*

$$\begin{aligned}
&\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_0,p_0,p_1}\cdots\mathcal{P}_{c_{k-2},p_{k-2},p_{k-1}}(\mathcal{P}_{c_{k-1},p_{k-1},p_k}\mathrm{D}\xi(p_k)\\
&\quad\times P_{c_{k-1},p_k,p_{k-1}} - \mathrm{D}\xi(p_{k-1}))\|\\
&\leq \int_{\sum_{i=0}^{k-2} l(c_i)}^{\sum_{i=0}^{k-1} l(c_i)} L(u)\mathrm{d}u.
\end{aligned} \tag{12}$$

Remark 4.1 (i) Clearly, the $(k+1)$-piece L-average Lipschitz condition implies the k-piece L-average Lipschitz condition.

(ii) The 1-piece L-average Lipschitz condition is equivalent to the center Lipschitz condition with the L-average.

Let $r_0 > 0$ and $b > 0$ be such that

$$\int_0^{r_0} L(u)\mathrm{d}u = 1 \quad \text{and} \quad b = \int_0^{r_0} L(u)u\mathrm{d}u. \tag{13}$$

The following proposition plays a key role in this section, which is taken from [31, Theorem 4.1].

Proposition 4.1 *Let* $p_0 \in M$ *be such that* $\mathrm{D}\xi(p_0)^{-1}$ *exists. Suppose that*

$$\beta := \|\mathrm{D}\xi(p_0)^{-1}\xi(p_0)\| \leq b \tag{14}$$

and that $\mathrm{D}\xi(p_0)^{-1}\mathrm{D}\xi$ *satisfies the* 2*-piece* L*-average Lipschitz condition on* $\mathbf{B}(p_0, r_0)$. *Then Newton's method* (7) *with initial point* p_0 *is well-defined and converges to a singular point* p^* *of* ξ *on* $\overline{\mathbf{B}(p_0, r_0)}$.

The following lemma estimates the value of the quantity $\|\mathrm{D}\xi(p_0)^{-1}\xi(p_0)\|$, which will be used in the proof of the main theorem of this section.

Lemma 4.1 *Let $0 < r \le r_0$ and let $p_0 \in \mathbf{B}(p^*, r)$. Suppose that $\mathrm{D}\xi(p^*)^{-1}\mathrm{D}\xi$ satisfies the center Lipschitz condition with the L-average on $\mathbf{B}(p^*, r)$. Then $\mathrm{D}\xi(p_0)^{-1}$ exists*,

$$\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,p^*}\mathrm{D}\xi(p^*)\| \le \frac{1}{1-\int_0^{d(p_0,p^*)} L(u)\mathrm{d}u} \tag{15}$$

and

$$\|\mathrm{D}\xi(p_0)^{-1}\xi(p_0)\| \le \frac{d(p_0, p^*) + \int_0^{d(p_0,p^*)} L(u)(u-d(p_0, p^*))\mathrm{d}u}{1-\int_0^{d(p_0,p^*)} L(u)\mathrm{d}u}. \tag{16}$$

Proof It follows from [31, Lemma 3.1] that $\mathrm{D}\xi(p_0)^{-1}$ exists and (15) holds. Below, we verify that (16) holds. Let $v \in T_{p^*}M$, and let c denote the geodesic connecting p^*, p and be defined by $c(t) := \exp_{p^*} tv$ for each $t \in [0, 1]$. Observe that

$$\begin{aligned}
&\mathcal{P}_{c,p^*,p_0}\xi(p_0)\\
&\quad = \mathcal{P}_{c,p^*,p_0}\xi(p_0) - \xi(p^*) - \mathcal{P}_{c,p^*,p_0}\mathrm{D}\xi(p_0)P_{c,p_0,p^*}v + \mathcal{P}_{c,p^*,p_0}\mathrm{D}\xi(p_0)P_{c,p_0,p^*}v\\
&\quad = \int_0^1 \mathcal{P}_{c,p^*,p_0}(\mathcal{P}_{c,p_0,c(\tau)}\mathrm{D}\xi(c(\tau))P_{c,c(\tau),p_0}\\
&\qquad - \mathrm{D}\xi(p_0))P_{c,p_0,p^*}v\mathrm{d}\tau + \mathcal{P}_{c,p^*,p_0}\mathrm{D}\xi(p_0)P_{c,p_0,p^*}v
\end{aligned} \tag{17}$$

where the second equality holds because of Lemma 2.1. This, together with (8) and (15), gives that

$$\begin{aligned}
&\|\mathrm{D}\xi(p_0)^{-1}\xi(p_0)\|\\
&\quad = \|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p^*}\mathcal{P}_{c,p^*,p_0}\xi(p_0)\|\\
&\quad \le \|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c,p_0,p^*}\mathrm{D}\xi(p^*)\|\\
&\qquad \cdot \int_0^1 \|\mathrm{D}\xi(p^*)^{-1}\mathcal{P}_{c,p^*,p_0}(\mathcal{P}_{c,p_0,c(\tau)}\mathrm{D}\xi(c(\tau))P_{c,c(\tau),p_0}\\
&\qquad - \mathrm{D}\xi(p_0))\|\|v\|\mathrm{d}\tau + \|v\|\\
&\quad \le \frac{1}{1-\int_0^{d(p_0,p^*)} L(u)\mathrm{d}u}\int_0^1\int_{\tau d(p_0,p^*)}^{d(p_0,p^*)} L(u)\mathrm{d}u\|v\|\mathrm{d}\tau + \|v\|\\
&\quad = \frac{\int_0^{d(p_0,p^*)} L(u)u\mathrm{d}u}{1-\int_0^{d(p_0,p^*)} L(u)\mathrm{d}u} + d(p_0, p^*).
\end{aligned} \tag{18}$$

Consequently, (16) is seen to hold and the proof is complete. □

We also need the following lemma whose proof is easy and so is omitted here.

Lemma 4.2 *Let* $\phi : [0, r_0] \to \mathbb{R}$ *be defined by*

$$\phi(t) = b - 2t + 2\int_0^t L(u)(t-u)\mathrm{d}u \quad \textit{for each } t \in [0, r_0], \tag{19}$$

where b *is given by* (13). *Then,* ϕ *is strictly decreasing on* $[0, r_0]$, *and has exactly one zero* $\hat{r}_0$ *in* $[0, r_0]$ *satisfying*

$$\frac{b}{2} < \hat{r}_0 < r_0. \tag{20}$$

Now, we are ready to give the main theorem of this section which shows that the radius of convergence ball of Newton's method is independent of the sectional curvature of the underlying Riemannian manifold.

