J Optim Theory Appl (2011) 148: 197–208
DOI 10.1007/s10957-010-9740-z

Optimality Conditions and Duality for Nondifferentiable Multiobjective Fractional Programming Problems with (C, α, ρ, d)-convexity

X.J. Long

Published online: 28 August 2010

Abstract The purpose of this paper is to consider a class of nondifferentiable multiobjective fractional programming problems in which every component of the objective function contains a term involving the support function of a compact convex set. Based on the (C, α, ρ, d)-convexity, sufficient optimality conditions and duality results for weakly efficient solutions of the nondifferentiable multiobjective fractional programming problem are established. The results extend and improve the corresponding results in the literature.

Keywords Nondifferentiable multiobjective fractional programming · Sufficient optimality condition · Duality · Weakly efficient solution · (C, α, ρ, d)-convexity

1 Introduction

In recent years, multiobjective fractional programming problems have received much attention by many authors due to in many practical optimization problems the objective functions are quotients of two functions (see, for example, [1–3, 6, 8–11, 14, 16, 17] and the references therein). In particular, Bector et al. [1] derived Fritz John and Karush-Kuhn-Tucker necessary and sufficient optimality conditions for a class of nondifferentiable convex multiobjective fractional programming problems and established some duality theorems for such problems. Following the

Communicated by P.M. Pardalos.

This work was supported by the National Natural Science Foundation of China (No. 11001287), the Education Committee Project Research Foundation of Chongqing (No. KJ100711), the Natural Science Foundation Project of Chongqing (CSTC 2009BB3372) and the Research Fund of Chongqing Technology and Business University (09-56-06).

X.J. Long (✉)
College of Mathematics and Statistics, Chongqing Technology and Business University, Chongqing 400067, P.R. China
e-mail: xianjunlong@hotmail.com

approaches of Bector et al. [1], Liu [9, 10] obtained some necessary and sufficient optimality conditions and duality theorems for a class of nonsmooth multiobjective fractional programming problems involving pseudoinvex functions or (F,ρ)-convex functions. Kuh et al. [6] established generalized Karush-Kuhn-Tucker necessary and sufficient optimality conditions and duality theorems for nonsmooth multiobjective fractional programming problems involving V-ρ-invex functions. Liang et al. [7] introduced the concept of (F,α,ρ,d)-convexity and obtained some optimality conditions and duality results for nonlinear fractional programming problems. Liang et al. [8] further obtained some optimality conditions and duality results for multiobjective fractional programming problems in the framework of (F,α,ρ,d)-convexity. Later, Liu and Feng [11] extended the results of Liang et al. [8] to nonsmooth case.

On the other hand, Mond and Schechter [13] studied non-differentiable symmetric duality, in which the objective functions contain a support function. Based on the ideas of Mond and Schechter [13], Yang et al. [18] studied generalized dual problems for a class of nondifferentiable multiobjective programs. Recently, under assumption of V-ρ-invexity, Kim et al. [4] derived some necessary and sufficient optimality conditions and duality results for nondifferentiable multiobjective fractional programming problems in which the objective function contains a support function. Very recently, Long et al. [12] extended their results to (F,α,ρ,d)-convexity.

Convexity and generalized convexity play an central role in mathematical economics, engineering, management science, and optimization theory. Therefore, the research on convexity and generalized convexity is one of the most important aspects in mathematical programming. In a recent paper [19], Yuan et al. introduced a class of functions, which called (C,α,ρ,d)-convex function and which includes (F,α,ρ,d)-convexity [7], V-ρ-invexity [5], (F,ρ)-convexity [15] as special cases. They obtained sufficient optimality conditions for nondifferentiable minimax fractional programming problems. Chinchuluun et al. [3] later studied nonsmooth multiobjective fractional programming problems in the framework of (C,α,ρ,d)-convexity.

In this paper, we are motivated by [4, 12, 19] to consider a class of nondifferentiable multiobjective fractional programming problems in which each component of the objective function contains a term involving the support function of a compact convex set. We derive some sufficient optimality conditions and duality results for weakly efficient solutions of nondifferentiable multiobjective fractional programming problems under the assumptions of (C,α,ρ,d)-convexity. The results extend and improve the corresponding results in the literature.

2 Preliminaries

Throughout this paper, let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^n_+$ be nonnegative orthant of $\mathbb{R}^n$. Let X be an open subset of $\mathbb{R}^n$. Assume that $\alpha : X \times X \to \mathbb{R}_+ \backslash \{0\}$, $\rho \in \mathbb{R}$ and $d : X \times X \to \mathbb{R}_+$ satisfies $d(x,x_0)=0 \Leftrightarrow x=x_0$. Let $C : X \times X \times \mathbb{R}^n \to \mathbb{R}$ be a function which satisfies $C_{(x,x_0)}(0)=0$ for any $(x,x_0)\in X\times X$.

Definition 2.1 A function $C : X \times X \times \mathbb{R}^n \to \mathbb{R}$ is said to be convex on $\mathbb{R}^n$ iff for any fixed $(x,x_0)\in X\times X$ and for any $y_1,y_2\in R^n$, one has

$$C_{(x,x_0)}\big(\lambda y_1+(1-\lambda)y_2\big)\le \lambda C_{(x,x_0)}(y_1)+(1-\lambda)C_{(x,x_0)}(y_2),\quad \forall\lambda\in]0,1[.$$

Using the convexity of C, Yuan et al. [19] introduced the following definition.

