J Optim Theory Appl (2010) 147: 113–124
DOI 10.1007/s10957-010-9710-5

Structure and Weak Sharp Minimum of the Pareto Solution Set for Piecewise Linear Multiobjective Optimization

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Published online: 12 May 2010

Abstract In this paper, the Pareto solution set of a piecewise linear multiobjective optimization problem in a normed space is shown to be the union of finitely many semiclosed polyhedra. If the problem is further assumed to be cone-convex, then it has the global weak sharp minimum property.

Keywords Piecewise linear functions · Multiobjective optimization problems · Pareto solution sets · Global weak sharp minimum · Image space analysis

1 Introduction

Let $f : X \to Y$ be a mapping from a normed space X to a normed space Y of finite dimension. As in [1], f is said to be a *piecewise linear function* (or a piecewise affine function), iff there exist families $\{P_1, \dots, P_k\}$, $\{T_1, \dots, T_k\}$, and $\{b_1, \dots, b_k\}$ of polyhedral convex sets in X, continuous linear operators from X to Y, and points in Y, respectively, such that $X = \bigcup_{i=1}^{k} P_i$ and

$$f(x) = f_i(x) := T_i(x) + b_i \quad \forall x \in P_i,\ \forall i \in \{1, \dots, k\}. \tag{1}$$

In order to have (1), we must have

$$f_i(x) = f_j(x) \quad \forall x \in P_i \cap P_j,\ \forall i, j \in \{1, \dots, k\}.$$

Communicated by F. Giannessi.

This work was partially supported by the Research Grants Council of Hong Kong (PolyU5303/05E) and National Foundation for Science & Technology Development (Vietnam).

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Given a finite family of continuous real-valued linear functions, it is easy to see that the operation of taking the maximum or the minimum of that family leads us to a continuous piecewise linear function. More generally, the maximum or the minimum of a finite family of arbitrary continuous real-valued piecewise linear functions is again a continuous real-valued piecewise linear function. As usual, we call a subset of a normed space a *polyhedral convex set* (a *polyhedron*, for brevity), iff it is the entire space or it is the intersection of finitely many closed half-spaces. Note that continuous piecewise linear functions and related multiobjective optimization problems have been studied from different points of view. For instance, Gowda and Sznajder [2] studied the pseudo-Lipschitz property of f^{-1} and showed that the obtained results can be applied to affine variational inequalities and linear complementarity problems. It is worth mentioning that the definition of a piecewise linear function recalled above is weaker than that in [2], where it was assumed that, for every pair (i, j), $P_i \cap P_j$ is either empty or a common face of P_i and P_j. For bicriteria linear programming problems with special network or financial structures, the Pareto solution set was found in [3, 4], respectively.

Let $D \subset X$ be a polyhedron, let $C \subset Y$ be a polyhedral convex cone and let $f : X \to Y$ be a piecewise linear function. Consider the following piecewise linear multiobjective optimization problem:

$$\text{(P)} \qquad \min_C f(x), \qquad \text{s.t.} \quad x \in D.$$

Let $u \in D$. We say that u is a *Pareto solution* of (P) iff

$$\nexists x \in D, \qquad f(u) - f(x) \in C \setminus \{0_Y\}.$$

A point u is said to be a *weak Pareto solution* of (P), iff there does not exist $x \in D$ with $f(u) - f(x) \in \operatorname{int} C$, where $\operatorname{int} C$ denotes the topological interior of C. The set of all the Pareto solutions (resp., the set of all the weak Pareto solutions) is denoted by S (resp., S_w). The study of characteristic properties of these solution sets is useful in the design of efficient algorithms for solving (P).

Following [5, p. 341], we say that $f : X \to Y$ is a *C-function* on D iff for any $x^1, x^2 \in D$ and $t \in]0, 1[$, it holds that

$$(1-t)f(x^1) + tf(x^2) - f\big((1-t)x^1 + tx^2\big) \in C.$$

In the case $Y = \mathbb{R}^n$ and $C = \mathbb{R}^n_+$ (the nonnegative orthant in $\mathbb{R}^n$), $f = (f_1, \ldots, f_n)$ is called a *convex function* on D whenever it is a C-function on D. Hence the convexity of f on D is equivalent to the convexity of the components f_i on D.

By abuse of terminology, if f is a C-function on D then we say that (P) is a *convex problem*.