Theorem 4.1 *Suppose that* $\xi(p^*) = 0$ *and that* $\mathrm{D}\xi(p^*)^{-1}\mathrm{D}\xi$ *satisfies the* 3*-piece* L*-average Lipschitz condition on* $\mathbf{B}(p^*, r_0)$. *If* $d(p^*, p_0) \le \hat{r}_0$ *with* $\hat{r}_0$ *given by Lemma* 4.2, *then the sequence* $\{p_n\}$ *generated by Newton's method* (7) *with initial point* p_0 *is well-defined and converges to* p^*.

Proof Write

$$\overline{L}(u) = \frac{L(u + d(p^*, p_0))}{1 - \int_0^{d(p_0,p^*)} L(u)\mathrm{d}u}. \tag{21}$$

Let $\bar{r}_0$, $\bar{b}$ be such that

$$\int_0^{\bar{r}_0} \bar{L}(u)\mathrm{d}u = 1 \quad \text{and} \quad \bar{b} = \int_0^{\bar{r}_0} \bar{L}(u)u\mathrm{d}u. \tag{22}$$

Clearly, $0 < \bar{r}_0 \le r_0$. Then, by the definition of r_0 and $\bar{r}_0$, one has

$$\begin{aligned}
\int_0^{\bar{r}_0} L(u + d(p_0, p^*))\mathrm{d}u &= 1 - \int_0^{d(p_0,p^*)} L(u)\mathrm{d}u \\
&= \int_0^{r_0} L(u)\mathrm{d}u - \int_0^{d(p_0,p^*)} L(u)\mathrm{d}u \\
&= \int_{d(p_0,p^*)}^{r_0} L(u)\mathrm{d}u.
\end{aligned}$$

Hence,

$$\int_{d(p_0,p^*)}^{\bar{r}_0 + d(p_0,p^*)} L(u)\mathrm{d}u = \int_{d(p_0,p^*)}^{r_0} L(u)\mathrm{d}u.$$

As $L(\cdot)$ is a nondecreasing and positive integrable function, one has

$$\bar{r}_0 + d(p_0, p^*) = r_0. \tag{23}$$

Therefore,

$$\bar{b} = \int_0^{\bar{r}_0} \bar{L}(u)u\mathrm{d}u = \int_0^{r_0 - d(p_0,p^*)} \bar{L}(u)u\mathrm{d}u$$
$$= \frac{\int_{d(p_0,p^*)}^{r_0} L(u)(u - d(p_0, p^*))\mathrm{d}u}{1 - \int_0^{d(p_0,p^*)} L(u)\mathrm{d}u}. \tag{24}$$

Below, we show that

$$\beta = \|\mathrm{D}\xi(p_0)^{-1}\xi(p_0)\| \le \bar{b}. \tag{25}$$

Since, $d(p_0, p^*) < \hat{r}_0 < r_0$, it follows from Lemma 4.1 that $\mathrm{D}\xi(p_0)^{-1}$ exists and

$$\|\mathrm{D}\xi(p_0)^{-1}\xi(p_0)\| \le \frac{d(p_0, p^*) + \int_0^{d(p_0,p^*)} L(u)(u - d(p_0, p^*))\mathrm{d}u}{1 - \int_0^{d(p_0,p^*)} L(u)\mathrm{d}u}. \tag{26}$$

Thus, by (24) and (26), to complete the proof of (25), it's sufficient to prove that

$$d(p_0, p^*) + \int_0^{d(p_0,p^*)} L(u)(u - d(p_0, p^*))\mathrm{d}u$$
$$\le \int_{d(p_0,p^*)}^{r_0} L(u)(u - d(p_0, p^*))\mathrm{d}u. \tag{27}$$

Noting that $b = \int_0^{r_0} L(u)u\mathrm{d}u$ and $\int_0^{r_0} L(u)\mathrm{d}u = 1$, we have

$$b - 2d(p_0, p^*) + 2\int_0^{d(p_0,p^*)} L(u)(d(p_0, p^*) - u)\mathrm{d}u$$
$$= \int_0^{r_0} L(u)u\mathrm{d}u - d(p_0, p^*) - d(p_0, p^*)\int_0^{r_0} L(u)\mathrm{d}u$$
$$- 2\int_0^{d(p_0,p^*)} L(u)(u - d(p_0, p^*))\mathrm{d}u$$
$$= \int_{d(p_0,p^*)}^{r_0} L(u)(u - d(p_0, p^*))\mathrm{d}u - d(p_0, p^*)$$
$$- \int_0^{d(p_0,p^*)} L(u)(u - d(p_0, p^*))\mathrm{d}u.$$

This means that (27) is equivalent to

$$b - 2d(p_0, p^*) + 2\int_0^{d(p_0,p^*)} L(u)(d(p_0, p^*) - u)\mathrm{d}u = \phi(d(p_0, p^*)) \ge 0. \tag{28}$$

Since $d(p_0, p^*) \le \hat{r}_0$ and ϕ is strictly decreasing, we have $\phi(d(p_0, p^*)) \ge \phi(\hat{r}_0) = 0$, that is, (28) holds. Therefore, (25) is true. Then in order to ensure that Proposition 4.1 is applicable, we have to show the following assertion: $\mathrm{D}\xi(p_0)^{-1}\mathrm{D}\xi$ satisfies the

2-piece $\bar{L}$-average Lipschitz condition on $\mathbf{B}(p_0,\bar{r}_0)$. Indeed, for any two points $p,q\in\mathbf{B}(p_0,\bar{r}_0)$, let c_1 be a minimizing geodesic connecting p_0 and p, and c_2 a geodesic connecting p and q such that $l(c_1)+l(c_2)<\bar{r}_0$. Since $\mathrm{D}\xi(p^*)^{-1}\mathrm{D}\xi$ satisfies the 3-piece L-average Lipschitz condition on $\mathbf{B}(p^*,r_0)$ and

$$d(p^*,p_0)+l(c_1)+l(c_2)<d(p_0,p^*)+\bar{r}_0=r_0$$

thanks to (23), we obtain that

$$\begin{aligned}&\|\mathrm{D}\xi(p^*)^{-1}\mathcal{P}_{p^*,p_0}\mathcal{P}_{c_1,p_0,p}(\mathcal{P}_{c_2,p,q}\mathrm{D}\xi(q)P_{c_2,q,p}-\mathrm{D}\xi(p))\|\\&\quad\le\int_{d(p^*,p_0)+l(c_1)}^{d(p^*,p_0)+l(c_1)+l(c_2)}L(u)\mathrm{d}u.\end{aligned}\tag{29}$$

