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Synchronization Criterion for Lur'e Systems via Delayed PD Controller

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Abstract In this paper, the effects of a time varying delay on a chaotic drive-response synchronization are considered. Using a delayed feedback proportional-derivative (PD) controller scheme, a delay-dependent synchronization criterion is derived for chaotic systems represented by the Lur'e system with sector and slope restricted nonlinearities. The derived criterion is a sufficient condition for the absolute stability of the error dynamics between the drive and the response systems. By the use of a convex representation of the nonlinearity and the discretized Lyapunov-Krasovskii functional, stability condition is obtained via the LMI formulation. The condition represented in the terms of linear matrix inequalities (LMIs) can be solved by the application of convex optimization algorithms. The effectiveness of the work is verified through numerical examples.

Keywords Lur'e systems · Synchronization · Absolute stability · PD controller · LMIs · Convex optimization

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1 Introduction

During the past several decades, the problem of stability and stabilization of dynamical systems is a most important issue in the real world [1, 2]. One of the hot issues of the nonlinear control society is the synchronization of systems. Synchronization problems are commonly found in many systems such as heart beat regulation, walking tempo, coordinated robot motion and so on. The synchronization of chaotic systems is a more challenging issue, because it is difficult to predict the behavior of chaotic systems, and they are very sensitive to initial conditions and noise. Since the pioneering work of Pecora and Carroll [3] on the synchronization of two identical chaotic systems, chaotic synchronization has received much attention because of its theoretical and practical importance. Recently, many researches on the chaotic synchronization of Lur'e systems have been presented, since various chaotic systems such as Chua's circuit, n-scroll attractors and hyperchaotic attractors can be modeled as Lur'e systems [4, 9–19, 24, 25]. The Lur'e system consists of a linear system and a feedback nonlinearity satisfying sector bound constraints. The stability of the Lur'e system is called absolute stability, which means global asymptotic stability. For this reason, many researchers have studied the chaotic synchronization of Lur'e systems and applied these studies to various fields. Recently, practical issues of the synchronization such as propagation delay, noise and model uncertainty, which are the source of instability or poor performance [5–8, 21], have been considered. Especially, many research efforts have been focused on the effect of the propagation delay of the chaotic synchronization, since Chen and Liu introduced a delay into chaotic synchronization and showed that the delay can break the synchronization [9]. The first report considering the effect of time delay in the chaotic synchronization of Lur'e system was presented by Yalcin et al. [10] and sufficient conditions for stability were included. Liao et al. proposed a synchronization method for the Lur'e systems with time delay using a feedback controller [11]. After the research of Liao et al. [11], various synchronization schemes for the Lur'e systems with time delay have been studied [12–16].

Also, there are several studies that consider the effect of delay on the chaotic synchronization [17–19]. In those researches, synchronization criteria were derived for given gain matrices of the synchronization controller and time delay, which are sufficient conditions for absolute stability of the synchronization. In Ref. [17], delay-independent and delay-dependent stability criteria for master-slave synchronization schemes for Lur'e systems with time delay, were derived through LMI formulation. However, the model transformation technique used in Refs. [10] and [17] can lead to conservative conditions by inducing additional dynamics. In order to derive less conservative conditions for synchronization, stability criteria that does not use the model transformation were presented independently in Ref. [18]. In Ref. [18], a more general Lur'e-Postnikov Lyapunov functional was presented to derive a less conservative criterion. Furthermore, a synchronization method for the chaotic Lur'e system was extended for a time-varying delay in Ref. [19]. In [20], Guo et al. proposed a synchronization method for the Lur'e system using a delayed feedback PD controller. As it is well known, the PD controller can provide a fast response in synchronization and enhance its stability properties. In the real world, since the time delay is inevitable and affects systems in various ways, it is very important to consider the time delays in implementing controllers. However, Guo et al. [20] showed only the method of finding the gain matrices K_p of the PD controller and did not analyze the effect of delay on stability.

In this paper, we analyze the effect of a time-varying delay from the synchronization criterion for master-slave Lur'e systems with a delayed feedback PD controller. A delay-dependent synchronization criterion is presented for the delayed feedback PD controller. The nonlinearity of the Lur'e system is represented by the convex combination of its lower and upper bounds. Then, the nonlinear constraint is converted to an equality constraint. The projection lemma [22] is utilized for the purpose of handling the equality constraint so that a delay-dependent synchronization criterion is obtained. By construction of the augmented Lyapunov functional that employs redundant state of error dynamics shifted in time by a fraction of the time delay, a novel condition for the stability of the error dynamics is given in terms of a linear matrix inequality. Finally, the maximum allowable delay in the synchronization of chaotic Lur'e systems is found using LMIs [23] and is compared to those of the previous studies throughout numerical examples.

In this presentation, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices. For a real matrix X, $X>0$ and $X<0$ mean that X is a positive/negative definite symmetric matrix, respectively; I is an identity matrix with the appropriate dimensions and 0 is a null matrix with the appropriate dimensions. For a given matrix $A\in\mathbb{R}^{m\times n}$, such that $rank(A)=r$, we define $A^{\perp}\in\mathbb{R}^{n\times(n-r)}$ as the right orthogonal complement of A; namely, $AA^{\perp}=0$. The block diag$(\cdots)$ represents a block diagonal matrix.

2 Problem Formulation

Consider the following drive-response synchronization scheme of chaotic Lur'e systems with a drive system $\mathcal{D}$, a response system $\mathcal{R}$ and time delayed PD output feedback controller $\mathcal{C}$:

$$\begin{aligned}
&\mathcal{D}: \quad \begin{cases}\dot{x}(t)=Ax(t)+B\varphi(\mu(t)),\\ \mu(t)=Cx(t),\end{cases}\\
&\mathcal{R}: \quad \begin{cases}\dot{y}(t)=Ay(t)+B\varphi(\kappa(t))+u(t),\\ \kappa(t)=Cy(t),\end{cases}\\
&\mathcal{C}: \quad u(t)=L(\mu(t-\tau(t))-\kappa(t-\tau(t)))+M(\dot{\mu}(t-h(t))-\dot{\kappa}(t-h(t))),
\end{aligned} \tag{1}$$

where $\tau(t)$ and $h(t)$ satisfying $0\le\tau(t)\le\tau_M$, $\dot{\tau}(t)\le\tau_d$, $0\le h(t)\le h_M$, $\dot{h}(t)\le h_d$ are the time-varying delays, $x(t)$, $y(t)\in\mathbb{R}^n$ and $\mu(t),\kappa(t)\in\mathbb{R}^p$ are state vectors and the output vectors of the Lur'e systems respectively, $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times p}$, and $C\in\mathbb{R}^{p\times n}$ are constant matrices, the controller gain matrix $L\in\mathbb{R}^{n\times p}$, $M\in\mathbb{R}^{n\times p}$ is a given constant matrix. The nonlinearity of the Lur'e system $\varphi(\cdot):\mathbb{R}^p\to\mathbb{R}^p$ is a memoryless vector-valued function, whose ith element $\varphi_i(\cdot)$ is in a certain sector, such that

$$\bar{\alpha}_i \le \frac{\varphi_i(\sigma_i(t))}{\sigma_i(t)} \le \bar{\beta}_i, \tag{2}$$

where $\sigma_i(t)$ is the ith element of $\sigma(\cdot)$, $\sigma(\cdot)$ is the input vector of $\varphi_i(\cdot)$, $\bar{\alpha}_i$ and $\bar{\beta}_i$ are lower and upper bounds of the sector, respectively.