Definition 2.2 A differentiable function $h : X \to \mathbb{R}$ is said to be (C, α, ρ, d)-convex at $x_0 \in X$ iff for any $x \in X$,

$$\frac{h(x) - h(x_0)}{\alpha(x, x_0)} \geq C_{(x,x_0)}\big(\nabla h(x_0)\big) + \rho \frac{d(x, x_0)}{\alpha(x, x_0)}.$$

The function h is said to be (C, α, ρ, d)-convex on X iff it is (C, α, ρ, d)-convex at every point in X. In particular, h is said to be strongly (C, α, ρ, d)-convex on X iff $\rho > 0$.

Remark 2.1 If the function C is sublinear with respect to the third argument, then the (C, α, ρ, d)-convexity is the same as the (F, α, ρ, d)-convexity introduced by Liang et al. [7].

Remark 2.2 Every (F, α, ρ, d)-convex function is (C, α, ρ, d)-convex. However, the converse is not true. This can be seen from the following example.

Example 2.1 Let $X = \{x : \frac{\pi}{4} \leq x \leq \frac{\pi}{2}\}$, $\rho = -1$, $\alpha(x, x_0) = 1$, $d(x, x_0) = \sqrt{(x - x_0)^2}$ and $C(x, x_0; a) = a^2(x - x_0)$ for any $(x, x_0) \in X \times X$. Let $h(x) = \sin^2 x$. Obviously, the function C is not sublinear with respect to the third argument. Then, h is not (F, α, ρ, d)-convex at $x_0 = \frac{\pi}{4}$. It is easy to prove that h is (C, α, ρ, d)-convex at $x_0 = \frac{\pi}{4}$.

In this paper, we consider the following multiobjective fractional programming problem:

$$\text{(MFP)} \qquad \min\left(\frac{f_1(x) + s(x|C_1)}{g_1(x)}, \frac{f_2(x) + s(x|C_2)}{g_2(x)}, \ldots, \frac{f_p(x) + s(x|C_p)}{g_p(x)}\right),$$
$$\text{s.t.} \quad h(x) \leq 0,$$

where $f_i : X \to \mathbb{R}$, $g_i : X \to \mathbb{R}$, $i = 1, 2, \ldots, p$ and $h = (h_1, h_2, \ldots, h_m) : X \to \mathbb{R}^m$, are continuously differentiable functions over X. Suppose that $f_i(x) + s(x|C_i) \geq 0$ and $g_i(x) > 0$ for any $x \in X$, $i = 1, 2, \ldots, p$; C_i, for each $i \in \{1, 2, \ldots, p\}$, is a compact convex set of $\mathbb{R}^n$, and $s(x|C_i)$ denotes the support function of C_i evaluated at x, defined by

$$s(x|C_i) = \max\big\{\langle x, w\rangle | w \in C_i\big\}.$$

Let $S = \{x \in X : h(x) \leq 0\}$ be the set of all feasible solutions and let $I(x) := \{j : h_j(x) = 0\}$ for any $x \in X$.

Let

$$k_i(x) = s(x|C_i), \quad i = 1, 2, \ldots, p.$$

Then, k_i is a convex function and

$$\partial k_i(x) = \big\{w \in C_i \mid \langle w, x\rangle = s(x|C_i)\big\},$$

where ∂k_i is the subdifferentiable of k_i (see [13]).

Theorem 2.1 *Let f and g be two real-valued differentiable functions defined on X, such that $f(x)+\langle w,x\rangle \geq 0$ and $g(x)>0$ for all $x\in X$. If $f(\cdot)+\langle w,\cdot\rangle$ and $-g(\cdot)$ are (C,α,ρ,d)-convex at $x_0\in X$, then $\frac{f(\cdot)+\langle w,\cdot\rangle}{g(\cdot)}$ is $(\overline{C},\overline{\alpha},\overline{\rho},\overline{d})$-convex at x_0, where*

$$\overline{\alpha}(x,x_0)=\frac{g(x_0)\alpha(x,x_0)}{g(x)},\qquad \overline{\rho}=\rho\left(1+\frac{f(x_0)+\langle w,x_0\rangle}{g(x_0)}\right),$$
$$\overline{d}(x,x_0)=\frac{d(x,x_0)}{g(x)}$$

and

$$\overline{C}_{(x,x_0)}(a)=\frac{f(x_0)+\langle w,x_0\rangle+g(x_0)}{g^2(x_0)}\cdot C_{(x,x_0)}\left(\frac{g^2(x_0)}{f(x_0)+\langle w,x_0\rangle+g(x_0)}\cdot a\right),$$