Therefore, using (15) and (29), we conclude that

$$\begin{aligned}&\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}(\mathcal{P}_{c_2,p,q}\mathrm{D}\xi(q)P_{c_2,q,p}-\mathrm{D}\xi(p))\|\\&\quad\le\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,p^*}\mathrm{D}\xi(p^*)\|\\&\qquad\cdot\|\mathrm{D}\xi(p^*)^{-1}\mathcal{P}_{p^*,p_0}\mathcal{P}_{c_1,p_0,p}(\mathcal{P}_{c_2,p,q}\mathrm{D}\xi(q)P_{c_2,q,p}-\mathrm{D}\xi(p))\|\\&\quad\le\frac{1}{1-\int_0^{d(p_0,p^*)}L(u)\mathrm{d}u}\int_{d(p^*,p_0)+l(c_1)}^{d(p^*,p_0)+l(c_1)+l(c_2)}L(u)\mathrm{d}u\\&\quad=\int_{l(c_1)}^{l(c_1)+l(c_2)}\frac{L(u+d(p_0,p^*))}{1-\int_0^{d(p_0,p^*)}L(u)\mathrm{d}u}\mathrm{d}u\\&\quad=\int_{l(c_1)}^{l(c_1)+l(c_2)}\bar{L}(u)\mathrm{d}u.\end{aligned}$$

Hence, $\mathrm{D}\xi(p_0)^{-1}\mathrm{D}\xi$ satisfies the 2-piece $\bar{L}$-average Lipschitz condition on $\mathbf{B}(p_0,\bar{r}_0)$. Thus, by applying Proposition 4.1, we conclude that the sequence $\{p_n\}$ generated by Newton's method (7) with the initial point p_0 converges to a singular point q^* of ξ on $\overline{\mathbf{B}(p_0,\bar{r}_0)}$. Since $r_0\le r_u$ and

$$d(p^*,q^*)\le d(p^*,p_0)+d(p_0,q^*)\le d(p^*,p_0)+\bar{r}_0=r_0,$$

it follows from Theorem 3.1 that $q^*=p^*$. This completes the proof of the theorem. □

5 Theorems Under the Kantorovich's Condition and the γ-condition

This section is devoted to the study of some applications of the results obtained in the preceding sections. At first, in the case when the Lipschitz conditions with the Lipschitz constant $L>0$ in $\mathbf{B}(p^*,r)$ are satisfied, it is easy to get, by (9), (13) and Lemma 4.2, $r_u=\frac{2}{L}$, $r_0=\frac{1}{L}$ and $\hat{r}_0=\frac{2-\sqrt{2}}{2L}$. Hence, by Theorems 3.1 and 4.1, we have the following corollaries.

Corollary 5.1 *Suppose that $\xi(p^*)=0$ and $\mathrm{D}\xi(p^*)^{-1}\mathrm{D}\xi$ satisfies the center Lipschitz condition with the Lipschitz constant L on $\mathbf{B}(p^*, \frac{2}{L})$. Then p^* is the unique singular pint of ξ on $\mathbf{B}(p^*, \frac{2}{L})$.*

Corollary 5.2 *Suppose that $\mathrm{D}\xi(p^*)^{-1}\mathrm{D}\xi$ satisfies the 3-piece Lipschitz condition with the Lipschitz constant L on $\mathbf{B}(x^*, \frac{1}{L})$. If $d(p_0, p^*) \le \frac{2-\sqrt{2}}{2L}$, then Newton's method (7) with initial point p_0 is well-defined and converges to p^*.*

The γ-condition for operators in Banach spaces was first introduced by Wang [29] for the study of Smale's point estimate theory and extended to vector fields and sections on Riemannian manifold in [28] and [31], respectively.

In the remainder of this section, we always assume that ξ is a C^2-section. Let $r>0$ and $\gamma>0$ be such that $r\gamma<1$. Let $p_0\in M$ be such that $\mathrm{D}\xi(p_0)^{-1}$ exists. Let k be a positive integer.

Definition 5.1 *ξ is said to satisfy the k-piece γ-condition at p_0 on $\mathbf{B}(p_0,r)$ iff for any k points $p_1,\dots,p_k\in\mathbf{B}(p_0,r)$ and for any geodesics c_i connecting p_i, p_{i+1} with $i=0,1,\dots,k-1$, c_0 a minimizing geodesic connecting p_0, p_1, and $\sum_{i=0}^{k-1} l(c_i)<r$, we have*

$$\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_0,p_0,p_1}\cdots\mathcal{P}_{c_{k-1},p_{k-1},p_k}\mathcal{D}^2\xi(p_k)\| \le \frac{2\gamma}{(1-\gamma\sum_{i=0}^{k-1} l(c_i))^3}. \tag{30}$$

Obviously, the $(k+1)$-piece γ-condition implies the k-piece γ-condition.

Let $\gamma>0$ and let L be the function defined by

$$L(u)=\frac{2\gamma}{(1-\gamma u)^3} \quad \text{for each } 0<u<\frac{1}{\gamma}. \tag{31}$$

The following proposition shows that the γ-condition implies the L-average Lipschitz condition.

Proposition 5.1 *Suppose that ξ satisfies the 3-piece γ-condition at p_0 on $\mathbf{B}(p_0,r)$. Then $\mathrm{D}\xi(p_0)^{-1}\mathrm{D}\xi$ satisfies the 3-piece L-average Lipschitz condition on $\mathbf{B}(p_0,r)$ with L given by (31).*

Proof For any three points $p_1,p_2,p_3\in\mathbf{B}(p_0,r)$ and each $i=0,1,2$, let c_i be a geodesic connecting p_i, p_{i+1} such that c_0 is a minimizing geodesic and $l(c_0)+l(c_1)+l(c_2)<r$. To complete the proof, it is sufficient to prove that

$$\begin{aligned}&\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_0,p_0,p_1}\mathcal{P}_{c_1,p_1,p_2}(\mathcal{P}_{c_2,p_2,p_3}\mathrm{D}\xi(p_3)P_{c_2,p_3,p_2}-\mathrm{D}\xi(p_2))\|\\ &\quad\le\int_{l(c_0)+l(c_1)}^{l(c_0)+l(c_1)+l(c_2)}\frac{2\gamma}{(1-\gamma u)^3}\mathrm{d}u.\end{aligned} \tag{32}$$