We assume that the nonlinearity $\varphi(\cdot)$ also satisfies a slope constraint:

$$\alpha_i \le \frac{d\varphi_i(\sigma_i(t))}{d\sigma_i(t)} \le \beta_i. \tag{3}$$

The synchronization scheme (1) achieves the synchronization of states between two systems by utilizing a time delayed output feedback as an input to the response system $\mathcal{R}$.

Let us define an error of synchronization as

$$e(t) = x(t) - y(t). \tag{4}$$

Then, the following error dynamics of the synchronization can be obtained:

$$\begin{aligned} &\dot{e}(t) - D\dot{e}(t-h(t)) = Ae(t) + Ee(t-\tau(t)) + B(\varphi(\mu(t)) - \varphi(\kappa(t))), \\ &e(\theta) = \psi(\theta), \quad \forall\theta \in [-\tau_M, 0] \end{aligned} \tag{5}$$

where $E = -LC$, $D = -MC$ and $\psi(\cdot)$ is a continuous vector valued function of the initial values.

Using the slope constraint of the nonlinear function in (3), we can derive new sector bounds for the error of the nonlinear functions $\varphi(\mu(t))$ and $\varphi(\kappa(t))$. By mean value theorem, there exists a constant $\delta \in (\mu_i(t), \kappa_i(t))$ such that

$$\varphi_i(\mu_i(t)) - \varphi_i(\kappa_i(t)) = \frac{d\varphi_i(\delta)}{d\sigma_i}(\mu_i(t) - \kappa_i(t)) \tag{6}$$

where $\mu_i(t)$ and $\kappa_i(t)$ are ith element of $\mu(t)$ and $\kappa(t)$, respectively. From the slope bounds in (3), we have

$$\alpha_i \le \frac{d\varphi_i(\delta)}{d\sigma_i} \le \beta_i.$$

Since $\mu(t) - \kappa(t) = C(x(t) - y(t)) = Ce(t)$, we also have

$$\alpha_i c_i e(t) \le \varphi_i(\mu_i(t)) - \varphi_i(\kappa_i(t)) \le \beta_i c_i e(t) \tag{7}$$

where c_i is ith row vector of the matrix C.

Let us set $\nu_i(t) = c_i e(t)$, then we obtain a new nonlinear function $\phi_i(\nu_i(t))$ bounded by a sector that belongs to $[\alpha_i, \beta_i]$ such that

$$\alpha_i \le \frac{\phi_i(\nu_i(t))}{\nu_i(t)} \le \beta_i, \tag{8}$$

where $\phi_i(\nu_i(t)) \triangleq \varphi_i(\mu_i(t)) - \varphi_i(\kappa_i(t))$.

Therefore, the error dynamics (5) can be represented as a Lur'e system with the new sector bounded nonlinear function $\phi(\nu(t))$ as follows:

$$\mathcal{E}: \quad \dot{e}(t) - D\dot{e}(t-h(t)) = Ae(t) + Ee(t-\tau(t)) + B\phi(\nu(t)) \tag{9}$$

where $\nu(t) = Ce(t)$.

Since the nonlinear function $\phi(\cdot)$ can be represented by a convex combination of the sector bounds such as α_i and β_i, we can rewrite the $\phi_i(\cdot)$ as below

$$\phi_i(\nu_i(t)) = (\lambda_i^l(\nu_i(t))\alpha_i + \lambda_i^u(\nu_i(t))\beta_i)\nu_i(t) \tag{10}$$

where

$$\lambda_i^l(\nu_i(t)) = \frac{\phi_i(\nu_i(t)) - \alpha_i\nu_i(t)}{(\beta_i - \alpha_i)\nu_i(t)}, \qquad \lambda_i^u(\nu_i(t)) = \frac{\beta_i\nu_i(t) - \phi_i(\nu_i(t))}{(\beta_i - \alpha_i)\nu_i(t)}. \tag{11}$$

From the relationship $\lambda_i^l(\nu_i) + \lambda_i^u(\nu_i) = 1$, $\lambda_i^l(\nu_i) \geq 0$ and $\lambda_i^u(\nu_i) \geq 0$, the $\phi_i(\cdot)$ can be represented using a convex hull:

$$\phi_i(\nu_i(t)) = \Delta_i(\nu_i(t))\nu_i(t) \tag{12}$$

where $\Delta_i(\nu_i(t))$ is an element of a convex hull $\mathrm{Co}\{\alpha_i, \beta_i\}$.

Let us define some diagonal matrices as

$$\begin{aligned} \Delta(\nu) &\triangleq \text{block diag}(\Delta_1(\nu_1), \ldots, \Delta_p(\nu_p)), \\ \alpha &\triangleq \text{block diag}(\alpha_1, \ldots, \alpha_p), \\ \beta &\triangleq \text{block diag}(\beta_1, \ldots, \beta_p). \end{aligned} \tag{13}$$

Then, the nonlinear function $\phi(\cdot)$ can be represented by

$$\phi(\nu(t)) = \Delta(\nu(t))\nu(t) \tag{14}$$

where $\Delta(\nu(t))$ belongs to the following set

$$\Phi := \{\Delta(\nu) | \Delta(\nu) \in \mathrm{Co}\{\alpha, \beta\}\}. \tag{15}$$

Like many papers [10–19] for the synchronization criteria, we suppose that the gain matrix of the synchronization controller is given, because the purpose of this paper is to find the maximum allowable delay bound such that the error dynamics of the synchronization (9) is absolutely stable. The following lemmas are useful for deriving the synchronization criterion.

Lemma 1 [18] *For any constant matrix $W \in \mathbb{R}^{n\times n}$, $W > 0$, scalar $\tau > 0$ and a vector function $e(t): \mathbb{R} \to \mathbb{R}^n$ such that the following integration is well defined, then*

$$-\tau\int_{-\tau}^{0} \dot{e}^T(t+\xi)W\dot{e}(t+\xi)d\xi \leq \begin{bmatrix} e(t)^T & e(t-\tau)^T \end{bmatrix} \begin{bmatrix} -W & W \\ W & -W \end{bmatrix} \begin{bmatrix} e(t) \\ e(t-\tau) \end{bmatrix}. \tag{16}$$

The following Projection Lemma is useful for conversion of an inequality subject to an equality constraint.