The classical Arrow, Barankin and Blackwell theorem (ABB theorem) states that, for any linear multiobjective optimization problem in finite-dimensional normed spaces, S (resp., S_w) is the union of finitely many polyhedra and is connected by line segments. The last property means that, for any given pair $u, \widetilde{u} \in S$ there exist $u^1, \ldots, u^\ell \in S$ such that $u^1 = u$, $u^\ell = \widetilde{u}$ and the line segment $[u^r, u^{r+1}]$ belongs to S for $r = 1, \ldots, \ell - 1$; see [6] and [7, pp. 137/138] (note that the proof is valid also for the case where the constraint set D is an unbounded polyhedron). For (P), S_w is shown to be the union of finitely many polyhedra and, assuming in addition that f be a C-function, S_w is shown to be connected by line segments in Zheng and Yang [1].

The notion of global weak sharp minima of scalar optimization problems was introduced in Burke and Ferris [8] and has been used later for proving the finite termination of some algorithms, such as gradient methods and gradient projection methods. Later on, this important notion has been developed and applied in a series of works; see for instance [9–11]. A related notion, called *sharp minimum*, has also been studied by several authors (see [12]).

The set S_w of a linear multiobjective optimization problem in finite-dimensional normed spaces is shown to have the global weak sharp minimum property in Deng and Yang [13]. This result has been extended in [14] to the case of a convex piecewise linear multiobjective optimization problem in a normed space by virtue of the properties of S_w established in [1].

This paper aims at extending the above-cited ABB theorem to the case of the Pareto solution set S of (P) and to applying it so as to establish a global weak sharp minimum property for a convex piecewise linear multiobjective optimization problem. Our method of proof, which relies on the *image space approach* to optimization problems and variational systems [5, 15], is very different from that of [1]. In particular, we do not use decompositions of X into sums of two linear subspaces. It is noted that several auxiliary results used in [1] for the weak Pareto solutions are no longer valid for the Pareto solutions. In fact, this obstacle had forced us to find a new way of reasoning.

Information about the structure of S [7, Chap. 4, Theorem 3.3], and the proof scheme of [14] are very useful for our investigation in the establishment of a global weak sharp minimum property for a convex piecewise linear multiobjective optimization problem. As a byproduct, we obtain also a weak sharp minima property for the Pareto solution set of a linear multiobjective optimization problem in a normed space.

The subsequent sections are organized as follows. In Sect. 2 we prove that the Pareto solution set S of (P) is the union of finitely many semiclosed polyhedra. In the same section, it is shown that, if f is a C-function, then S is the union of finitely many polyhedra and is connected by line segments. In Sect. 3, we establish a global weak sharp minimum property for the Pareto solution set of a convex piecewise linear multiobjective optimization problem.

2 Structure of the Pareto Solution Set

2.1 Nonconvex Case

A subset of a normed space is called a *semiclosed polyhedron*, iff it is the intersection of a family consisting of finitely many closed half-spaces and finitely many open half-spaces. Here and in the sequel, the intersection of an empty family of subsets in a normed space is stipulated to be the entire space.

Theorem 2.1 *The Pareto solution set S of* (P) *is the union of finitely many semiclosed polyhedra.*

Proof Put $M = \bigcup_{i=1}^{k} M_i$, where $M_i := f_i(P_i \cap D)$ for $i = 1, \ldots, k$. By $E(M_i|C)$ we denote the Pareto solution set of M_i. From the definitions, it follows that

$$S = \{x \in D : f(x) \in E(M|C)\} = f^{-1}\big(E(M|C)\big) \cap D. \tag{2}$$

Formula (2) allows us to use the image space approach (see [5, 15]) to achieve our goal: We first show that, in the image space Y, the set $E(M|C)$ is the union of finitely many semiclosed polyhedra; then we deduce from (2) and the assumed piecewise linearity of f that S is the union of finitely many semiclosed polyhedra.