Let $v\in T_{p_2}M$ be arbitrary. Then there exists a unique vector field Y such that $Y(c_2(0))=v$ and $\nabla_{c_2'(t)}Y=0$. Then $Y(c_2(s))=P_{c_2,c_2(s),p_2}v$ for each $s\in[0,1]$. Thus we apply Lemma 2.1 (to $\zeta = \mathrm{D}_Y\xi$) to conclude that

$$\begin{aligned}
&\mathcal{P}_{c_2,p_2,p_3}\mathrm{D}\xi(p_3)P_{c_2,p_3,p_2}v - \mathrm{D}\xi(p_2)v \\
&\quad = \mathcal{P}_{c_2,p_2,p_3}\mathrm{D}\xi(p_3)Y(p_3) - \mathrm{D}\xi(p_2)Y(p_2) \\
&\quad = \mathcal{P}_{c_2,p_2,p_3}\mathrm{D}_Y\xi(p_3) - \mathrm{D}_Y\xi(p_2) \\
&\quad = \int_0^1 \mathcal{P}_{c_2,p_2,c_2(s)}(\mathrm{D}(\mathrm{D}_Y\xi(c_2(s)))c_2'(s))\mathrm{d}s.
\end{aligned} \tag{33}$$

Since $\nabla_{c_2'(s)}Y(c_2(s)) = 0$, it follows that

$$\begin{aligned}
\mathcal{D}^2\xi(c_2(s))Y(c_2(s))c_2'(s) &= \mathrm{D}_{c_2'(s)}(\mathrm{D}_Y\xi(c_2(s)) - \mathrm{D}\xi(\nabla_{c_2'(s)}Y(c_2(s))) \\
&= \mathrm{D}(\mathrm{D}_Y\xi(c_2(s)))c_2'(s).
\end{aligned}$$

Combining this with (33), we have that

$$\begin{aligned}
&\mathcal{P}_{c_2,p_2,p_3}\mathrm{D}\xi(p_3)P_{c_2,p_3,p_2}v - \mathrm{D}\xi(p_2)v \\
&\quad = \int_0^1 \mathcal{P}_{c_2,p_2,c_2(s)}(\mathcal{D}^2\xi(c_2(s))Y(c_2(s))c_2'(s))\mathrm{d}s.
\end{aligned} \tag{34}$$

Since c_2 is a geodesic connecting p_2 and p_3, there exists $\overline{v} \in T_{p_2}M$ such that $p_3 = \exp_{p_2}(\overline{v})$ and $l(c_2) = \|\overline{v}\|$. It follows from (34) and (30) that

$$\begin{aligned}
&\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_0,p_0,p_1}\mathcal{P}_{c_1,p_1,p_2}(\mathcal{P}_{c_2,p_2,p_3}\mathrm{D}\xi(p_3)P_{c_2,p_3,p_2} - \mathrm{D}\xi(p_2))v\| \\
&\quad \le \int_0^1 \|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_0,p_0,p_1}\mathcal{P}_{c_1,p_1,p_2}\mathcal{P}_{c_2,p_2,c_2(s)}\mathcal{D}^2\xi(c_2(s))\|\|Y(c_2(s))\|\|c_2'(s)\|\mathrm{d}s \\
&\quad \le \int_0^1 \frac{2\gamma}{(1-\gamma(l(c_0)+l(c_1)+s\|\overline{v}\|))^3}\|\overline{v}\|\|v\|\mathrm{d}s \\
&\quad = \int_{l(c_0)+l(c_1)}^{l(c_0)+l(c_1)+l(c_2)} \frac{2\gamma}{(1-\gamma u)^3}\mathrm{d}u\|v\|.
\end{aligned}$$

As $v \in T_pM$ is arbitrary, (32) is seen to hold and the proof is complete. □

Suppose that L is defined by (31). Then, by (9), we compute $r_u = \frac{1}{2\gamma}$. Similarly to Proposition 5.1, we can prove that the 1-piece γ-condition implies the center Lipschitz condition. Therefore, applying Theorem 3.1, we get the following corollary. Recall that $p^* \in M$ is such that $\mathrm{D}\xi(p^*)^{-1}$ exists and $\xi(p^*) = 0$.

Corollary 5.3 *Suppose that $\xi(p^*) = 0$ and ξ satisfies the 1-piece γ-condition at p^* on $\mathbf{B}(p^*, \frac{1}{2\gamma})$. Then p^* is the unique singular point of ξ in $\mathbf{B}(p^*, \frac{1}{2\gamma})$.*

Suppose that L is defined by (31). Then, by (13) and Lemma 4.2, one has

$$r_0 = \frac{2-\sqrt{2}}{2\gamma} \quad \text{and} \quad \hat{r}_0 = \frac{5-2\sqrt{2}-\sqrt{12\sqrt{2}-15}}{8\gamma}.$$

Then, applying Theorem 4.1, we have the following corollary which shows that the radius of the convergence ball is independent of the sectional curvature of the underlying manifold and hence improves the corresponding results in [28].

Corollary 5.4 *Suppose that $\xi(p^*)=0$ and ξ satisfies the 3-piece γ-condition at p^* on $\mathbf{B}(p^*, \frac{2-\sqrt{2}}{2\gamma})$. If $d(p_0, p^*) < \frac{5-2\sqrt{2}-\sqrt{12\sqrt{2}-15}}{8\gamma}$, then Newton's method (7) with initial point p_0 is well-defined and converges to p^*.*

6 Applications to Analytic Sections and Generalized Smale's γ-theory

Throughout this section, we assume that M be a real complete analytic Riemannian manifold, and ξ be an analytic section. Following [1], we define, for a point $p \in M$,

$$\gamma(\xi, p) = \sup_{k\geq 2}\left\| \mathrm{D}\xi(p)^{-1}\frac{\mathcal{D}^k\xi(p)}{k!}\right\|_p^{\frac{1}{k-1}}. \tag{35}$$

Also we adopt the convention that $\gamma(\xi, p)=\infty$ if $\mathrm{D}\xi(p)$ is not invertible. Note that this definition is justified and in the case where $\mathrm{D}\xi(p)$ is invertible, by analyticity, $\gamma(\xi, p)$ is finite. Recall that $p^* \in M$ is such that $\mathrm{D}\xi(p^*)^{-1}$ exists.

For the study in the remainder, we need the following two lemmas which are taken from [31]. For notational simplicity, for $p \in M$ and $v \in T_{p^*}M$, c denotes the geodesic connecting p^*, p and is defined by $c(t) := \exp_{p^*} tv$ for each $t \in [0, 1]$ in Lemma 6.1. We use the function ψ defined by

$$\psi(u) := 1-4u+2u^2 \quad \text{for each } u \in \left[0, 1-\frac{\sqrt{2}}{2}\right). \tag{36}$$

Note that ψ is strictly monotonically decreasing on $[0, 1-\frac{\sqrt{2}}{2})$.