Lemma 2 (Projection Lemma [22]) *Let $x \in \mathbb{R}^n$, $\Theta = \Theta^T \in \mathbb{R}^{n\times n}$, $\Gamma \in \mathbb{R}^{m\times n}$ such that rank$(\Gamma) < n$. The following statements are equivalent eqnarray*

i. $x^T \Theta x < 0 \quad s.t. \quad \Gamma x = 0,\ \forall x \neq 0,$
ii. ${\Gamma^{\perp}}^T \Theta \Gamma^{\perp} < 0.$

3 Main Results

In this section, we derive LMI conditions for the absolute stability of the Lur'e system in the form of (9) with a time-varying delay. A delay-dependent criterion will be proposed in the next theorem, which can be further simplified to other equivalent conditions.

The convex representation (14) of the nonlinearity can be used to establish equality constraints. From (14), we have the following equality constraint

$$\phi(\nu(t)) - \Delta(\nu(t))\nu(t) = \phi(\nu(t)) - \Delta C e(t), \quad \forall \Delta \in \Phi. \tag{17}$$

Furthermore, we can establish an additional equality constraint from the error dynamics (9) and the definition of $e(t)$ as follows:

$$Ae(t) + Ee(t-\tau(t)) + B\phi(\nu(t)) - \dot{e}(t) + D\dot{e}(t-h(t)) = 0. \tag{18}$$

Next, the augmented vectors are defined for simplicity,

$$\begin{aligned}\zeta(t) = \big[\dot{e}^T(t) \quad & e^T(t) \quad e^T(t-\tau(t)) \quad e^T(t-\tfrac{\tau_M}{2}) \\ & e^T(t-\tau_M) \quad \phi^T(\nu(t)) \quad \dot{e}(t-h(t))\big]^T.\end{aligned} \tag{19}$$

In the following theorem, a synchronization criterion for the Lur'e system (9) is presented by the use of Lemma 2.

Theorem 1 *The error system described as (9) is absolutely stable for any delay $\tau(t)$ such that $0 \le \tau(t) \le \tau_M$, $\dot{\tau}(t) \le \tau_d$, $0 \le h(t) \le h_M$, $\dot{h}(t) \le h_d$ if there exist positive definite matrices P, $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ \star & G_{22}\end{smallmatrix}\right]$, Q_1, Q_2, Q_3, R_1 and $R_2 \in \mathbb{R}^{n\times n}$ and a positive definite diagonal matrix $S \in \mathbb{R}^{p\times p}$ satisfying the following LMI*:

$${\Gamma^{\perp}_j}^T \Pi_i \Gamma^{\perp}_j < 0, \quad i = 1,2,\ j = 1,2 \tag{20}$$

where $\Gamma^{\perp}(\Delta)$ is a right orthogonal complement of

$$\Gamma = \begin{bmatrix} I & -A & -E & 0 & 0 & -B & -D \\ 0 & \Lambda_j C & 0 & 0 & 0 & -I & 0 \end{bmatrix}, \tag{21}$$

Π_1 *is*

$$\Pi_1=\begin{bmatrix}\Phi_{11} & \Phi_{12} & 0 & 0 & 0 & 0 & 0\\ 0 & \Phi_{22} & \Phi_{23} & \Phi_{24} & 0 & \Phi_{26} & 0\\ 0 & 0 & \Phi_{33} & \Phi_{34} & 0 & 0 & 0\\ 0 & 0 & 0 & \Phi_{44} & \Phi_{45} & 0 & 0\\ 0 & 0 & 0 & 0 & \Phi_{55} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \Phi_{66} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \Phi_{77}\end{bmatrix}, \tag{22}$$

Π_2 *is*

$$\Pi_2=\begin{bmatrix}\Xi_{11} & \Xi_{12} & 0 & 0 & 0 & 0 & 0\\ 0 & \Xi_{22} & 0 & \Xi_{24} & 0 & \Xi_{26} & 0\\ 0 & 0 & \Xi_{33} & \Xi_{34} & \Xi_{35} & 0 & 0\\ 0 & 0 & 0 & \Xi_{44} & \Xi_{45} & 0 & 0\\ 0 & 0 & 0 & 0 & \Xi_{55} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \Xi_{66} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \Xi_{77}\end{bmatrix}, \tag{23}$$

and $\Phi_{11}=Q_3+\frac{\tau_M}{2}(R_1+R_2)$, $\Phi_{12}=P$, $\Phi_{22}=Q_1+Q_2+Q_4+G_{11}-\frac{2}{\tau_M}R_1$, $\Phi_{23}=\frac{2}{\tau_M}R_1$, $\Phi_{24}=G_{12}$, $\Phi_{26}=C^T(\alpha+\beta)^TS$, $\Phi_{33}=-(1-\tau_D)Q_1-\frac{4}{\tau_M}R_1$, $\Phi_{34}=\frac{2}{\tau_M}R_1$, $\Phi_{44}=-Q_4+G_{22}-G_{11}-\frac{2}{\tau_M}(R_1+R_2)$, $\Phi_{45}=-G_{12}+\frac{2}{\tau_M}R_2$, $\Phi_{55}=-Q_2-G_{22}-\frac{2}{\tau_M}R_2$, $\Phi_{66}=-2S$, $\Phi_{77}=-(1-h_D)Q_3$, $\Xi_{11}=Q_3+\frac{\tau_M}{2}(R_1+R_2)$, $\Xi_{12}=P$, $\Xi_{22}=Q_1+Q_2+Q_4+G_{11}-\frac{2}{\tau_M}R_1$, $\Xi_{24}=G_{12}+\frac{2}{\tau_M}R_1$, $\Xi_{26}=C^T(\alpha+\beta)^TS$, $\Xi_{33}=-(1-\tau_D)Q_1-\frac{4}{\tau_M}R_2$, $\Xi_{34}=\frac{2}{\tau_M}R_2$, $\Xi_{35}=\frac{2}{\tau_M}R_2$, $\Xi_{44}=-Q_4+G_{22}-G_{11}-\frac{2}{\tau_M}(R_1+R_2)$, $\Xi_{45}=-G_{12}$, $\Xi_{55}=-Q_2-G_{22}-\frac{2}{\tau_M}R_2$, $\Xi_{66}=-2S$, $\Xi_{77}=-(1-h_D)Q_3$.