$a:=\nabla(\frac{f(x_0)+\langle w,x_0\rangle}{g(x_0)})$, *for all $x\in X$.*

Proof Let $k(x)=s(x|C)$ and $w\in\partial k(x_0)$. Then, for any $x,x_0\in X$,

$$\frac{f(x)+\langle w,x\rangle}{g(x)}-\frac{f(x_0)+\langle w,x_0\rangle}{g(x_0)}=\frac{f(x)+\langle w,x\rangle-f(x_0)-\langle w,x_0\rangle}{g(x)}$$
$$-\left[f(x_0)+\langle w,x_0\rangle\right]\cdot\frac{g(x)-g(x_0)}{g(x)g(x_0)}.$$

By the definition of (C,α,ρ,d)-convexity and the above equation, we have

$$\begin{aligned}
&\frac{1}{\alpha(x,x_0)}\left(\frac{f(x)+\langle w,x\rangle}{g(x)}-\frac{f(x_0)+\langle w,x_0\rangle}{g(x_0)}\right)\\
&\quad\geq\frac{1}{g(x)}\left(C_{(x,x_0)}\left(\nabla\left[f(x_0)+\langle w,x_0\rangle\right]\right)+\rho\frac{d(x,x_0)}{\alpha(x,x_0)}\right)\\
&\qquad+\frac{f(x_0)+\langle w,x_0\rangle}{g(x)g(x_0)}\left(C_{(x,x_0)}\left(-\nabla g(x_0)\right)+\rho\frac{d(x,x_0)}{\alpha(x,x_0)}\right)\\
&\quad=\frac{f(x_0)+\langle w,x_0\rangle+g(x_0)}{g(x)g(x_0)}\cdot\frac{g(x_0)}{f(x_0)+\langle w,x_0\rangle+g(x_0)}\\
&\qquad\times\left(C_{(x,x_0)}\left(\nabla\left[f(x_0)+\langle w,x_0\rangle\right]\right)\right)\\
&\qquad+\frac{f(x_0)+\langle w,x_0\rangle+g(x_0)}{g(x)g(x_0)}\cdot\frac{f(x_0)+\langle w,x_0\rangle}{f(x_0)+\langle w,x_0\rangle+g(x_0)}\left(C_{(x,x_0)}\left(-\nabla g(x_0)\right)\right)\\
&\qquad+\rho\frac{f(x_0)+\langle w,x_0\rangle+g(x_0)}{g(x)g(x_0)}\cdot\frac{d(x,x_0)}{\alpha(x,x_0)}\\
&\quad\geq\frac{f(x_0)+\langle w,x_0\rangle+g(x_0)}{g(x)g(x_0)}\left(C_{(x,x_0)}\left(\frac{g^2(x_0)}{f(x_0)+\langle w,x_0\rangle+g(x_0)}\right.\right.\\
&\qquad\times\left.\left.\left(\frac{g(x_0)\nabla[f(x_0)+\langle w,x_0\rangle]}{g^2(x_0)}-\frac{f(x_0)+\langle w,x_0\rangle}{g^2(x_0)}\cdot\nabla g(x_0)\right)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\rho\frac{f(x_0)+\langle w,x_0\rangle+g(x_0)}{g(x)g(x_0)}\cdot\frac{d(x,x_0)}{\alpha(x,x_0)}\\
&=\frac{f(x_0)+\langle w,x_0\rangle+g(x_0)}{g(x)g(x_0)}\left\{C_{(x,x_0)}\left(\frac{g^2(x_0)}{f(x_0)+\langle w,x_0\rangle+g(x_0)}\right.\right.\\
&\quad\left.\left.\cdot\nabla\left[\frac{f(x_0)+\langle w,x_0\rangle}{g(x_0)}\right]\right)\right\}+\rho\frac{f(x_0)+\langle w,x_0\rangle+g(x_0)}{g(x)g(x_0)}\cdot\frac{d(x,x_0)}{\alpha(x,x_0)}.
\end{aligned}
$$

Denote

$$
\begin{aligned}
&\overline{\alpha}(x,x_0)=\frac{g(x_0)\alpha(x,x_0)}{g(x)}, \qquad \overline{\rho}=\rho\left(1+\frac{f(x_0)+\langle w,x_0\rangle}{g(x_0)}\right),\\
&\overline{d}(x,x_0)=\frac{d(x,x_0)}{g(x)}
\end{aligned}
$$

and

$$
\overline{C}_{(x,x_0)}(a)=\frac{f(x_0)+\langle w,x_0\rangle+g(x_0)}{g^2(x_0)}\left(C_{(x,x_0)}\left(\frac{g^2(x_0)}{f(x_0)+\langle w,x_0\rangle+g(x_0)}\cdot a\right)\right),
$$

where

$$
a:=\nabla\left(\frac{f(x_0)+\langle w,x_0\rangle}{g(x_0)}\right).
$$

It is easy to prove that $\overline{C}$ is a convex function with respect to variable a. By Definition 2.2, we have that $\frac{f(\cdot)+\langle w,\cdot\rangle}{g(\cdot)}$ is $(\overline{C},\overline{\alpha},\overline{\rho},\overline{d})$-convex at x_0. This completes of the proof. □

3 Optimality Conditions

In this section, we derive sufficient optimality conditions for a weakly efficient solutions of (MFP) under the assumption of (C,α,ρ,d)-convexity.