Lemma 6.1 *Let $r=\frac{1}{\gamma(\xi,p^*)}$. Let $p \in M$ and $v \in T_{p^*}M$ be such that $\|v\|<r$ and $p=\exp_{p^*} v$. Then*

$$\mathcal{D}^j\xi(p) = \mathcal{P}_{c,p,p^*}\left(\sum_{k=0}^{\infty}\frac{1}{k!}\mathcal{D}^{k+j}\xi(p^*)v^k\right)P^j_{c,p^*,p} \quad \textit{for each } j=0,1,2,\ldots, \tag{37}$$

where $P^j_{c,p^*,p}$ stands for the map from $(T_pM)^j$ to $(T_{p^*}M)^j$ defined by

$$P^j_{c,p^*,p}(v_1,\ldots,v_j) = (P_{c,p^*,p}v_1,\ldots,P_{c,p^*,p}v_j) \quad \textit{for each } (v_1,\ldots,v_j)\in(T_pM)^j.$$

Lemma 6.2 *Let $p \in M$ be such that*

$$u=\gamma(\xi,p^*)\,d(p^*,p) < 1-\frac{\sqrt{2}}{2}. \tag{38}$$

Then $\mathrm{D}\xi(p)^{-1}$ *exists, and*

$$\gamma(\xi,p)\le\frac{\gamma(\xi,p^*)}{(1-u)\psi(u)}.\tag{39}$$

Let $\gamma=\gamma(\xi,p^*)$. We show that any analytic section satisfies the γ-condition. For this purpose, we need a simple known fact (cf. [36, p. 150]):

$$\sum_{j=0}^{\infty}\frac{(k+j)!}{k!\,j!}t^j=\frac{1}{(1-t)^{k+1}}\quad\text{for each } t\in[-1.1] \text{ and } k=0,1,\ldots.\tag{40}$$

Proposition 6.1 *Let* $p^*\in M$ *and* $0<r\le\frac{2-\sqrt{2}}{2\gamma}$. *Suppose that* ξ *is analytic at* p^*. *Then* ξ *satisfies the* 3-*piece* γ-*condition at* p^* *on* $\mathbf{B}(p^*,r)$.

Proof Let $p_1,p,q\in\mathbf{B}(p^*,r)$. Let c_0 be a minimizing geodesic connecting p^*,p_1. Let c_1,c_2 be geodesics connecting p_1,p and p,q respectively such that $l(c_0)+l(c_1)+l(c_2)<r$. Then, there exist $v_1\in T_{p^*}M$, $v_2\in T_{p_1}M$ and $v_3\in T_pM$ satisfying $p_1=\exp_{p^*}v_1$, $p=\exp_{p_1}v_2$, $q=\exp_p v_3$, $l(c_0)=\|v_1\|$, $l(c_1)=\|v_2\|$ and $l(c_2)=\|v_3\|$. We claim that

$$\|v_3\|<\frac{1}{\gamma(\xi,p)},\qquad\|v_2\|<\frac{1}{\gamma(\xi,p_1)}\quad\text{and}\quad\|v_1\|<\frac{1}{\gamma(\xi,p^*)}.\tag{41}$$

We only show that $\|v_3\|<\frac{1}{\gamma(\xi,p)}$ since the proofs for $\|v_2\|<\frac{1}{\gamma(\xi,p_1)}$ and $\|v_1\|<\frac{1}{\gamma(\xi,p^*)}$ are similar. Write $u=\gamma(\xi,p^*)d(p^*,p)$. Since $p\in\mathbf{B}(p^*,r)$, Lemma 6.2 is applicable. It follows that

$$\frac{1}{\gamma(\xi,p)}\ge\frac{(1-u)\psi(u)}{\gamma(\xi,p^*)}.\tag{42}$$

By a simple calculation, we see that

$$\frac{(1-u)\psi(u)}{\gamma(\xi,p^*)}\ge\frac{2-\sqrt{2}}{2\gamma(\xi,p^*)}-d(p^*,p).\tag{43}$$

Noting that

$$d(p^*,p)+\|v_3\|\le\|v_1\|+\|v_2\|+\|v_3\|<r\le\frac{2-\sqrt{2}}{2\gamma(\xi,p^*)},$$

it follows from (42) and (43) that $\|v_3\|\le\frac{1}{\gamma(\xi,p)}$; hence our claim stands. Thus, by (41), Lemma 6.1 is applicable to concluding that

$$\begin{aligned}&\mathrm{D}\xi(p^*)^{-1}\mathcal{P}_{c_0,p^*,p_1}\mathcal{P}_{c_1,p_1,p}\mathcal{P}_{c_2,p,q}\mathcal{D}^2\xi(q)\\&\quad=\mathrm{D}\xi(p^*)^{-1}\mathcal{P}_{c_0,p^*,p_1}\mathcal{P}_{c_1,p_1,p}\sum_{l=0}^{\infty}\frac{1}{l!}\mathcal{D}^{l+2}\xi(p)v_3^lP^2_{c_2,p,q}\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{D}\xi(p^*)^{-1}\mathcal{P}_{c_0,p^*,p_1}\sum_{l=0}^{\infty}\frac{1}{l!}\sum_{j=0}^{\infty}\frac{1}{j!}\mathcal{D}^{l+j+2}\xi(p_1)v_2^j P_{c_1,p_1,p}^{l+2}v_3^l P_{c_2,p,q}^2 \\
&= \sum_{l=0}^{\infty}\frac{1}{l!}\sum_{j=0}^{\infty}\frac{1}{j!}\sum_{k=0}^{\infty}\frac{1}{k!}\mathrm{D}\xi(p^*)^{-1}\mathcal{D}^{l+j+k+2}\xi(p^*)v_1^k P_{c_0,p^*,p_1}^{l+j+2}v_2^j \\
&\quad \times P_{c_1,p_1,p}^{l+2}v_3^l P_{c_2,p,q}^2.
\end{aligned} \tag{44}
$$

Since

$$
\frac{\|\mathrm{D}\xi(p^*)^{-1}\mathcal{D}^{l+j+k+2}\xi(p^*)\|}{(l+j+k+2)!} \le \gamma(\xi,p^*)^{l+j+k+1},
$$

one has from (44) that

$$
\begin{aligned}
&\|\mathrm{D}\xi(p^*)^{-1}\mathcal{P}_{c_0,p^*,p_1}\mathcal{P}_{c_1,p_1,p}\mathcal{P}_{c_2,p,q}\mathcal{D}^2\xi(q)\| \\
&\quad \le \sum_{l=0}^{\infty}\frac{1}{l!}\sum_{j=0}^{\infty}\frac{(l+j+2)!}{j!}\sum_{k=0}^{\infty}\frac{(l+j+k+2)!}{k!(l+j+2)!} \\
&\quad \times \gamma(\xi,p^*)^{l+j+k+1}\|v_1\|^k\|v_2\|^j\|v_3\|^l.
\end{aligned} \tag{45}
$$