Proof Consider the following Lyapunov-Krasovskii functional:

$$V(e(t))=V_1(e(t))+V_2(e(t))+V_3(e(t))+V_4(e(t)) \tag{24}$$

where

$$\begin{aligned}V_1(e(t))&=e(t)^TPe(t),\\ V_2(e(t))&=\int_{t-\tau(t)}^{t}e^T(\xi)Q_1e(\xi)d\xi+\int_{t-\tau_M}^{t}e^T(\xi)Q_2e(\xi)d\xi\\ &\quad+\int_{t-h(t)}^{t}\dot{e}^T(\xi)Q_3\dot{e}(\xi)d\xi,\\ V_3(e(t))&=\int_{t-\tau_M/2}^{t}\begin{bmatrix}e(\xi)\\ e(\xi-\frac{\tau_M}{2})\end{bmatrix}^T\begin{bmatrix}G_{11} & G_{12}\\ \star & G_{22}\end{bmatrix}\begin{bmatrix}e(\xi)\\ e(\xi-\frac{\tau_M}{2})\end{bmatrix}d\xi,\end{aligned}$$

$$V_4(e(t)) = \int_{t-\tau_M/2}^{t} \left(\frac{\tau_M}{2} - t + \xi\right)\dot{e}^T(\xi)R_1\dot{e}(\xi)d\xi + \int_{t-\tau_M}^{t-\tau_M/2} (\tau_M - t + \xi)\dot{e}^T(\xi)R_2\dot{e}(\xi)d\xi.$$

The time-derivative of V_1 can be calculated as

$$\dot{V}_1(e(t)) = \dot{e}^T(t)Pe(t) + e^T(t)P\dot{e}(t) \triangleq \zeta^T(t)\Omega_1\zeta(t) \tag{25}$$

where

$$\Omega_1 = \begin{bmatrix} 0 & P & 0 & 0 & 0 & 0 & 0 \\ \star & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & \star & 0 \end{bmatrix}. \tag{26}$$

An upper bound of the time-derivative of V_2 can be obtained as

$$\begin{aligned}\dot{V}_2(e(t)) &= \begin{bmatrix} e(t) \\ e(t-\tau(t)) \end{bmatrix}^T \begin{bmatrix} Q_1 & 0 \\ 0 & -(1-\tau_D)Q_1 \end{bmatrix} \begin{bmatrix} e(t) \\ e(t-\tau(t)) \end{bmatrix} \\ &\quad + \begin{bmatrix} e(t) \\ e(t-\tau_M) \end{bmatrix}^T \begin{bmatrix} Q_2 & 0 \\ 0 & -Q_2 \end{bmatrix} \begin{bmatrix} e(t) \\ e(t-\tau_M) \end{bmatrix} \\ &\quad + \begin{bmatrix} \dot{e}(t) \\ \dot{e}(t-h(t)) \end{bmatrix}^T \begin{bmatrix} Q_3 & 0 \\ 0 & -(1-h_D)Q_3 \end{bmatrix} \begin{bmatrix} \dot{e}(t) \\ \dot{e}(t-h(t)) \end{bmatrix} = \zeta^T(t)\Omega_2\zeta(t)\end{aligned} \tag{27}$$

where

$$\Omega_2 = \begin{bmatrix} Q_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & Q_1+Q_2 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & -(1-\tau_d)Q_1 & 0 & 0 & 0 & 0 \\ \star & \star & \star & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -Q_2 & 0 & 0 \\ \star & \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -(1-h_D)Q_3 \end{bmatrix}. \tag{28}$$

Calculation of V_3 leads to

$$\begin{aligned}\dot{V}_3(e(t)) &\le \begin{bmatrix} e(t) \\ e(t-\frac{\tau_M}{2}) \end{bmatrix}^T \begin{bmatrix} G_{11} & G_{12} \\ \star & G_{22} \end{bmatrix} \begin{bmatrix} e(t) \\ e(t-\frac{\tau_M}{2}) \end{bmatrix} \\ &\quad - \begin{bmatrix} e(t-\frac{\tau_M}{2}) \\ e(t-\tau_M) \end{bmatrix}^T \begin{bmatrix} G_{11} & G_{12} \\ \star & G_{22} \end{bmatrix} \begin{bmatrix} e(t-\frac{\tau_M}{2}) \\ e(t-\tau_M) \end{bmatrix} \\ &\triangleq \zeta^T(t)\Omega_3\zeta(t)\end{aligned} \tag{29}$$

where

$$\Omega_3=\begin{bmatrix}0&0&0&0&0&0&0\\ \star&G_{11}&0&G_{12}&0&0&0\\ \star&\star&0&0&0&0&0\\ \star&\star&\star&G_{22}-G_{11}&-G_{12}&0&0\\ \star&\star&\star&\star&-G_{22}&0&0\\ \star&\star&\star&\star&\star&0&0\\ \star&\star&\star&\star&\star&\star&0\end{bmatrix}. \tag{30}$$

Also, an upper bound of the time derivative of V_4 can be obtained as

$$\begin{aligned}\dot V_4(e(t))\le{}&\frac{\tau_M}{2}\dot e^T(\xi)R_1\dot e(\xi)-\int_{t-\frac{\tau_M}{2}}^{t}\dot e^T(t)R_1\dot e(t)d\xi+\frac{\tau_M}{2}\dot e^T(t)R_2\dot e(t)\\ &-\int_{t-\tau_M}^{t-\frac{\tau_M}{2}}\dot e^T(\xi)R_2\dot e(\xi)d\xi.\end{aligned} \tag{31}$$

Next, the time-varying delays $\tau(t)$ can be considered in two intervals $[0,\ \tau_M/2)$ and $[\tau_M/2,\ \tau_M]$, thus the time derivative of V_4 is obtained using Lemma 1 for two cases at each interval.

Case I: $0\le\tau(t)<\tau_M/2$

$$\begin{aligned}\dot V_4(e(t))\le{}&\frac{\tau_M}{2}\dot e^T(t)R_1\dot e(t)+\frac{\tau_M}{2}\dot e^T(t)R_2\dot e(t)\\ &+\frac{2}{\tau_M}\begin{bmatrix}e(t)\\ e(t-\tau(t))\end{bmatrix}^T\begin{bmatrix}-R_1&R_1\\ R_1&-R_1\end{bmatrix}\begin{bmatrix}e(t)\\ e(t-\tau(t))\end{bmatrix}\\ &+\frac{2}{\tau_M}\begin{bmatrix}e(t-\tau(t))\\ e(t-\frac{\tau_M}{2})\end{bmatrix}^T\begin{bmatrix}-R_1&R_1\\ R_1&-R_1\end{bmatrix}\begin{bmatrix}e(t-\tau(t))\\ e(t-\frac{\tau_M}{2})\end{bmatrix}\\ &+\frac{2}{\tau_M}\begin{bmatrix}e(t-\frac{\tau_M}{2})\\ e(t-\tau_M)\end{bmatrix}^T\begin{bmatrix}-R_2&R_2\\ R_2&-R_2\end{bmatrix}\begin{bmatrix}e(t-\frac{\tau_M}{2})\\ e(t-\tau_M)\end{bmatrix}\triangleq\zeta^T(t)\Omega_4\zeta(t)\end{aligned} \tag{32}$$