Theorem 3.1 *Let $x_0\in S$ be a feasible solution of* (*MFP*). *Assume that there exist $\lambda_i>0$, $i=1,2,\ldots,p$, and $\mu_j\geq 0$, $j=1,2,\ldots,m$, such that*

$$
\sum_{i=1}^{p}\lambda_i\nabla\left(\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)}\right)+\sum_{j=1}^{m}\mu_j\nabla h_j(x_0)=0, \tag{1}
$$

$$
\langle w_i,x_0\rangle=s(x_0|C_i),\quad w_i\in C_i,\quad i=1,2,\ldots,p, \tag{2}
$$

$$
\sum_{j=1}^{m}\mu_j h_j(x_0)=0. \tag{3}
$$

Let $f_i(\cdot)+\langle w_i,\cdot\rangle$, $-g_i(\cdot)$, $i=1,2,\ldots,p$, *be* (C,α_i,ρ_i,d_i)-*convex at* x_0, *and let* $h_j(\cdot)$, $j=1,2,\ldots,m$, *be* (C,β_j,η_j,c_j)-*convex at* x_0, *and*

$$\sum_{i=1}^{p}\lambda_i\overline{\rho}_i\frac{\overline{d}_i(x,x_0)}{\overline{\alpha}_i(x,x_0)}+\sum_{j=1}^{m}\mu_j\eta_j\frac{c_j(x,x_0)}{\beta_j(x,x_0)}\geq 0, \tag{4}$$

where

$$\overline{\alpha}_i(x,x_0)=\frac{g_i(x_0)\alpha_i(x,x_0)}{g_i(x)},\qquad \overline{\rho}_i=\rho_i\left(1+\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)}\right),$$
$$\overline{d}_i(x,x_0)=\frac{d_i(x,x_0)}{g_i(x)}.$$

Then x_0 *is a weakly efficient solution of* (*MFP*).

Proof Let x_0 be not a weakly efficient solution of (MFP). Then, there exists $x\in S$, such that

$$\frac{f_i(x)+s(x|C_i)}{g_i(x)}<\frac{f_i(x_0)+s(x_0|C_i)}{g_i(x_0)},\quad i=1,2,\ldots,p.$$

From (3.2) and $\langle w_i,x\rangle\leq s(x|C_i)$ for $i=1,2,\ldots,p$, we get

$$\frac{f_i(x)+\langle w_i,x\rangle}{g_i(x)}\leq\frac{f_i(x)+s(x|C_i)}{g_i(x)}<\frac{f_i(x_0)+s(x_0|C_i)}{g_i(x_0)}=\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)}. \tag{5}$$

By Theorem 2.1, for each i, $1\leq i\leq p$, $\frac{f_i(\cdot)+\langle w_i,\cdot\rangle}{g_i(\cdot)}$ is $(\overline{C},\overline{\alpha}_i,\overline{\rho}_i,\overline{d}_i)$-convex at x_0, i.e.,

$$\begin{aligned}&\frac{1}{\overline{\alpha}_i(x,x_0)}\left(\frac{f_i(x)+\langle w_i,x\rangle}{g_i(x)}-\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)}\right)\\&\quad\geq\overline{\rho}_i\frac{\overline{d}_i(x,x_0)}{\overline{\alpha}_i(x,x_0)}+\frac{f_i(x_0)+\langle w_i,x_0\rangle+g_i(x_0)}{g_i^2(x_0)}\\&\qquad\times\left\{C_{(x,x_0)}\left(\frac{g_i^2(x_0)}{f_i(x_0)+\langle w_i,x_0\rangle+g_i(x_0)}\cdot\nabla\left[\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)}\right]\right)\right\},\end{aligned} \tag{6}$$

where

$$\overline{\alpha}_i(x,x_0)=\frac{g_i(x_0)\alpha_i(x,x_0)}{g_i(x)},\qquad \overline{\rho}_i=\rho_i\left(1+\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)}\right),$$
$$\overline{d}_i(x,x_0)=\frac{d_i(x,x_0)}{g_i(x)}.$$

By the (C,β_j,η_j,c_j)-convexity of $h_j(\cdot)$ $(j=1,2,\ldots,m)$, one has

$$\frac{h_j(x)-h_j(x_0)}{\beta_j(x,x_0)}\geq C_{(x,x_0)}\big(\nabla h_j(x_0)\big)+\eta_j\frac{c_j(x,x_0)}{\beta_j(x,x_0)}. \tag{7}$$

Denote

$$\tau = \sum_{i=1}^{p} \lambda_i \frac{f_i(x_0) + \langle w_i, x_0\rangle + g_i(x_0)}{g_i^2(x_0)} + \sum_{j=1}^{m} \mu_j .$$