Claim 2.1 *$E(M|C)$ is the union of finitely many semiclosed polyhedra.*

Note that

$$E(M|C) \subset E(M_1|C) \cup E(M_2|C) \cup \cdots \cup E(M_k|C). \tag{3}$$

For any $i \in \{1, \ldots, k\}$, $M_i = \{T_i(x) + b_i : x \in P_i \cap D\}$ is a polyhedral convex set in Y. Since Y is a finite-dimensional normed space and $C \subset Y$ is a polyhedral convex cone, the ABB theorem asserts that $E(M_i|C)$ is the union of finitely many polyhedra (which are faces of M_i). Suppose that

$$E(M_i|C) = \bigcup_{r=1}^{m_i} Q_{i,r}, \quad i = 1, \ldots, k, \tag{4}$$

where each $Q_{i,r}$ is a polyhedron in Y. Using (3) and (4), we now describe an algorithm for finding the whole set $E(M|C)$.

Step 1. Consider the Pareto solution set $E(M_1|C)$. By (4) we have,

$$E(M_1|C) = \bigcup_{r=1}^{m_1} Q_{1,r}. \tag{5}$$

Clearly, a point $v \in Q_{1,r}$, with $r \in \{1, \ldots, m_1\}$, falls into $E(M|C)$, if and only if

$$v \notin M_i + \big(C \setminus \{0_Y\}\big), \quad \forall i \in \{2, \ldots, k\}. \tag{6}$$

Take any $v \in Q_{1,r}$. Let $i = 2$. If $v \in M_2 + C$, then there are two possibilities:

$$\text{(a)} \quad v \in M_2 + \big(C \setminus \{0_Y\}\big), \qquad \text{(b)} \quad v \in M_2.$$

In case (a), by the criterion (6), v cannot belong to $E(M|C)$. Hence we can exclude v from $Q_{1,r}$. In case (b), there are two subcases:

$$\text{(b1)} \quad v \in E(M_2|C), \qquad \text{(b2)} \quad v \notin E(M_2|C).$$

The subcase (b1) will be analyzed in next step when we deal with the set $E(M_2|C)$. In the subcase (b2), we can ignore this v because, according to (3), $v \notin E(M|C)$. Therefore, if the case (b2) occurs, then we can also exclude v from $Q_{1,r}$. By Theorem 19.1 in [16], $M_2 + C$ is a polyhedral convex set. Observe that, after excluding from $Q_{1,r}$ all the points belonging to $M_2 + C$, we obtain the union of finitely many semiclosed polyhedra. Indeed, if $M_2 + C = \bigcap_{s=1}^{\ell_2} \Pi_s$, where ℓ_2 is an integer and each Π_s is a closed half-space, then

$$Q_{1,r}^{(2)} := Q_{1,r} \setminus [M_2 + C] = \bigcup_{s=1}^{\ell_2} (Q_{1,r} \setminus \Pi_s) = \bigcup_{s=1}^{\ell_2} \big[Q_{1,r} \cap (Y \setminus \Pi_s)\big].$$

Since $Q_{1,r} \cap (Y \setminus \Pi_s)$ $(s = 1, \dots, \ell_2)$ are semiclosed polyhedra, our observation is justified. Repeating the procedure done for $i = 2$ with $i = 3$, we obtain the set

$$Q_{1,r}^{(3)} := Q_{1,r}^{(2)} \setminus [M_3 + C]$$

which is the union of finitely many semiclosed polyhedra. Continue the above recursive process until reaching the set

$$Q_{1,r}^{(k)} := Q_{1,r}^{(k-1)} \setminus [M_k + C].$$

Note that $Q_{1,r}^{(k)}$ is contained in $E(M|C)$ and can be represented as the union of finitely many semiclosed polyhedra. Taking account of (5) and the above construction, we see that $\bigcup_{r=1}^{m_1} Q_{1,r}^{(k)} \subset E(M|C)$.

Step 2. We repeat the procedure done for $E(M_1|C)$ with the next Pareto solution set $E(M_2|C)$, letting $i = 1$ play the role of $i = 2$ in Step 1. We put

$$Q_{2,r}^{(1)} := \big(Q_{2,r} \setminus [M_1 + C]\big) \cup \big(Q_{2,r} \cap E(M_1|C)\big)$$

whenever $Q_{2,r} \cap E(M_1|C) \neq \emptyset$. As a result, we obtain the sets $Q_{2,r}^{(k)}$ $(r = 1, \dots, m_2)$ such that each of them is the union of finitely many semiclosed polyhedra. By the second equality in (4), we have $\bigcup_{r=1}^{m_2} Q_{2,r}^{(k)} \subset E(M|C)$.