where

$$\Omega_4=\begin{bmatrix}\frac{\tau_M}{2}(R_1+R_2)&0&0&0&0&0&0\\ \star&-\frac{2}{\tau_M}R_1&\frac{2}{\tau_M}R_1&0&0&0&0\\ \star&\star&-\frac{4}{\tau_M}R_1&\frac{2}{\tau_M}R_1&0&0&0\\ \star&\star&\star&-\frac{2}{\tau_M}(R_1+R_2)&\frac{2}{\tau_M}R_2&0&0\\ \star&\star&\star&\star&-\frac{2}{\tau_M}R_2&0&0\\ \star&\star&\star&\star&\star&0&0\\ \star&\star&\star&\star&\star&\star&0\end{bmatrix}. \tag{33}$$

Case II: $\tau_M/2 \le \tau(t) \le \tau_M$

$$\begin{aligned}\dot V_4(e(t)) &\le \frac{\tau_M}{2}\dot e^T(t)R_1\dot e(t) + \frac{\tau_M}{2}\dot e^T(t)R_2\dot e(t)\\ &\quad + \frac{2}{\tau_M}\begin{bmatrix} e(t) \\ e(t-\frac{\tau_M}{2})\end{bmatrix}^T \begin{bmatrix} -R_1 & R_1 \\ R_1 & -R_1\end{bmatrix}\begin{bmatrix} e(t) \\ e(t-\frac{\tau_M}{2})\end{bmatrix}\\ &\quad + \frac{2}{\tau_M}\begin{bmatrix} e(t-\frac{\tau_M}{2}) \\ e(t-\tau(t))\end{bmatrix}^T \begin{bmatrix} -R_2 & R_2 \\ R_2 & -R_2\end{bmatrix}\begin{bmatrix} e(t-\frac{\tau_M}{2}) \\ e(t-\tau(t))\end{bmatrix}\\ &\quad + \frac{2}{\tau_M}\begin{bmatrix} e(t-\tau(t)) \\ e(t-\tau_M)\end{bmatrix}^T \begin{bmatrix} -R_2 & R_2 \\ R_2 & -R_2\end{bmatrix}\begin{bmatrix} e(t-\tau(t)) \\ e(t-\tau_M)\end{bmatrix} \triangleq \zeta^T(t)\Omega_5\zeta(t)\end{aligned} \tag{34}$$

where

$$\Omega_5 = \begin{bmatrix} \frac{\tau_M}{2}(R_1+R_2) & 0 & 0 & 0 & 0 & 0 & 0\\ \star & -\frac{2}{\tau_M}R_1 & 0 & \frac{2}{\tau_M}R_1 & 0 & 0 & 0\\ \star & \star & -\frac{4}{\tau_M}R_2 & \frac{2}{\tau_M}R_2 & \frac{2}{\tau_M}R_2 & 0 & 0\\ \star & \star & \star & -\frac{4}{\tau_M}(R_1+R_2) & 0 & 0 & 0\\ \star & \star & \star & \star & -\frac{2}{\tau_M}R_2 & 0 & 0\\ \star & \star & \star & \star & \star & 0 & 0\\ \star & \star & \star & \star & \star & \star & 0\end{bmatrix}. \tag{35}$$

From the sector constraint of the nonlinearity $\phi(\cdot)$, the following inequality is obtained

$$2(\phi(\nu(t)-\alpha\nu(t))^T S(\phi(\nu(t))-\beta\nu(t)) \le 0. \tag{36}$$

By applying the well-known S-procedure [23] to (36) and utilizing (25)–(35), we have the following inequalities at each interval:
(i) $0 \le \tau(t) \le \tau_M/2$

$$\begin{aligned}\dot V(e(t)) &\le \zeta^T(t)(\Omega_1+\Omega_2+\Omega_3+\Omega_4)\zeta(t)\\ &\quad - 2(\phi(\nu(t)-\alpha\nu(t))^T S(\phi(\nu(t))-\beta\nu(t))\\ &\le 0.\end{aligned} \tag{37}$$

(ii) $\tau_M/2 \le \tau(t) \le \tau_M$

$$\begin{aligned}\dot V(e(t)) &\le \zeta^T(t)(\Omega_1+\Omega_2+\Omega_3+\Omega_5)\zeta(t)\\ &\quad - 2(\phi(\nu(t)-\alpha\nu(t))^T S(\phi(\nu(t))-\beta\nu(t))\\ &\le 0.\end{aligned} \tag{38}$$

By rewriting (37) and (38) for $\zeta(t)$ at each interval, we have

$$\begin{aligned}\dot{V}(e(t)) &\le \zeta^T(t)\Pi_1\zeta(t), \quad \text{for } 0 \le \tau(t) \le \tau_M/2,\\ \dot{V}(e(t)) &\le \zeta^T(t)\Pi_2\zeta(t), \quad \text{for } \tau_M/2 \le \tau(t) \le \tau_M.\end{aligned} \tag{39}$$

Finally, the equality constraints (17)–(18) for the augmented state (19) and the non-linear function $\Phi(\cdot)$ are rewritten with the extended vector $\zeta(t)$ as below

$$\Gamma\zeta(t) = 0 \tag{40}$$

where

$$\Gamma = \begin{bmatrix} I & -A & -E & 0 & 0 & -B & -D \\ 0 & \Delta C & 0 & 0 & 0 & -I & 0 \end{bmatrix}. \tag{41}$$

Therefore, a sufficient condition for stability of the error dynamics (9) with a given time delay is that

$$\dot{V}(e(t)) \le \zeta^T(t)\Pi_i\zeta(t) < 0 \quad \forall\zeta(t) \ne 0,\ i = 1, 2 \tag{42}$$

subject to

$$\Gamma\zeta(t) = 0. \tag{43}$$

Applying the projection lemma to (42) and (43), we obtain the LMIs:

$$\zeta^T(t)\Gamma^{\perp T}\Pi_i\Gamma^{\perp}\zeta(t) < 0, \quad \forall\zeta(t) \ne 0,\ i = 1, 2. \tag{44}$$

Since Δ is element of the convex set Φ, (20) is satisfied if the inequality (44) holds for each point pairs $j = 1, 2$. This completes the proof. □

In the following corollary, we derive an LMI condition for the stability of the error system (9) with fixed delay.