It is easy to see that $\tau > 0$. Multiplying both side of (6) by $\frac{\lambda_i}{\tau}$ and of (7) by $\frac{\mu_j}{\tau}$, respectively, adding them and using the convexity of $C_{(x,x_0)}(\cdot)$, we get

$$\begin{aligned}
&\sum_{i=1}^{p} \frac{\lambda_i}{\tau\overline{\alpha}_i(x,x_0)}\left(\frac{f_i(x) + \langle w_i, x\rangle}{g_i(x)} - \frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0)}\right) + \sum_{j=1}^{m} \frac{\mu_j}{\tau}\frac{h_j(x) - h_j(x_0)}{\beta_j(x,x_0)} \\
&\quad \geq \sum_{i=1}^{p} \frac{\lambda_i}{\tau}\frac{f_i(x_0) + \langle w_i, x_0\rangle + g_i(x_0)}{g_i^2(x_0)} \\
&\quad\quad \times \left\{C_{(x,x_0)}\left(\frac{g_i^2(x_0)}{f_i(x_0) + \langle w_i, x_0\rangle + g_i(x_0)} \cdot \nabla\left[\frac{f_i(x_0) + \langle w, x_0\rangle}{g_i(x_0)}\right]\right)\right\} \\
&\quad\quad + \sum_{j=1}^{m} \frac{\mu_j}{\tau} C_{(x,x_0)}\big(\nabla h_j(x_0)\big) + \sum_{i=1}^{p} \frac{\lambda_i}{\tau}\overline{\rho}_i \frac{\overline{d}_i(x,x_0)}{\overline{\alpha}_i(x,x_0)} + \sum_{j=1}^{m} \frac{\mu_j}{\tau}\eta_j \frac{c_j(x,x_0)}{\beta_j(x,x_0)} \\
&\quad \geq C_{(x,x_0)}\left(\frac{1}{\tau}\left[\sum_{i=1}^{p} \lambda_i \nabla\left(\frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0)}\right) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0)\right]\right) \\
&\quad\quad + \sum_{i=1}^{p} \frac{\lambda_i}{\tau}\overline{\rho}_i \frac{\overline{d}_i(x,x_0)}{\overline{\alpha}_i(x,x_0)} + \sum_{j=1}^{m} \frac{\mu_j}{\tau}\eta_j \frac{c_j(x,x_0)}{\beta_j(x,x_0)}.
\end{aligned}$$

This fact together with (1) and (4) yields

$$\begin{aligned}
&\sum_{i=1}^{p} \frac{\lambda_i}{\tau\overline{\alpha}_i(x,x_0)}\left(\frac{f_i(x) + \langle w_i, x\rangle}{g_i(x)} - \frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0)}\right) \\
&\quad + \sum_{j=1}^{m} \frac{\mu_j}{\tau}\frac{h_j(x) - h_j(x_0)}{\beta_j(x,x_0)} \geq 0.
\end{aligned} \tag{8}$$

Since x_0 is a feasible solution of (MPF), it follows from (3.3) that

$$\sum_{j=1}^{m} \mu_j \frac{h_j(x) - h_j(x_0)}{\beta_j(x,x_0)} \leq 0. \tag{9}$$

Combining (5) and (9) yields

$$\sum_{i=1}^{p} \frac{\lambda_i}{\tau\overline{\alpha}_i(x,x_0)}\left(\frac{f_i(x) + \langle w_i, x\rangle}{g_i(x)} - \frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0)}\right)$$

$$+\sum_{j=1}^{m}\frac{\mu_j}{\tau}\frac{h_j(x)-h_j(x_0)}{\beta_j(x,x_0)}<0,$$

which contradicts to (8). Therefore, x_0 is a weakly efficient solution of (MFP). □

Remark 3.1 Theorem 3.1 generalizes Theorem 2.3 of Kim et al. [4] and Theorem 3.2 of Long et al. [12].

Corollary 3.1 *Let $x_0 \in S$ be a feasible solution of (MFP). Assume that there exist $\lambda_i > 0$, $i = 1, 2, \dots, p$, and $\mu_j \geq 0$, $j = 1, 2, \dots, m$, such that*

$$\sum_{i=1}^{p}\lambda_i\nabla\left(\frac{f_i(x_0)+\langle w_i,x_0\rangle}{g_i(x_0)}\right)+\sum_{j=1}^{m}\mu_j\nabla h_j(x_0)=0,$$

$$\langle w_i,x_0\rangle = s(x_0|C_i),\quad w_i\in C_i,\quad i=1,2,\dots,p,$$

$$\sum_{j=1}^{m}\mu_j h_j(x_0)=0.$$

If $f_i(\cdot)+\langle w_i,\cdot\rangle$, $-g_i(\cdot)$, $(i=1,2,\dots,p)$, are strongly (C,α_i,ρ_i,d_i)-convex at x_0, $h_j(\cdot)$, $(j=1,2,\dots,m)$, are strongly (C,β_j,η_j,c_j)-convex at x_0. Then x_0 is a weakly efficient solution of (MFP).

Proof We can easily check that (4) holds under the assumptions of the corollary. □

4 Duality Results

In this section, we consider the following Mond-Weir type dual (MFD) to the primal problem (MFP):

$$\begin{cases} \max\ \left(\frac{f_1(u)+\langle w_1,u\rangle}{g_1(u)},\dots,\frac{f_p(u)+\langle w_p,u\rangle}{g_p(u)}\right),\\ \text{s.t.}\ \ \sum_{i=1}^{p}\lambda_i\nabla\left(\frac{f_i(u)+\langle w_i,u\rangle}{g_i(u)}\right)+\sum_{j=1}^{m}\mu_j\nabla h_j(u)=0,\\ \qquad \sum_{j=1}^{m}\mu_j h_j(u)\geq 0,\\ \qquad w:=(w_1,w_2,\dots,w_p),\quad w_i\in C_i,\quad i=1,2,\dots,p,\ u\in X,\\ \qquad \mu_j\geq 0,\quad j=1,2,\dots,m, \lambda=(\lambda_1,\lambda_2,\dots,\lambda_p)\in\Lambda^+, \end{cases} \tag{10}$$

where

$$\Lambda^+=\left\{\lambda\in R_+^p:\lambda_i>0\right\}.$$

In the following, we shall prove the weak duality and strong duality results.