Step j ($j \in \{3, \dots, k-1\}$). We let $i = 1$ (resp., $i = 2, \dots, i = j-1$) play the role of $i = 2$ (resp., $i = 3, \dots, i = j$) in Step 1. We put

$$Q_{j,r}^{(i)} := \big(Q_{j,r} \setminus [M_i + C]\big) \cup \big(Q_{j,r} \cap E(M_i|C)\big)$$

whenever $Q_{j,r} \cap E(M_i|C) \neq \emptyset$ for $i \in \{1, \dots, j-1\}$. As a result, we get the inclusion $\bigcup_{r=1}^{m_j} Q_{j,r}^{(k)} \subset E(M|C)$, where each of the sets $Q_{j,r}^{(k)}$ is the union of finitely many semiclosed polyhedra.

Step k. We let $i = 1$ (resp., $i = 2, \dots, i = k-1$) play the role of $i = 2$ (resp., $i = 3, \dots, k$) in Step 1. We put

$$Q_{k,r}^{(i)} := \big(Q_{k,r} \setminus [M_i + C]\big) \cup \big[Q_{k,r} \cap E(M_i|C)\big],$$

whenever $Q_{k,r} \cap E(M_i|C) \neq \emptyset$ for $i \in \{1, \dots, k-1\}$. As a result, we get the inclusion $\bigcup_{r=1}^{m_k} Q_{k,r}^{(k-1)} \subset E(M|C)$, where each of the sets $Q_{k,r}^{(k-1)}$ is the union of finitely many semiclosed polyhedra.

After completing the above k steps, we obtain the inclusion

$$\left[\bigcup_{i=1}^{k-1}\left(\bigcup_{r=1}^{m_i} Q_{i,r}^{(k)}\right)\right] \cup \left(\bigcup_{r=1}^{m_k} Q_{k,r}^{(k-1)}\right) \subset E(M|C).$$

Moreover, from (3) and from our exhausting construction, it follows that the last inclusion holds as an equality, that is,

$$E(M|C) = \left[\bigcup_{i=1}^{k-1}\left(\bigcup_{r=1}^{m_i} Q_{i,r}^{(k)}\right)\right] \cup \left(\bigcup_{r=1}^{m_k} Q_{k,r}^{(k-1)}\right). \tag{7}$$

Clearly, the set on the right-hand side of this equality is the union of finitely many semiclosed polyhedra.

Claim 2.2 *S is the union of finitely many semiclosed polyhedra.*

From (2) and (7) we can deduce that S is the union of finitely many semiclosed polyhedra. Leaving the details to the reader, we observe only that, for an open half-space

$$\Omega := \{y \in Y : \langle y^*, y\rangle < \beta\} \quad (y^* \in Y^*,\ \beta \in \mathbb{R})$$

and for any $i \in \{1, \ldots, k\}$, the formulas

$$\begin{aligned} f_i^{-1}(\Omega) \cap P_i &= \{x \in P_i : f_i(x) \in \Omega\} \\ &= \{x \in P_i : \langle y^*, T_i(x) + b_i\rangle < \beta\} \\ &= \{x \in P_i : \langle T_i^*(y^*), x\rangle < \beta - \langle y^*, b_i\rangle\} \end{aligned}$$

show that $f_i^{-1}(\Omega) \cap P_i$ is a semiclosed polyhedron in X.

The proof is complete. □

The above proof scheme is also valid for establishing Theorem 3.1 of [1], whose details are omitted. We illustrate the result in Theorem 2.1 by an example.

Example 2.1 Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $D = \{x : 0 \le x \le 1\}$, $C = \mathbb{R}^2_+$, and

$$f(x) = \begin{cases} f_1(x) := (2x, 1 - 2x) & \text{if } x \in P_1 :=]-\infty, \frac{1}{2}] \\ f_2(x) := (\frac{3}{2} - x, 0) & \text{if } x \in P_2 := [\frac{1}{2}, \infty[. \end{cases}$$

We have

$$M_1 := f_1(P_1 \cap D) = f_1\left(\left[0, \frac{1}{2}\right]\right) = \operatorname{conv}\big\{(0, 1), (1, 0)\big\},$$

where conv is the operation of taking a convex hull. So,

$$E\big(f_1(P_1 \cap D)|\mathbb{R}^2_+\big) = \operatorname{conv}\big\{(0, 1), (1, 0)\big\}.$$