Corollary 1 *For fixed delay $\tau > 0$, with the absence of neutral type control, M, the error system (9) is absolutely stable if there exist positive matrices P, $G = \left[\begin{smallmatrix} G_{11} & G_{12} \\ \star & G_{22} \end{smallmatrix}\right]$, Q, R_1, $R_2 \in \mathbb{R}^{n\times n}$ and a positive definite diagonal matrix $S \in \mathbb{R}^{p\times p}$ satisfying the following LMI*:

$$N_i^{\perp T}\Theta N_i^{\perp} < 0, \quad i = 1, 2, \tag{45}$$

where $N^{\perp}$ is a right orthogonal complement of

$$N_i = \begin{bmatrix} I & -A & 0 & -E & -B & 0 & 0 & 0 \\ 0 & \Delta_i C & 0 & 0 & -I & 0 & 0 & 0 \\ 0 & -I & I & 0 & 0 & I & 0 & 0 \\ 0 & 0 & -I & I & 0 & 0 & I & 0 \\ 0 & -I & 0 & I & 0 & 0 & 0 & I \end{bmatrix} \tag{46}$$

and $\Theta(\tau)$ *is*

$$\Theta(\tau)=\begin{bmatrix}\Phi_{11} & \Phi_{12} & 0 & 0 & 0 & 0 & 0 & 0\\ \star & \Phi_{22} & \Phi_{23} & 0 & \Phi_{25} & 0 & 0 & 0\\ \star & \star & \Phi_{33} & \Phi_{34} & 0 & 0 & 0 & 0\\ \star & \star & \star & \Phi_{44} & 0 & 0 & 0 & 0\\ \star & \star & \star & \star & \Phi_{55} & 0 & 0 & 0\\ \star & \star & \star & \star & \star & \Phi_{66} & 0 & 0\\ \star & \star & \star & \star & \star & \star & \Phi_{77} & 0\\ \star & \star & \star & \star & \star & \star & \star & \Phi_{88}\end{bmatrix} \tag{47}$$

where

$$\Phi_{11}=\tau(R_1+R_2),\ \Phi_{12}=P,\ \Phi_{22}=Q+G_{11},\ \Phi_{23}=G_{12},\ \Phi_{25}=C^T(\alpha+\beta)^TS,$$
$$\Phi_{33}=G_{22}-G_{11},\ \Phi_{34}=-G_{12},\ \Phi_{44}=-Q-G_{22},\ \Phi_{55}=-2S,$$
$$\Phi_{66}=-\frac{2}{\tau}R_1,\ \Phi_{77}=-\frac{2}{\tau}R_1,\ \Phi_{88}=-\frac{2}{\tau}R_2.$$

Proof First of all, let us define the augmented vector,

$$\zeta(t)=\Big[\dot e^T(t)e^T(t)e^T\Big(t-\frac{\tau}{2}\Big)e^T(t-\tau)\phi^T(\nu(t))$$
$$\times\Big(\int_{t-\frac{\tau}{2}}^{t}e(\xi)d\xi\Big)^T\Big(\int_{t-\tau}^{t-\frac{\tau}{2}}e(\xi)d\xi\Big)^T\Big(\int_{t-\tau}^{t}e(\xi)d\xi\Big)^T\Big].$$

Next, Consider the following Lyapunov-Krasovskii functional:

$$V(e(t))=V_1(e(t))+V_2(e(t)) \tag{48}$$

where

$$V_1(e(t))=e(t)^TPe(t)+\int_{t-\tau}^{t}e^T(t)Qe(t)dt$$
$$+\int_{t-\frac{\tau}{2}}^{t}\begin{bmatrix}e(\xi)\\ e(\xi-\frac{\tau}{2})\end{bmatrix}^T\begin{bmatrix}G_{11} & G_{12}\\ \star & G_{22}\end{bmatrix}\begin{bmatrix}e(\xi)\\ e(\xi-\frac{\tau}{2})\end{bmatrix}d\xi,$$
$$V_2(e(t))=\int_{t-\frac{\tau}{2}}^{t}\Big(\frac{\tau}{2}-t+\xi\Big)\dot e^T(\xi)R_1\dot e(\xi)d\xi$$
$$+\int_{t-\tau}^{t-\frac{\tau}{2}}(\tau-t+\xi)\dot e^T(\xi)R_2\dot e(\xi)d\xi.$$

The time-derivative of $V_1(e(t))$ can be calculated as

$$\begin{aligned}\dot{V}_1(e(t)) &= \dot{e}^T(t)Pe(t) + e^T(t)P\dot{e}(t) + e^T(t)Qe(t) - e^T(t-\tau)Qe(t-\tau)\\ &\quad + \begin{bmatrix} e(t) \\ e(t-\frac{\tau}{2}) \end{bmatrix}^T \begin{bmatrix} G_{11} & G_{12} \\ \star & G_{22} \end{bmatrix} \begin{bmatrix} e(t) \\ e(t-\frac{\tau}{2}) \end{bmatrix}\\ &\quad - \begin{bmatrix} e(t-\frac{\tau}{2}) \\ e(t-\tau) \end{bmatrix}^T \begin{bmatrix} G_{11} & G_{12} \\ \star & G_{22} \end{bmatrix} \begin{bmatrix} e(t-\frac{\tau}{2}) \\ e(t-\tau) \end{bmatrix}. \end{aligned} \tag{49}$$

Next, the time-derivative of $V_2(e(t))$ is found as

$$\begin{aligned}\dot{V}_2 &\le \tau\dot{e}^T(t)(R_1+R_2)\dot{e}(t) - \frac{2}{\tau}\left(\int_{t-\frac{\tau}{2}}^{t} e(\xi)d\xi\right)^T R_1 \left(\int_{t-\frac{\tau}{2}}^{t} e(\xi)d\xi\right)\\ &\quad - \frac{2}{\tau}\left(\int_{t-\tau}^{t-\frac{\tau}{2}} e(\xi)d\xi\right)^T R_1 \left(\int_{t-\tau}^{t-\frac{\tau}{2}} e(\xi)d\xi\right)\\ &\quad - \frac{1}{\tau}\left(\int_{t-\tau}^{t} e(\xi)d\xi\right)^T R_2 \left(\int_{t-\tau}^{t} e(\xi)d\xi\right). \end{aligned} \tag{50}$$

From the sector constraint of the nonlinearity $\phi(\cdot)$, the following inequality is obtained

$$2(\phi(\nu(t) - \alpha\nu(t))^T S(\phi(\nu(t)) - \beta\nu(t)) \le 0. \tag{51}$$

Analogously to Theorem 1, one obtains the inequality (47) by applying the S-procedure [23]. Next, the equality constraint and the nonlinear function $\phi(\nu(t))$ are written as

$$\begin{bmatrix} I & -A & 0 & -E & -B & 0 & 0 & 0 \\ 0 & \Delta_i C & 0 & 0 & -I & 0 & 0 & 0 \end{bmatrix}. \tag{52}$$