Using (40) to calculate the quantity on the right-hand side of the inequality (45), we get that

$$
\begin{aligned}
&\|\mathrm{D}\xi(p^*)^{-1}\mathcal{P}_{c_0,p^*,p_1}\mathcal{P}_{c_1,p_1,p}\mathcal{P}_{c_2,p,q}\mathcal{D}^2\xi(q)\| \\
&\quad \le \frac{2\gamma(\xi,p^*)}{(1-\gamma(\xi,p^*)(\|v_1\|+\|v_2\|+\|v_3\|)^3}.
\end{aligned} \tag{46}
$$

Since $l(c_0)+l(c_1)+l(c_2)=\|v_1\|+\|v_2\|+\|v_3\|$, it follows from (46) that

$$
\begin{aligned}
&\|\mathrm{D}\xi(p^*)^{-1}\mathcal{P}_{c_0,p^*,p_1}\mathcal{P}_{c_1,p_1,p}\mathcal{P}_{c_2,p,q}\mathcal{D}^2\xi(q)\| \\
&\quad \le \frac{2\gamma(\xi,p^*)}{(1-\gamma(\xi,p^*)(l(c_0)+l(c_1)+l(c_2)))^3}.
\end{aligned}
$$

Hence ξ satisfies 3-piece γ-condition at p^* on $\mathbf{B}(p^*,r)$ and the proof is complete. □

Since the 3-piece γ-condition implies the 1-piece γ-condition, one has that an analytic section satisfies the 1-piece γ-condition. Hence, applying Corollary 5.3, we get the following Corollary 6.1, which shows that the radius of the uniqueness ball is independent of the sectional curvature of the underlying manifold and hence improves significantly the corresponding results of [1] and [2]. Recall that ξ is analytic on M and $p^*\in M$ is such that $\mathrm{D}\xi(p^*)^{-1}$ exists. Write $\gamma=\gamma(\xi,p^*)$.

Corollary 6.1 *Suppose $\xi(p^*)=0$. Then p^* is the unique singular point of ξ on $\mathbf{B}(p^*,\frac{1}{2\gamma})$.*

Using Corollary 5.4 and Proposition 6.1, we get the following corollary which shows that the radius of the convergence ball is independent of the sectional curvature of the underlying manifold and so improves the corresponding results in Li and Wang [28] and Dedieu et al. [1].

Corollary 6.2 *Suppose that* $\xi(p^*) = 0$. *If*

$$d(p_0, p^*) < \frac{5 - 2\sqrt{2} - \sqrt{12\sqrt{2} - 15}}{8\gamma},$$

then Newton's method (7) *with the initial point* p_0 *is well-defined and converges to* p^*.

7 Approximate Singular Point

This section is devoted to the study of a criterion to determine a point being an approximate singular point of an analytic section. In [31], the notion of the approximate singular point has been extended to the case of sections on Riemannian manifolds as follows.

Definition 7.1 *Suppose* $p_0 \in M$ *is such that Newton's method* (7) *is well-defined for* ξ *and satisfies*

$$\Theta(p_n) \le \left(\frac{1}{2}\right)^{2^{n-1}} \Theta(p_{n-1}) \quad \textit{for all } n = 1, 2, \ldots,$$

where $\Theta(p_n)$ *denotes some measurement of the approximation degree between* p_n *and the singular point* p^*. *Then* p_0 *is said to be an approximate singular point of* ξ *in the sense of* $\Theta(p_n)$.

The following proposition is useful, which is taken from [31].

Proposition 7.1 *Let* $\gamma > 0$ *and* $\beta = \|\mathrm{D}\xi(p_0)^{-1}\xi(p_0)\|$. *Suppose that*

$$\alpha = \beta\gamma \le \frac{13 - 3\sqrt{17}}{4} \approx 0.157671$$

and that ξ *satisfies the* 2*-piece* γ*-condition at* p_0 *in* $\mathbf{B}(p_0, \frac{2-\sqrt{2}}{2\gamma})$. *Then Newton's method* (7) *with initial point* p_0 *is well-defined and the generated sequence* $\{p_n\}$ *converges to a singular point* p^* *of* ξ *in* $\overline{\mathbf{B}(p_0, \frac{2-\sqrt{2}}{2\gamma})}$. *Moreover, there hold for each* $n = 2, 3, \ldots,$

$$\begin{aligned} &\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,p_{n-1}}\mathcal{P}_{c_n,p_{n-1},p_n}\xi(p_n)\| \\ &\quad \le \left(\frac{1}{2}\right)^{2^{n-1}} \|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,p_{n-2}}\mathcal{P}_{c_{n-1},p_{n-2},p_{n-1}}\xi(p_{n-1})\|, \end{aligned} \tag{47}$$

and for each $n = 1, 2, \ldots,$

$$\|\mathrm{D}\xi(p_n)^{-1}\xi(p_n)\| \leq \left(\frac{1}{2}\right)^{2^{n-1}} \|\mathrm{D}\xi(p_{n-1})^{-1}\xi(p_{n-1})\|, \tag{48}$$

where c_n *is defined by*

$$c_n(\lambda) := \exp_{p_{n-1}}(-\lambda \mathrm{D}\xi(p_{n-1})^{-1}\xi(p_{n-1})) \quad \textit{for each } \lambda \in [0, 1]. \tag{49}$$

Below, we assume that ξ be an analytic section at p^*,

$$\xi(p^*) = 0 \quad \text{and} \quad \gamma = \gamma(\xi, p^*).$$

Recall that $\psi(w) = 1 - 4w + 2w^2$ for each $w \in [0, \frac{2-\sqrt{2}}{2})$. Let $w_0 = 0.0858167\cdots$ be the smallest positive root of the equation

$$\frac{w - 2w^2}{\psi(w)^2} = \frac{13 - 3\sqrt{17}}{4}.$$

Then, we have the following corollary.

Corollary 7.1 *Suppose* $\xi(p^*) = 0$ *and*

$$d(p_0, p^*) < \frac{w_0}{\gamma} = \frac{0.0858167\cdots}{\gamma}.$$

For each $n = 1, 2, \ldots,$ *let* c_n *be the geodesic defined by* (49). *Then* p_0 *is an approximate singular point of* ξ *in the senses of* $\|\mathrm{D}\xi(p_n)^{-1}\xi(p_n)\|$ *and* $\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,p_{n-1}} \mathcal{P}_{c_n,p_{n-1},p_n}\xi(p_n)\|$.