Similarly,

$$M_2 := f_2(P_2 \cap D) = f_2\left(\left[\frac{1}{2}, 1\right]\right) = \operatorname{conv}\left\{(1,0), \left(\frac{1}{2}, 0\right)\right\}$$

and

$$E\big(f_2(P_2 \cap D)|\mathbb{R}^2_+\big) = \left\{\left(\frac{1}{2}, 0\right)\right\}.$$

Setting $M := M_1 \cup M_2 = f(D)$, we deduce that

$$E(M|\mathbb{R}^2_+) = \left\{\left(\frac{t}{2}, 1 - \frac{t}{2}\right) : 0 \le t < 1\right\} \cup \left\{\left(\frac{1}{2}, 0\right)\right\}, \quad S = \left[0, \frac{1}{4}\right[\cup \{1\},$$

and

$$E^w(M|\mathbb{R}^2_+) = \left\{\left(\frac{t}{2}, 1 - \frac{t}{2}\right) : 0 \le t \le 1\right\} \cup \left\{\left(\frac{1}{2}, 0\right)\right\}, \quad S_w = \left[0, \frac{1}{4}\right] \cup \{1\}.$$

Observe that both the sets $E(M|\mathbb{R}^2_+)$ and S are disconnected, not closed, and each is the union of two semiclosed polyhedra.

2.2 Convex Case

The conclusion of Theorem 2.1 can be improved, if (P) is a convex problem.

Theorem 2.2 *If f is a C-function on D, then the Pareto solution set S of* (P) *is the union of finitely many polyhedra and is connected by line segments.*

Proof Again, following the image space approach, we focus our analysis on a family of subsets of the image space Y. Let M_i $(i = 1, \ldots, k)$ and M be the same as in the proof of Theorem 2.1. First, we show that $E(M|C)$ is the union of finitely many polyhedra and is connected by line segments. Clearly, $E(M|C) = E(M + C|C)$. By the ABB theorem, it suffices to show that $M + C$ is a polyhedron. From the assumption that f is a C-function on D, it follows that $M + C = f(D) + C$ is a convex set. Since C is a polyhedral convex cone and each $M_i = f_i(P_i \cap D)$ is a polyhedron, by Theorem 19.1 in [16], we have the representation

$$C = \left\{\sum_{r=1}^{s} \mu_r v_r : \mu_r \ge 0 \ \forall r\right\}$$

and

$$M_i = \left\{\sum_{p=1}^{p_i} \lambda_i^{(p)} a_{i,p} + \sum_{q=1}^{q_i} \mu_i^{(q)} v_{i,q} : \lambda_i^{(p)} \ge 0 \ \forall p, \ \sum_{p=1}^{p_i} \lambda_i^{(p)} = 1, \ \mu_i^{(q)} \ge 0 \ \forall q\right\}$$

for some suitably chosen vectors v_r $(r = 1, \dots, s)$, $a_{i,p}$ $(p = 1, \dots, p_i)$, and $v_{i,q}$ $(q = 1, \dots, q_i)$. Then, taking account of the convexity of $M + C$, we see that

$$\begin{aligned} M + C &= \left(\bigcup_{i=1}^{k} M_i\right) + C \\ &\subset \left\{ \sum_{i=1}^{k}\sum_{p=1}^{p_i} \lambda_i^{(p)} a_{i,p} + \sum_{i=1}^{k}\sum_{q=1}^{q_i} \mu_i^{(q)} v_{i,q} + \sum_{r=1}^{s} \mu_r v_r \ : \right. \\ &\qquad \left. \lambda_i^{(p)} \geq 0\ \forall i\ \forall p,\ \sum_{i=1}^{k}\sum_{p=1}^{p_i} \lambda_i^{(p)} = 1,\ \mu_i^{(q)} \geq 0\ \forall i\ \forall q,\ \mu_r \geq 0\ \forall r \right\} \\ &= \operatorname{co}\left\{ \left(\bigcup_{i=1}^{k} M_i\right) + C \right\} = M + C. \end{aligned}$$

This implies that the inclusion in the last expression must be an equality. Applying Theorem 19.1 in [16] once more, we conclude that $M + C$ is a polyhedral convex set.