Also there are three additional equality constraints derived from $(\int_{t-\frac{\tau}{2}}^{t} e(\xi)d\xi)$, $(\int_{t-\tau}^{t-\frac{\tau}{2}} e(\xi)d\xi)$ and $(\int_{t-\tau}^{t} e(\xi)d\xi)$ are as below

$$\begin{bmatrix} 0 & -I & I & 0 & 0 & I & 0 & 0 \\ 0 & 0 & -I & I & 0 & 0 & I & 0 \\ 0 & -I & 0 & I & 0 & 0 & 0 & I \end{bmatrix}. \tag{53}$$

Combining (52) and (53) yields the following equality constraint

$$N_i\zeta(t) = 0 \tag{54}$$

where

$$N_i = \begin{bmatrix} I & -A & 0 & -E & -B & 0 & 0 & 0 \\ 0 & \Delta_i C & 0 & 0 & -I & 0 & 0 & 0 \\ 0 & -I & I & 0 & 0 & I & 0 & 0 \\ 0 & 0 & -I & I & 0 & 0 & I & 0 \\ 0 & -I & 0 & I & 0 & 0 & 0 & I \end{bmatrix}. \tag{55}$$

Therefore, a sufficient condition for stability of the error dynamics (9) with the absence of neutral type control and a given fixed time delay is that

$$\dot{V}(e(t)) \le \zeta^T(t)\Theta\zeta(t) < 0 \quad \forall\zeta(t) \neq 0 \tag{56}$$

subject to

$$N_i\zeta(t) = 0. \tag{57}$$

Applying the projection lemma to (56) and (57), we obtain the LMIs:

$$\zeta^T(t)N_i^{\perp T}\Theta N_i^{\perp}\zeta(t) < 0, \quad \forall\zeta(t) \neq 0, \tag{58}$$

Since Δ is element of the convex set Φ, Equation (45) is satisfied if the inequality (58) holds for each point pair $i = 1, 2$. □

4 Numerical Examples

In this section, Chua's circuit is used to illustrate the effectiveness of the proposed synchronization criterion given in Theorem 1.

Example 1 Consider the following Chua's circuit as shown in [10]

$$\begin{cases} \dot{x} = a(y - h(x)), \\ \dot{y} = x - y + z, \\ \dot{z} = -by \end{cases} \tag{59}$$

with the nonlinear characteristic

$$h(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(|x + c| - |x - c|)$$

and the parameters $m_0 = -1/7$, $m_1 = 2/7$, $a = 9$, $b = 14.28$ and $c = 1$. The circuit can be represented in Lur'e system by

$$\begin{aligned} &A = \begin{bmatrix} -am_1 & a & 0 \\ 1 & -1 & 1 \\ 0 & -b & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} -a(m_0 - m_1) \\ 0 \\ 0 \end{bmatrix}, \\ &C = H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \end{aligned} \tag{60}$$

and $\varphi(\mu) = \frac{1}{2}(|\mu + c| - |\mu - c|)$ belonging to sector [0, 1].

Table 1 shows that the maximum allowable delay bound τ_M which is obtained by Corollary 1 with the gain matrices in Table 2 and the synchronization criterion of Corollary 1 is less conservative than the ones given in previous studies. Figure 1 shows synchronization errors with the maximum allowable delay bound $\tau_M = 0.203$ and gain matrices as shown in Table 2. Moreover, through numerical simulation, we

Table 1 The maximum allowable time delay τ_M for each gain matrix

Gain matrix L		The maximum allowable delay τ_M
L_1	Yalcin et al. [10]	0.039
	Han [18]	0.1418
	Souza et al. [19]	0.141
	Corollary 1	**0.203**

Table 2 Gain matrices of the synchronization scheme used in delay effect analysis for Chua's circuit

Number	Gain matrices	Real delay
Case 1	$M_1 = [0.1 \quad 0 \quad 0]$, $L_1 = [6.0229 \quad 1.3367 \quad -2.2164]^T$	$\tau_M^* = 0.225$
Case 2	$M_2 = [0.5 \quad 0 \quad 0]$, $L_1 = [6.0229 \quad 1.3367 \quad -2.2164]^T$	$\tau_M^* = 0.234$

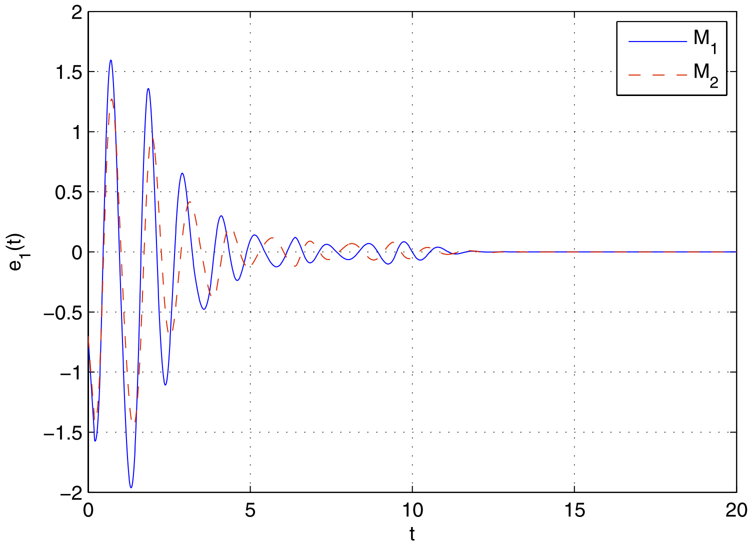
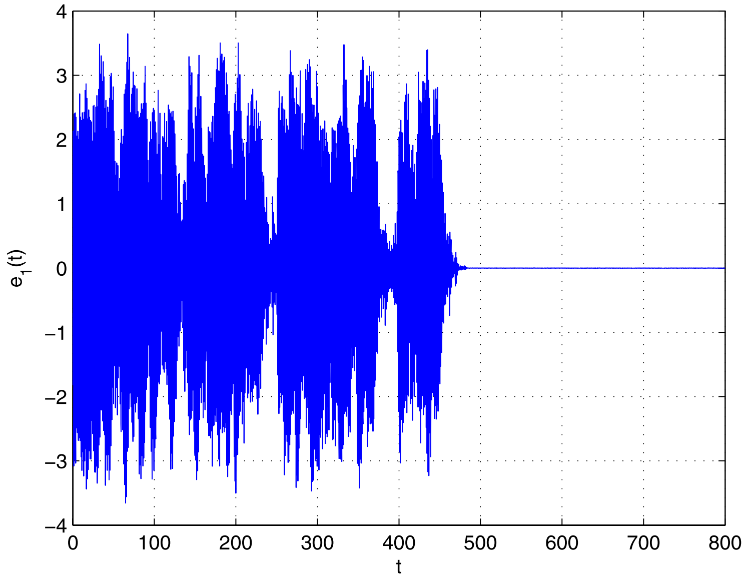
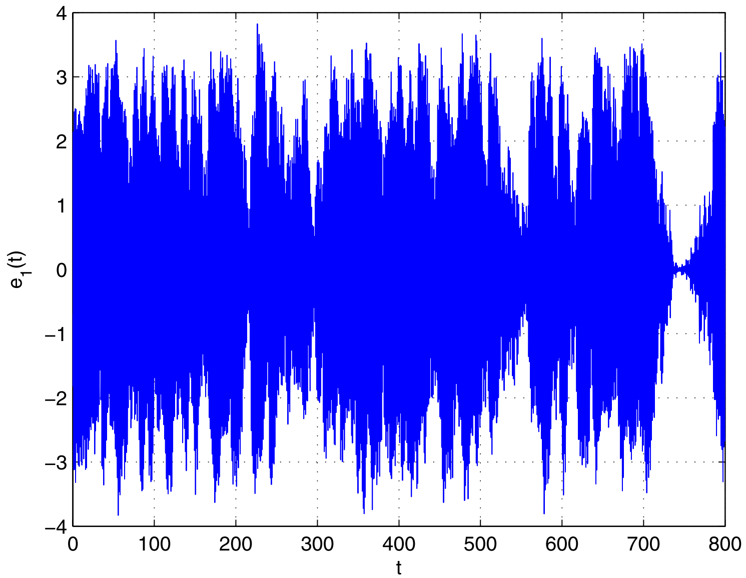