Theorem 4.1 (Weak Duality) *Let x and (u,λ,w,μ) be feasible solutions of (MFP) and (MFD), respectively. Let $f_i(\cdot)+\langle w_i,\cdot\rangle$ and $-g_i(\cdot)$ $(i=1,2,\dots,p)$ be*

$(C, \alpha_i, \rho_i, d_i)$-convex at u, and let $h_j(\cdot)$ $(j = 1, 2, \ldots, m)$ be $(C, \beta_j, \eta_j, c_j)$-convex at u. If

$$\sum_{i=1}^{p} \lambda_i \overline{\rho}_i \frac{\overline{d}_i(x,u)}{\overline{\alpha}_i(x,u)} + \sum_{j=1}^{m} \mu_j \eta_j \frac{c_j(x,u)}{\beta_j(x,u)} \geq 0, \tag{11}$$

where

$$\overline{\alpha}_i(x,u) = \frac{g_i(u)\alpha_i(x,u)}{g_i(x)}, \qquad \overline{\rho}_i = \rho_i\left(1 + \frac{f_i(u) + \langle w_i, u\rangle}{g_i(u)}\right),$$
$$\overline{d}_i(x,u) = \frac{d_i(x,u)}{g_i(x)}.$$

Then the following cannot hold:

$$\left(\frac{f_1(x) + s(x|C_1)}{g_1(x)}, \ldots, \frac{f_p(x) + s(x|C_p)}{g_p(x)}\right) < \left(\frac{f_1(u) + \langle w_1, u\rangle}{g_1(u)}, \ldots, \frac{f_p(u) + \langle w_p, u\rangle}{g_p(u)}\right). \tag{12}$$

Proof Let x and (u, λ, w, μ) be feasible solutions of (MFP) and (MFD), respectively. It follows that

$$\sum_{j=1}^{m} \mu_j h_j(x) \leq 0 \leq \sum_{j=1}^{m} \mu_j h_j(u).$$

By the $(C, \beta_j, \eta_j, c_j)$-convexity of h_j $(j = 1, 2, \ldots, m)$, one has

$$0 \geq \sum_{j=1}^{m} \mu_j \frac{h_j(x) - h_j(u)}{\beta_j(x,u)} \geq \sum_{j=1}^{m} \mu_j C_{(x,u)}\big(\nabla h_j(u)\big) + \sum_{j=1}^{m} \mu_j \eta_j \frac{c_j(x,u)}{\beta_j(x,u)}. \tag{13}$$

We now suppose that (12) holds. Since $s(x|C_i) \geq \langle w_i, x\rangle$, $i = 1, 2, \ldots, p$,

$$\frac{f_i(x) + \langle w_i, x\rangle}{g_i(x)} \leq \frac{f_i(x) + s(x|C_i)}{g_i(x)} < \frac{f_i(u) + \langle w_i, u\rangle}{g_i(u)}. \tag{14}$$

By Theorem 2.1, we get

$$\begin{aligned}
&\frac{1}{\overline{\alpha}_i(x,u)}\left(\frac{f_i(x) + \langle w_i, x\rangle}{g_i(x)} - \frac{f_i(u) + \langle w_i, u\rangle}{g_i(u)}\right) \\
&\quad \geq \overline{\rho}_i \frac{\overline{d}_i(x,u)}{\overline{\alpha}_i(x,u)} + \frac{f_i(u) + \langle w_i, u\rangle + g_i(u)}{g_i^2(u)} \\
&\qquad \times \left\{C_{(x,u)}\left(\frac{g_i^2(u)}{f_i(u) + \langle w_i, u\rangle + g_i(u)} \cdot \nabla\left[\frac{f_i(u) + \langle w_i, u\rangle}{g_i(u)}\right]\right)\right\},
\end{aligned} \tag{15}$$

where

$$\overline{\alpha}_i(x,u)=\frac{\alpha_i(x,u)g_i(u)}{g_i(x)},\qquad \overline{\rho}_i=\rho_i\left(1+\frac{f_i(u)+\langle w_i,u\rangle}{g_i(u)}\right),$$
$$\overline{d}_i(x,u)=\frac{d_i(x,u)}{g_i(x)}.$$

Denote

$$\tau=\sum_{i=1}^{p}\lambda_i\frac{f_i(u)+\langle w_i,u\rangle+g_i(u)}{g_i^2(u)}+\sum_{j=1}^{m}\mu_j.$$