Since $E(M|C)$ can be represented as the union of finitely many polyhedra, from (2) it follows that S has the same property. In order to show that S is connected by line segments, we fix two points $u, \widetilde{u} \in S$ and put $v = f(u)$, $\widetilde{v} = f(\widetilde{u})$. As $E(M|C)$ is connected by line segments, there exist $v^1, \dots, v^\ell \in Y$ such that $v^1 = v$, $v^\ell = \widetilde{v}$ and the line segment $[v^r, v^{r+1}]$ belongs to $E(M|C)$ for $r = 1, \dots, \ell - 1$. The inclusions

$$[v^1, v^2] \subset E(M|C) \subset \bigcup_{i=1}^{k} M_i$$

imply that there exist $v^{1,p}$ $(p = 1, \dots, p_1 + 1)$ in $[v^1, v^2]$ and indexes $i_1, \dots, i_{p_1} \subset \{1, \dots, k\}$ such that $v^{1,1} = v^1$, $v^{1,p_1+1} = v^2$, and

$$\left[v^{1,p}, v^{1,p+1}\right] \subset M_{i_p} \quad \forall p \in \{1, \dots, p_1\}.$$

For each $p \in \{1, \dots, p_1\}$, we can select a point $u^{1,p} \in P_{i_p}$ satisfying $f_{i_p}\left(u^{1,p}\right) = v^{1,p+1}$ and a point $\widetilde{u}^{1,p} \in P_{i_{p+1}}$ satisfying $f_{i_{p+1}}\left(\widetilde{u}^{1,p}\right) = v^{1,p+1}$. Since $(1-t)u + tu^{1,1} \in P_{i_1}$ and since

$$f_{i_1}\left((1-t)u + tu^{1,1}\right) = (1-t)v^{1,1} + tv^{1,2} \in E(M|C),$$

for every $t \in [0, 1]$, it follows that $\left[u, u^{1,1}\right] \subset S$. Similarly,

$$\left[\widetilde{u}^{1,p}, u^{1,p+1}\right] \subset S, \quad \forall p \in \{1, \dots, p_1\}.$$

For any $p \in \{1, \dots, p_1\}$, by the assumption that f is a C-function, we have

$$\begin{aligned} &v^{1,p+1} - f\left((1-t)u^{1,p} + t\widetilde{u}^{1,p}\right) \\ &\quad = (1-t)f\left(u^{1,p}\right) + tf\left(\widetilde{u}^{1,p}\right) - f\left((1-t)u^{1,p} + t\widetilde{u}^{1,p}\right) \in C. \end{aligned}$$

Since $v^{1,p+1} \in E(M|C)$, this implies that

$$f\big((1-t)u^{1,p} + t\widetilde{u}^{1,p}\big) = v^{1,p+1} \quad \forall t \in [0,1].$$

Hence $[u^{1,p}, \widetilde{u}^{1,p}] \subset S$. We have shown that there exists a continuous curve in S which consists of a finite number of line segments and joins u and u^{1,p_1}, where $f(u) = v = v^1$ and $f(u^{1,p_1}) = v^2$. Arguing similarly we can find a continuous curve in S which consists of a finite number of line segments and joins u^{1,p_1} and a point $u^{2,p_2} \in S$ satisfying $f(u^{2,p_2}) = v^3$. We continue the process until reaching a point $u^{\ell-1,p_{\ell-1}} \in S$ satisfying $f(u^{\ell-1,p_{\ell-1}}) = v^\ell = \widetilde{v}$. Of course, at the last step, we can select $u^{\ell-1,p_{\ell-1}} = \widetilde{u}$. We have shown that u can be joined with $\widetilde{u}$ by a continuous curve in S which consists of a finite number of line segments. The proof is complete. □

It is easy to see that, if $f : X \to Y$ is a continuous linear operator, then for any convex cone $C \subset Y$, f is a C-function on X. Hence, from Theorem 2.1 we can derive the following statement on the structure of the Pareto solution set in linear vector optimization in an infinite-dimensional normed space.

Theorem 2.3 (Infinite-dimensional version of the ABB theorem) *If $f : X \to Y$ is a continuous linear operator, then S is the union of finitely many polyhedra and is connected by line segments.*

3 Global Weak Sharp Minimum Property

Adopting a terminology used in [14] for the weak Pareto solution set S_w, we say that the Pareto solution set S of (P) has a *global weak sharp minimum property*, iff there exists a constant $\gamma \in [0, +\infty[$ such that

$$d(x,S) \le \gamma\big[d(f(x), f(S)) + d(x,D)\big], \quad \forall x \in X, \tag{8}$$

where $d(u,\Omega) := \inf\{\|u-\omega\| : \omega \in \Omega\}$ is the distance from u to Ω.