Proof Since ξ is analytic at p^*, one has from Proposition 6.1 that ξ satisfies the 3-piece γ-condition at p^* in $\mathbf{B}(p^*, \frac{2-\sqrt{2}}{2\gamma})$. Thus, by Proposition 5.1, $\mathrm{D}\xi(p^*)^{-1}\mathrm{D}\xi$ satisfies the 3-piece L-average Lipschitz condition in $\mathbf{B}(p^*, r)$ with L given by (31). Write

$$w = d(p_0, p^*)\gamma.$$

We apply Lemma 4.1 (with L given by (31)) to concluding that

$$\begin{aligned}\beta &= \|\mathrm{D}\xi(p_0)^{-1}\xi(p_0)\| \\ &\leq \frac{d(p_0, p^*) + \int_0^{d(p_0,p^*)} \frac{2\gamma(u - d(p_0,p^*))}{(1-\gamma u)^3}\mathrm{d}u}{1 - \int_0^{d(p_0,p^*)} \frac{2\gamma}{(1-\gamma u)^3}\mathrm{d}u} \\ &= \frac{(1-w)(1-2w)}{\psi(w)} \frac{w}{\gamma}.\end{aligned} \tag{50}$$

Set

$$\bar{\gamma} = \frac{\gamma}{\psi(w)(1-w)} \quad \text{and} \quad \bar{\alpha} = \beta\bar{\gamma}.$$

Then, one has from (50) that

$$\bar{\alpha} = \beta\bar{\gamma} \le \frac{w - 2w^2}{\psi(w)^2} < \frac{w_0 - 2w_0^2}{\psi(w_0)^2} = \frac{13 - 3\sqrt{17}}{4}, \tag{51}$$

where the second inequality holds because the function $w \to \frac{w-2w^2}{\psi(w)^2}$ is strictly increasing on $[0, \frac{2-\sqrt{2}}{2})$ and $w = d(p_0, p^*)\gamma < w_0$. Write $\bar{r}_0 = \frac{2-\sqrt{2}}{2\bar{\gamma}}$. Note by standard differential technique,

$$\begin{aligned}\frac{2-\sqrt{2}}{2\gamma} - d(p_0, p^*) &= \frac{2-\sqrt{2}}{2\gamma} - \frac{w}{\gamma} \ge \frac{2-\sqrt{2}}{2} \cdot \frac{\psi(w)(1-w)}{\gamma} \\ &= \frac{2-\sqrt{2}}{2\bar{\gamma}} = \bar{r}_0.\end{aligned} \tag{52}$$

Thus in order that the Proposition 7.1 is applicable, it's sufficient to verify that ξ satisfies the 2-piece $\bar{\gamma}$-condition at p_0 in $\mathbf{B}(p_0, \bar{r}_0)$. Indeed, for any two points $p, q \in \mathbf{B}(p_0, \bar{r}_0)$, let c_1 be a minimizing geodesic connecting p_0, p and c_2 a geodesic connecting p, q such that $l(c_1) + l(c_2) < \bar{r}_0$. Let c_0 be a minimizing geodesic connecting p^*, p_0. Since ξ satisfies 3-piece γ-condition at p^* in $\mathbf{B}(p^*, \frac{2-\sqrt{2}}{2\gamma})$ and $l(c_0) + l(c_1) + l(c_2) < d(p^*, p_0) + \bar{r}_0 \le \frac{2-\sqrt{2}}{2\gamma}$ thanks to (52), we obtain that

$$\|\mathrm{D}\xi(p^*)^{-1}\mathcal{P}_{c_0,p^*,p_0}\mathcal{P}_{c_1,p_0,p}\mathcal{P}_{c_2,p,q}\mathrm{D}^2\xi(q)\| \le \frac{2\gamma}{(1-\gamma(l(c_0)+l(c_1)+l(c_2)))^3}.$$

This, together with (15) (with L given by (31)), gives that

$$\begin{aligned}&\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_1,p_0,p}\mathcal{P}_{c_2,p,q}\mathrm{D}^2\xi(p)\| \\ &\quad= \|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{c_0,p_0,p^*}\mathrm{D}\xi(p^*)\|\|\mathrm{D}\xi(p^*)^{-1}\mathcal{P}_{c_0,p^*,p_0}\mathcal{P}_{c_1,p_0,p}\mathcal{P}_{c_2,p,q}\mathrm{D}^2\xi(q)\| \\ &\quad\le \frac{(1-w)^2}{\psi(w)}\frac{2\gamma}{(1-\gamma(l(c_0)+l(c_1)+l(c_2)))^3} \\ &\quad= \frac{2\gamma}{\psi(w)(1-w)}\frac{(1-w)^3}{(1-w-\gamma(l(c_1)+l(c_2)))^3} \\ &\quad= \frac{2\bar{\gamma}}{(1-\frac{\gamma}{1-w}(l(c_1)+l(c_2)))^3} \\ &\quad\le \frac{2\bar{\gamma}}{(1-\frac{\gamma}{\psi(w)(1-w)}(l(c_1)+l(c_2)))^3} \\ &\quad= \frac{2\bar{\gamma}}{(1-\bar{\gamma}(l(c_1)+l(c_2)))^3},\end{aligned}$$

because $l(c_0) = d(p^*, p_0)$ and $0 < \psi(w) < 1$ for each $w \in (0, 1 - \frac{\sqrt{2}}{2})$. Therefore, ξ satisfies 2-piece $\bar{\gamma}$-condition at p_0 in $\mathbf{B}(p_0, \bar{r}_0)$. Combining this and (51) yields that Proposition 7.1 is applicable to concluding that Newton's method (7) with initial point p_0 is well-defined and the generated sequence $\{p_n\}$ converges to a singular point q^* of ξ in $\overline{\mathbf{B}(p_0, \bar{r}_0)}$. Moreover, (47) and (48) hold. Noting by (52) that

$$d(p^*, q^*) \le d(p^*, p_0) + d(p_0, q^*) \le d(p^*, p_0) + \bar{r}_0 \le \frac{2-\sqrt{2}}{2\gamma} < \frac{1}{2\gamma},$$

it follows from Corollary 6.1 that $q^* = p^*$. Hence, p_0 is an approximate singular point of ξ in the senses of $\|\mathrm{D}\xi(p_n)^{-1}\xi(p_n)\|$ and

$$\|\mathrm{D}\xi(p_0)^{-1}\mathcal{P}_{p_0,p_{n-1}}\mathcal{P}_{c_n,p_{n-1},p_n}\xi(p_n)\|. \qquad \square$$

8 Conclusion

We have explored in the present paper the local behavior of Newton's method for sections on Riemannian manifolds. Under the assumption that the covariant derivatives of the sections satisfy one kind of Lipschitz condition with L-average, new estimates of the radii of convergence balls of Newton's method and the radii of uniqueness balls of singular points of sections on Riemannian manifolds are given. In particular, our estimates are completely independent of the sectional curvature and hence improve the corresponding results due to [2]. Applications to special cases, which include the Kantorovich's condition and the γ-condition, as well as the Smale's γ-theory for sections on Riemannian manifolds, are provided, which consequently improve the corresponding result in [1].