It follows from (10), (11), (13)–(15) and the convexity of $C_{(x,u)}(\cdot)$ that

$$\begin{aligned}
0&>\sum_{i=1}^{p}\frac{\lambda_i}{\tau\overline{\alpha}_i(x,u)}\left(\frac{f_i(x)+\langle w_i,x\rangle}{g_i(x)}-\frac{f_i(u)+\langle w_i,u\rangle}{g_i(u)}\right)+\sum_{j=1}^{m}\frac{\mu_j}{\tau}\frac{h_j(x)-h_j(u)}{\beta_j(x,u)}\\
&\geq\sum_{i=1}^{p}\frac{\lambda_i}{\tau}\frac{f_i(u)+\langle w_i,u\rangle+g_i(u)}{g_i^2(u)}\\
&\quad\times\left(C_{(x,u)}\left(\frac{g_i^2(u)}{f_i(u)+\langle w_i,u\rangle+g_i(u)}\cdot\nabla\left[\frac{f_i(u)+\langle w_i,u\rangle}{g_i(u)}\right]\right)\right)\\
&\quad+\sum_{j=1}^{m}\frac{\mu_j}{\tau}C_{(x,u)}\big(\nabla h_j(u)\big)+\sum_{i=1}^{p}\frac{\lambda_i}{\tau}\overline{\rho}_i\frac{\overline{d}_i(x,u)}{\overline{\alpha}_i(x,u)}+\sum_{j=1}^{m}\frac{\mu_j}{\tau}\eta_j\frac{c_j(x,u)}{\beta_j(x,u)}\\
&\geq C_{(x,u)}\left(\frac{1}{\tau}\left(\sum_{i=1}^{p}\lambda_i\nabla\left(\frac{f_i(u)+\langle w_i,u\rangle}{g_i(u)}\right)+\sum_{j=1}^{m}\mu_j\nabla h_j(u)\right)\right)\\
&\quad+\sum_{i=1}^{p}\frac{\lambda_i}{\tau}\overline{\rho}_i\frac{\overline{d}_i(x,u)}{\overline{\alpha}_i(x,u)}+\sum_{j=1}^{m}\frac{\mu_j}{\tau}\eta_j\frac{c_j(x,u)}{\beta_j(x,u)}\geq 0,
\end{aligned}$$

which gives a contradiction. This completes the proof. □

Corollary 4.1 (Weak Duality) *Let x and (u,λ,w,μ) be feasible solutions of (MFP) and (MFD), respectively. Let $f_i(\cdot)+\langle w_i,\cdot\rangle$ and $-g_i(\cdot)$ $(i=1,2,\ldots,p)$ be strongly (C,α_i,ρ_i,d_i)-convex at u, and let $h_j(\cdot)(j=1,2,\ldots,m)$ be strongly (C,β_j,η_j,c_j)-convex at u. Then the following cannot hold*:

$$\left(\frac{f_1(x)+s(x|C_1)}{g_1(x)},\ldots,\frac{f_p(x)+s(x|C_p)}{g_p(x)}\right)$$
$$<\left(\frac{f_1(u)+\langle w_1,u\rangle}{g_1(u)},\ldots,\frac{f_p(u)+\langle w_p,u\rangle}{g_p(u)}\right).$$

Proof We can easily check that (11) holds under the assumptions of the corollary. □

Theorem 4.2 (Strong Duality) *Let* $\overline{x}$ *be a weakly efficient solution of* (*MFP*). *If there exists* $z^* \in \mathbb{R}^n$, *such that* $\langle \nabla h_j(\overline{x}), z^* \rangle > 0$, $j \in I(\overline{x})$, *then there exist* $\overline{\lambda} \in \mathbb{R}^p_+$, $\overline{\lambda} \in \Lambda^+$, $\overline{\mu} \in \mathbb{R}^m_+$, *and* $\overline{w} := (\overline{w}_1, \overline{w}_2, \dots, \overline{w}_p) \in (C_1 \times C_2 \times \cdots C_p)$, *such that* $(\overline{x}, \overline{\lambda}, \overline{w}, \overline{\mu})$ *is a feasible solution for* (*MFD*) *and* $\langle \overline{x}, \overline{w}_i \rangle = s(\overline{x}|C_i)$, $i = 1, \dots, p$. *If the assumptions in Theorem* 4.1 *are satisfied, then* $(\overline{x}, \overline{\lambda}, \overline{w}, \overline{\mu})$ *is a weakly efficient solution of* (*MFD*).

Proof Since $\overline{x}$ is a weakly efficient solution of (MFP), by Theorem 2.2 in [4], there exist $\overline{\lambda} \in \mathbb{R}^p_+$, $\overline{\lambda} \in \Lambda^+$, $\overline{\mu} \in \mathbb{R}^m_+$, and $\overline{w} := (\overline{w}_1, \overline{w}_2, \dots, \overline{w}_p) \in (C_1 \times C_2 \times \cdots C_p)$, such that $(\overline{x}, \overline{\lambda}, \overline{w}, \overline{\mu})$ is a feasible solution for (MFD) and $\langle \overline{x}, \overline{w}_i \rangle = s(\overline{x}|C_i)$, $i = 1, \dots, p$. If $(\overline{x}, \overline{\lambda}, \overline{w}, \overline{\mu})$ is not a weakly efficient solution of (MFD), then there exists a feasible solution $(x^*, \lambda^*, w^*, \mu^*)$ of (MFD), such that