Introducing several suitable modifications to the proof scheme of Zheng and Yang [14], we are able to establish the following result.

Theorem 3.1 *If f is a C-function on X, then the Pareto solution set S of* (P) *has a global weak sharp minimum property.*

Next three lemmas, which were employed in [14], are useful for the proof of this result.

Lemma 3.1 (See [17, Theorem 2.49]) *If $\varphi : X \to \mathbb{R}$ is a convex and piecewise linear function, then there exist $a_1^*, \dots, a_\ell^* \in X^*$ and $\alpha_1, \dots, \alpha_\ell \in \mathbb{R}$, such that*

$$\varphi(x) = \max_{1\le j\le \ell}\big[\langle a_j^*, x\rangle + \alpha_j\big], \quad \forall x \in X.$$

Lemma 3.2 (See [14, Lemma 3.2]) *Let P and Q be polyhedra in X. If $P \cap Q \neq \emptyset$, then there exists a constant $\tau \in [0, +\infty[$ such that*

$$d(x, P \cap Q) \leq \tau\big[d(x, P) + d(x, Q)\big], \quad \forall x \in X.$$

Lemma 3.3 *Let $a_1^*, \ldots, a_\ell^* \in X^*$, $\alpha_1, \ldots, \alpha_\ell \in \mathbb{R}$, and*

$$P := \{x \in X : \langle a_j^*, x\rangle + \alpha_j \leq 0, j = 1, \ldots, \ell\}.$$

If $P \neq \emptyset$, then there exists $\gamma \in [0, +\infty[$ such that

$$d(x, P) \leq \gamma \max_{1\leq j\leq \ell} \big[\langle a_j^*, x\rangle + \alpha_j\big]_+ \quad \forall x \in X,$$

where $\mu_+ := \max\{\mu, 0\}$ for every $\mu \in \mathbb{R}$.

Lemma 3.3 states the well-known Hoffman error bound for systems of linear inequalities; see [18] and the references therein.

Proof of Theorem 3.1 From the assumptions it follows that $f(D) + C$ is a polyhedron (see the proof of Theorem 2.2). Therefore, by Theorem 3.3 in Chap. 4 of [7], there exists a finite number of vectors $y_1^*, \ldots, y_q^* \in \operatorname{ri} C^*$ (where $\operatorname{ri} C^*$ denotes the relative interior of the positive dual cone C^* of C), such that

$$f(S) = E(f(D) + C|C) = \bigcup_{r=1}^{q} Q_r, \tag{9}$$

where each

$$Q_r := \Big\{v \in f(D) + C : \langle y_r^*, v\rangle = \min_{y\in f(D)+C} \langle y_r^*, y\rangle\Big\} \tag{10}$$

is a face of $f(D) + C$. Without loss of generality, we can assume that $\|y_r^*\| = 1$ for all r. Given any $r \in \{1, \ldots, q\}$, we put $\delta_r = \min_{y\in f(D)+C}\langle y_r^*, y\rangle$ and let $\varphi_r(x) := \langle y_r^*, f(x)\rangle$ for all $x \in X$. Since f is a C-function on X and $y_r^* \in C^*$, φ_r is a convex function. Besides, φ is a piecewise linear function. Hence, by Lemma 3.1, there exist $a_{r,1}^*, \ldots, a_{r,\ell_r}^* \in X^*$ and $\alpha_{r,1}, \ldots, \alpha_{r,\ell_r} \in \mathbb{R}$, such that

$$\varphi_r(x) = \max_{1\leq j\leq \ell_r} \big[\langle a_{r,j}^*, x\rangle + \alpha_{r,j}\big] \quad \forall x \in X. \tag{11}$$

Let

$$\begin{aligned} X_r &:= \{x \in X : \varphi_r(x) \leq \delta_r\} = \{x \in X : \langle y_r^*, f(x)\rangle \leq \delta_r\} \\ &= \{x \in X : \langle a_{r,j}^*, x\rangle + \alpha_{r,j} \leq \delta_r, \quad j = 1, \ldots, \ell_r\}. \end{aligned}$$