References

1. Dedieu, J.P., Priouret, P., Malajovich, G.: Newton's method on Riemannian manifolds: covariant alpha theory. IMA J. Numer. Anal. **23**, 395–419 (2003)
2. Li, C., Wang, J.H.: Convergence of Newton's method and uniqueness of zeros of vector fields on Riemannian manifolds. Sci. China Ser. A **48**(11), 1465–1478 (2005)
3. Adler, R., Dedieu, J.P., Margulies, J., Martens, M., Shub, M.: Newton method on Riemannian manifolds and a geometric model for human spine. IMA J. Numer. Anal. **22**, 1–32 (2002)
4. Burke, J.V., Lewis, A., Overton, M.: Optimal stability and eigenvalue multiplicity. Found. Comput. Math. **1**, 205–225 (2001)
5. Mahony, R.E.: The constrained Newton method on a Lie group and the symmetric eigenvalue problem. Linear Algebra Appl. **248**, 67–89 (1996)
6. Smith, S.T.: Optimization techniques on Riemannian manifolds. In: Fields Institute Communications, vol. 3, pp. 113–146. Am. Math. Soc., Providence (1994)
7. Da Cruz Neto, J.X., Ferreira, O.P., Lucambio Pérez, L.R.: Monotone point-to-set vector fields. Balk. J. Geom. Appl. **5**, 69–79 (2000)
8. Ferreira, O.P., Oliveira, P.R.: Proximal point algorithm on Riemannian manifolds. Optimization **51**, 257–270 (2002)
9. Ferreira, O.P., Lucambio Pérez, L.R., Nemeth, S.Z.: Singularities of monotone vector fields and an extragradient-type algorithm. J. Glob. Optim **31**, 133–151 (2005)
10. Rapcsk, T.: Smooth Nonlinear Optimization in $\mathbb{R}^n$. Nonconvex Optimization and Its Applications, vol. 19. Kluwer Academic, Dordrecht (1997)
11. Martín-Márquez, V.: Nonexpansive mappings and monotone vector fields in Hadamard manifold. Commun. Appl. Anal. **13**, 633–646 (2009)

12. Edelman, A., Arias, T.A., Smith, T.: The geometry of algorithms with orthogonality constraints. SIAM J. Matrix Anal. Appl. **20**, 303–353 (1998)
13. Smith, S.T.: Geometric optimization method for adaptive filtering. Ph.D. thesis, Harvard University, Cambridge, MA (1993)
14. Absil, P.A., Baker, C.G., Gallivan, K.A.: Trust-region methods on Riemannian manifolds. Found. Comput. Math. **7**, 303–330 (2007)
15. Udriste, C.: Convex Functions and Optimization Methods on Riemannian Manifolds. Mathematics and Its Applications, vol. 297. Kluwer Academic, Dordrecht (1994)
16. Li, S.L., Li, C., Liou, Y.C., Yao, J.C.: Existence of solutions for variational inequalities on Riemannian manifolds. Nonlinear Anal. **71**(11), 5695–5706 (2009)
17. Ledyaev, Y.S., Zhu, Q.J.: Nonsmooth analysis on smooth manifolds. Trans. Am. Math. Soc. **359**, 3687–3732 (2007)
18. Wang, J.H., Huang, S., Li, C.: Extended Newton's method for mappings on Riemannian manifolds with values in a cone. Taiwan. J. Math. **13**, 633–656 (2009)
19. Wang, J.H., Li, C.: Convergence of the family of Euler-Halley type methods on Riemannian manifolds under the γ-condition. Taiwan. J. Math. **13**(2), 585–606 (2009)
20. Wang, J.H., Li, C.: Uniqueness of the singular point of vector field on Riemannian manifold under the γ-condition. J. Complex. **22**, 533–548 (2006)
21. Li, C., Wang, J.H., Dedieu, J.P.: Smale's point estimate theory for Newton's method on Lie groups. J. Complex. **25**, 128–151 (2009)
22. Azagra, D., Ferrera, J., López-Mesas, F.: Nonsmooth analysis and Hamilton-Jacobi equations on Riemannian manifolds. J. Funct. Anal. **220**, 304–361 (2005)
23. Li, C., López, G., Martín-Márquez, V.: Monotone vector fields and the proximal point algorithm on Hadamard manifolds. J. Lond. Math. Soc. **79**(3), 663–683 (2009)
24. Németh, S.Z.: Monotone vector fields. Publ. Math. (Debr.) **54**(3–4), 437–449 (1999)
25. Kantorovich, L.V., Akilov, G.P.: Functional Analysis. Oxford, Pergamon (1982)
26. Smale, S.: Newton's method estimates from data at one point. In: Ewing, R., Gross, K., Martin, C. (eds.) The Merging of Disciplines: New Directions in Pure, Applied and Computational Mathematics, pp. 185–196. Springer, New York (1986)
27. Ferreira, O.P., Svaiter, B.F.: Kantorovich's Theorem on Newton's method in Riemannian manifolds. J. Complex. **18**, 304–329 (2002)
28. Li, C., Wang, J.H.: Newton's method on Riemannian manifolds: Smale's point estimate theory under the γ-condition. IMA J. Numer. Anal. **26**, 228–251 (2006)
29. Wang, X.H., Han, D.F.: Criterion α and Newton's method. Chin. J. Numer. Appl. Math. **19**, 96–105 (1997)
30. Alvarez, F., Bolte, J., Munier, J.: A unifying local convergence result for Newton's method in Riemannian manifolds. Found. Comput. Math. **8**, 197–226 (2008)
31. Li, C., Wang, J.H.: Newton's method for sections on Riemannian manifolds: generalized covariant α-theory. J. Complex. **24**, 423–451 (2008)
32. DoCarmo, M.P.: Riemannian Geometry. Birkhauser, Boston (1992)
33. Chern, S.S.: Vector bundle with connection. In: Selected Papers, vol. 4. pp. 245–268. Springer, New York (1989)
34. Wells, R.O.: Differential Analysis on Complex Manifolds. GTM, vol. 65. Springer, New York (1980)
35. Wang, X.H.: Convergence of Newton's method and uniqueness of the solution of equations in Banach space. IMA J. Numer. Anal. **20**(1), 123–134 (2000)
36. Blum, L., Cucker, F., Shub, M., Smale, S.: Complexity and Real Computation. Springer, New York (1997)