$$\left(\frac{f_1(\overline{x}) + s(\overline{x}|C_1)}{g_1(\overline{x})}, \dots, \frac{f_p(\overline{x}) + s(\overline{x}|C_p)}{g_p(\overline{x})}\right) < \left(\frac{f_1(x^*) + \langle w_1, x^* \rangle}{g_1(x^*)}, \dots, \frac{f_p(x^*) + \langle w_p, x^* \rangle}{g_p(x^*)}\right),$$

which contradicts the result of Theorem 4.1. This completes the proof. □

Remark 4.1 Theorems 4.1 and 4.2 generalize Theorems 3.1 and 3.2 of Kim et al. [4] and Theorems 4.1 and 4.2 of Long et al. [12].

5 Conclusions

In this paper, we consider a class of nondifferentiable multiobjective fractional programming problems in which each component of the objective function contains a term involving the support function of a compact convex set. We derive some sufficient optimality conditions and duality results for weakly efficient solutions of nondifferentiable multiobjective fractional programming problems under the assumptions of (C, α, ρ, d)-convexity.

It is well-known that (C, α, ρ, d)-convexity includes (F, α, ρ, d)-convexity [7], V-ρ-invexity [5], (F, ρ)-convexity [15] as special cases. Now one open problem arises in a natural way: How to characterize that a given function is a (C, α, ρ, d)-convex function?

As pointed out by Professor F. Giannessi, it is deserved to consider the above open problem in the future.

References

1. Bector, C.R., Chandra, S., Husain, I.: Optimality conditions and duality in subdifferentiable multiobjective fractional programming. J. Optim. Theory Appl. **79**, 105–125 (1993)
2. Chandra, S., Craven, B.D., Mond, B.: Vector-valued Lagrangian and multiobjective fractional programming duality. Numer. Funct. Anal. Optim. **11**, 239–254 (1990)

3. Chinchuluun, A., Yuan, D.H., Pardalos, P.M.: Optimality conditions and duality for nondifferentiable multiobjective fractional programming with generalized convexity. Ann. Oper. Res. **154**, 133–147 (2007)
4. Kim, D.S., Kim, S.J., Kim, M.H.: Optimality and duality for a class of nondifferentiable multiobjective fractional programming problems. J. Optim. Theory Appl. **129**, 131–146 (2006)
5. Kuk, H., Lee, G.M., Kim, D.S.: Nonsmooth multiobjective programs with (V, ϱ)-invexity. Ind. J. Pure Appl. Math. **29**, 405–412 (1998)
6. Kuk, H., Lee, G.M., Tanina, T.: Optimality and duality for nonsmooth multiobjective fractional programming with generalized invexity. J. Math. Anal. Appl. **262**, 365–375 (2001)
7. Liang, Z.A., Huang, H.X., Pardalos, P.M.: Optimality conditions and duality for a class of nonlinear fractional programming problems. J. Optim. Theory Appl. **110**, 611–619 (2001)
8. Liang, Z.A., Huang, H.X., Pardalos, P.M.: Efficiency conditions and duality for a class of multiobjective fractional programming problems. J. Global Optim. **27**, 447–471 (2003)
9. Liu, J.C.: Optimality and duality for multiobjective fractional programming involving nonsmooth pseudoinvex functions. Optimization **37**, 27–39 (1996)
10. Liu, J.C.: Optimality and duality for multiobjective fractional programming involving nonsmooth (F, ρ)-convex pseudoinvex functions. Optimization **36**, 333–346 (1996)
11. Liu, S.M., Feng, E.M.: Optimality conditions and duality for a class of nondifferentiable multiobjective fractional programming problems. J. Global Optim. **38**, 653–666 (2007)
12. Long, X.J., Huang, N.J., Liu, Z.B.: Optimality conditions, duality and saddle points for nondifferentiable multiobjective fractional programs. J. Ind. Manag. Optim. **4**, 287–298 (2008)
13. Mond, B., Schechter, M.: Nondifferentiable symmetric duality. Bull. Aust. Math. Soc. **53**, 177–187 (1996)
14. Mukherjee, R.N.: Generalized convex duality for multiobjective fractional programs. J. Math. Anal. Appl. **162**, 309–316 (1991)
15. Rreda, V.: On efficiency and duality for multiobjective programs. J. Math. Anal. Appl. **166**, 365–377 (1992)
16. Schaible, S.: Fractional programming. In: Horst, R., Pardalos, P.M. (eds.) Handbook of Global Optimization, pp. 495–608. Kluwer Academic, Dordrecht (1995)
17. Weir, T.: A dual for a multiobjective fractional programming problem. J. Inf. Optim. Sci. **7**, 261–269 (1986)
18. Yang, X.M., Teo, K.L., Yang, X.Q.: Duality for a class of nondifferentiable multiobjective programming problems. J. Math. Anal. Appl. **252**, 999–1005 (2000)
19. Yuan, D.H., Liu, X.L., Chinchuluun, A., Pardalos, P.M.: Nondifferentiable minimax fractional programming problems with (C, α, ρ, d)-convexity. J. Optim. Theory Appl. **129**, 185–199 (2006)