According to Lemma 3.3, there exists $\gamma_r \geq 0$, such that

$$d(x, X_r) \leq \gamma_r \max_{1\leq j\leq \ell_r} \big[\langle a_{r,j}^*, x\rangle + \alpha_{r,j} - \delta_r\big]_+, \quad \forall x \in X.$$

Then, by (11), we have

$$d(x, X_r) \leq \gamma_r\big[\varphi_r(x) - \delta_r\big]_+, \quad \forall x \in X. \tag{12}$$

If $x \in X_r \cap D$, then $f(x) \in f(D) \subset f(D) + C$ and $\delta_r \geq \varphi_r(x) = \langle y_r^*, f(x)\rangle$. By the choice of y_r^*, this implies $\langle y_r^*, f(x)\rangle = \delta_r$; hence $f(x) \in Q_r$ by (10). From (9) it follows that $x \in S$. Therefore, $\bigcup_{r=1}^q (X_r \cap D) \subset S$.

For every $r \in \{1, \ldots, q\}$, it holds $X_r \cap D \neq \emptyset$. Indeed, taking any $v \in Q_r$, by (9) we can find an $u \in S$ with $f(u) = v$. According to (10),

$$\langle y_r^*, f(u)\rangle = \langle y_r^*, v\rangle = \min_{y \in f(D)+C} \langle y_r^*, y\rangle = \delta_r,$$

hence $u \in X_r$. As $u \in X_r \cap D$, it follows that $X_r \cap D \neq \emptyset$. By Lemma 3.2, there is $\tau_r \in [0, +\infty)$ such that

$$d(x, X_r \cap D) \leq \tau_r\big[d(x, X_r) + d(x, D)\big], \quad \forall x \in X. \tag{13}$$

We put

$$\gamma := \max_{1 \leq r \leq q} \tau_r(\gamma_r + 1).$$

Let there be given any $x \in X$. For an arbitrarily chosen $\varepsilon > 0$, we can find $v \in f(S)$ such that

$$\|f(x) - v\| \leq d(f(x), f(S)) + \varepsilon.$$

By (9), there exists $r \in \{1, \ldots, q\}$ such that $v \in Q_r$. Let $u \in S$ be such that $f(u) = v$. Clearly,

$$\delta_r = \min_{y \in f(D)+C} \langle y_r^*, y\rangle = \langle y_r^*, v\rangle = \langle y_r^*, f(u)\rangle = \varphi_r(u).$$

Therefore, since $X_r \cap D \subset S$, invoking (12) and (13) we get

$$\begin{aligned}
d(x, S) \leq d(x, X_r \cap D) &\leq \tau_r\big[d(x, X_r) + d(x, D)\big] \\
&\leq \tau_r\big\{\gamma_r\big[\varphi_r(x) - \delta_r\big]_+ + d(x, D)\big\} \\
&\leq \tau_r\big\{\gamma_r\big[(\varphi_r(x) - \delta_r) - (\varphi_r(u) - \delta_r)\big]_+ + d(x, D)\big\} \\
&\leq \tau_r\big\{\gamma_r\big[\langle y_r^*, f(x) - f(u)\rangle\big]_+ + d(x, D)\big\} \\
&\leq \tau_r\big\{\gamma_r\|y_r^*\|\|f(x) - f(u)\| + d(x, D)\big\} \\
&\leq \gamma\big[\|f(x) - v\| + d(x, D)\big],
\end{aligned}$$

which implies that

$$d(x, S) \leq \gamma\big[d(f(x), f(S)) + d(x, D) + \varepsilon\big].$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, this establishes (8). □

The assumption that f is a C-function in the above theorem cannot be dropped. To see this, as in Example 3.1 of [14], one can choose $X = Y = \mathbb{R}$, $C = \mathbb{R}_+$, $D = \mathbb{R}$, and

$$f(x) = \begin{cases} |x|, & \text{if } x \in]-1, 1[\\ 1, & \text{if } x \in]-\infty, -1] \cup [1, \infty[. \end{cases}$$

From Theorem 3.1, we obtain the following analogue of Theorem 2.2 in [13], where the case of the weak Pareto solution set of a linear multiobjective optimization problem in Euclidean spaces was considered.

Theorem 3.2 *If $f : X \to Y$ is a continuous linear operator, then the Pareto solution set S of* (P) *has a global weak sharp minimum property.*

This theorem describes a global weak sharp minimum property of the Pareto solution set of linear multiobjective optimization in normed spaces.

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