Fig. 1 Synchronization errors for the Chua's circuits for $\tau_m = 0.203$ with gain matrices in Table 2

find delay bounds τ_M^* by which the drive-response of Chua's circuit can be synchronized with the delay bounds τ_M^* which are given in Table 2. Figure 2 shows one observation for Case 1. The output trajectory between the drive-response system does not synchronize for the gain matrix of Case 1 at $\tau_M = 0.225$. During the simulation, we take the initial values $x(0) = [-0.2 \ -0.33 \ 0.2]^T$ for the drive system and $y(0) = [0.5 \ -0.1 \ 0.66]^T$ for the response system. $h_M = 0.3$, $h_d = 0$, $\tau_d = 0$ are chosen.



(a) For $\tau_M = 0.224$



(b) For For $\tau_M = 0.225$

Fig. 2 Synchronization errors for the Chua's circuits for $h_M = 0.3$, $h_d = 0$ and $\tau_d = 0$

Table 3 Stability bounds τ_M for different varying rate of delay h_d and τ_d for Case 1

$h_d \diagdown \tau_d$	0.1	0.5	0.9
0.1	0.1372	0.1355	0.1354
0.5	0.1346	0.1328	0.1327
0.9	0.1194	0.1170	0.1169

Table 4 Stability bounds τ_M for different varying rate of delay h_d and τ_d for Case 2

$h_d \diagdown \tau_d$	0.1	0.5	0.9
0.1	0.1220	0.1194	0.1192
0.5	0.1093	0.1065	0.1064
0.9	0.0095	0.0094	0.0094

We have demonstrated the effectiveness of the proposed synchronization criterion compared with other synchronization schemes by paying attention to the design of a feedback controller for synchronization. Tables 3 and 4 shows the maximum allowable delay τ_M for each τ_D and h_D.

Example 2 Next, consider the following hyper-chaotic system which combines two Chua's circuits shown in [26, 27]

$$\begin{cases} \dot{x}_1 = a(x_2 - h(x_1)), \\ \dot{x}_2 = x_1 - x_2 + x_3, \\ \dot{x}_3 = -bx_2, \\ \dot{x}_4 = a(x_5 - h(x_4)) + K(x_4 - x_1), \\ \dot{x}_5 = x_4 - x_5 + x_6, \\ \dot{x}_6 = -bx_5 \end{cases} \tag{61}$$

with the nonlinear characteristic

$$h(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(|x + c| - |x - c|)$$

and the parameters $m_0 = -1/7$, $m_1 = 2/7$, $a = 9$, $b = 14.28$ and $c = 1$. The system can be represented in Lur'e system by

$$A = \begin{bmatrix} -am_1 & a & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -b & 0 & 0 & 0 & 0 \\ -K & 0 & 0 & -am_1 + K & a & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -b & 0 \end{bmatrix},$$

Table 5 Gain matrices of the synchronization scheme used in delay effect analysis for hyper-chaotic system

Number	Gain matrices
Case 3	$M_1 = \begin{bmatrix} 0.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0 \end{bmatrix}$, $L_1 = \begin{bmatrix} 7.6909 & 2.1313 & -3.9865 & -0.3491 & 0.1811 & -0.5180 \\ -1.052 & 0.0835 & 0.3455 & 8.0879 & 1.8021 & -4.8256 \end{bmatrix}^T$
Case 4	$M_1 = \begin{bmatrix} 0.3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0 \end{bmatrix}$, $L_1 = \begin{bmatrix} 7.6909 & 2.1313 & -3.9865 & -0.3491 & 0.1811 & -0.5180 \\ -1.052 & 0.0835 & 0.3455 & 8.0879 & 1.8021 & -4.8256 \end{bmatrix}^T$

Table 6 Stability bounds τ_M for different varying rate of delay h_d and τ_d for Case 3

$h_d \diagdown \tau_d$	0.1	0.5	0.9
0.1	0.1167	0.1152	0.0961
0.5	0.1138	0.1123	0.1123
0.9	0.0971	0.0951	0.0951

Table 7 Stability bounds τ_M for different varying rate of delay h_d and τ_d for Case 4

$h_d \diagdown \tau_d$	0.1	0.5	0.9
0.1	0.0984	0.0961	0.0961
0.5	0.0852	0.0826	0.0826
0.9	0.0066	0.0066	0.0066

$$B = \begin{bmatrix} -a(m_0 - m_1) & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -a(m_0 - m_1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{62}$$

$$C = H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and $\varphi(\mu) = \frac{1}{2}(|\mu + c| - |\mu - c|)$ belonging to sector $[0, 1]$. The gain matrices are shown in Table 5 and the delay bounds which are found using Theorem 1 are shown in Tables 6 and 7.

5 Conclusions

In this paper, we have analyzed the effect of a time-varying delay in a master-slave synchronization represented by Lur'e systems with a sector using delayed feedback PD controller. A new synchronization criterion was presented and it is a sufficient condition of the error dynamics for the given time-varying delay. A convex representation for the nonlinearity was derived, and then, equality constraints were employed so that a less conservative criterion was obtained by utilization of the projection lemma. The augmented Lyapunov-Krasovskii functional based on the delay discretization approach has also been used for the criterion. Through two numerical examples, we have demonstrated the effectiveness of the proposed methods.

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