J Optim Theory Appl (2010) 145: 543–578
DOI 10.1007/s10957-010-9696-z

Maximum Principle for Partially-Observed Optimal Control of Fully-Coupled Forward-Backward Stochastic Systems

J.T. Shi · Z. Wu

Published online: 20 April 2010

Abstract This paper is concerned with partially-observed optimal control problems for fully-coupled forward-backward stochastic systems. The maximum principle is obtained on the assumption that the forward diffusion coefficient does not contain the control variable and the control domain is not necessarily convex. By a classical spike variational method and a filtering technique, the related adjoint processes are characterized as solutions to forward-backward stochastic differential equations in finite-dimensional spaces. Then, our theoretical result is applied to study a partially-observed linear-quadratic optimal control problem for a fully-coupled forward-backward stochastic system and an explicit observable control variable is given.

Keywords Fully-coupled forward-backward stochastic systems · Partially-observed optimal control · Maximum principle · Adjoint equations · Linear-quadratic control

1 Introduction and Problem Formulation

Throughout this paper, $\mathfrak{R}^n$ denotes the n-dimensional Euclidean space. $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the scalar product and norm in the Euclidean space, respectively. $\top$ appearing

Communicated by F. Zirilli.

This work was supported by the National Basic Research Program of China (973 Program, Grant. 2007CB814904), the National Natural Science Foundations of China (Grants. 10921101, 10701050) and the Natural Science Foundation of Shandong Province (Grants. JQ200801 and 2008BS01024).

J.T. Shi · Z. Wu (✉)
School of Mathematics, Shandong University, Jinan 250100, People's Republic of China
e-mail: wuzhen@sdu.edu.cn

J.T. Shi
e-mail: shijingtao@sdu.edu.cn

in the superscripts denotes the transpose of a matrix. C always denotes some positive constant.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a complete filtered probability space equipped with a natural filtration

$$\mathcal{F}_t := \sigma\{W(s), Y(s); 0 \le s \le t\},$$

where $W(\cdot)$ and $Y(\cdot)$ are two independent standard Brownian motions valued in $\Re^d$ and $\Re^r$, respectively. Let $T > 0$ be a fixed time horizon and let $\mathcal{F} := \mathcal{F}_T$. $\mathbb{E}$ denotes the expectation on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Define

$$\mathcal{F}_t^W := \sigma\{W(s); 0 \le s \le t\}, \qquad \mathcal{F}_t^Y := \sigma\{Y(s); 0 \le s \le t\}.$$

$L^2(\Omega, \mathcal{F}_T^W; \Re^n)$ denotes the space of all $\Re^n$-valued $\mathcal{F}_T^W$-measurable random variables ξ such that $\mathbb{E}[|\xi|^2] < \infty$, $L^2_{\mathcal{F}^W}([0,T]; \Re^n)$ denotes the space of all $\Re^n$-valued $\mathcal{F}_t^W$-adapted processes ψ_t such that $\mathbb{E}[\int_0^T |\psi_t|^2 dt] < \infty$.

Let U be a nonempty subset of $\Re^k$. A control variable $v : [0,T] \times \Omega \to U$ is called admissible, if it is $\mathcal{F}_t^Y$-adapted and satisfies $\sup_{0 \le t \le T} \mathbb{E}|v(t)|^i < \infty, i = 1, 2, \cdots$. The set of the admissible control variables is denoted by $\mathcal{U}_{ad}$.

For given $v(\cdot) \in \mathcal{U}_{ad}$, consider the following fully-coupled forward-backward stochastic control system:

$$\begin{aligned} dx^v(t) &= b(t, x^v(t), y^v(t), z^v(t), v(t))dt + \sigma(t, x^v(t), y^v(t), z^v(t))dW(t), \\ -dy^v(t) &= f(t, x^v(t), y^v(t), z^v(t), v(t))dt - z^v(t)dW(t), \\ x^v(0) &= x_0, \quad y^v(T) = g(x^v(T)), \end{aligned} \tag{1}$$

here the state processes $(x^v(t), y^v(t), z^v(t)) \equiv (x^v(t,\omega), y^v(t,\omega), z^v(t,\omega)) \in \Re^n \times \Re^m \times \Re^{m\times d}, 0 \le t \le T, \omega \in \Omega$. $x_0 \in \Re^n$ is deterministic and

$$\begin{aligned} &b : [0,T] \times \Re^n \times \Re^m \times \Re^{m\times d} \times U \to \Re^n, \\ &\sigma : [0,T] \times \Re^n \times \Re^m \times \Re^{m\times d} \to \Re^{n\times d}, \\ &f : [0,T] \times \Re^n \times \Re^m \times \Re^{m\times d} \times U \to \Re^m, \quad g : \Re^n \to \Re^m. \end{aligned}$$

We assume that the state processes $(x^v(\cdot), y^v(\cdot), z^v(\cdot))$ cannot be observed directly, but the controllers can observe a related noisy process $Y(\cdot)$ of the state processes which is described by

$$dY(t) = h(t, x^v(t), y^v(t), v(t))dt + d\tilde{W}(t), \quad Y(0) = 0, \tag{2}$$

here $\tilde{W}(\cdot)$ is a stochastic process depending on the control variable $v(\cdot)$ and

$$h : [0,T] \times \Re^n \times \Re^m \times U \to \Re^r.$$

We are given an $m \times n$ full-rank matrix G. For any $v(\cdot) \in \mathcal{U}_{ad}$, we use the notations

$$\Gamma^v := \begin{pmatrix} x^v \\ y^v \\ z^v \end{pmatrix}, \qquad \mathcal{A}(t, \Gamma^v, v) := \begin{pmatrix} -G^\top f \\ Gb \\ G\sigma \end{pmatrix}(t, \Gamma^v, v),$$

here $G\sigma \equiv (G\sigma_1, G\sigma_2, \ldots, G\sigma_d)$. We assume that the following hypothesis holds.

(H1)

(i) $\forall \Gamma^v, v(\cdot) \in \mathcal{U}_{ad}, \mathcal{A}(\cdot, \Gamma^v, v) \in L^2_{\mathcal{F}^W}([0,T]; \Re^n \times \Re^m \times \Re^{m\times d})$;
(ii) b, f are continuously differentiable in Γ^v, and their partial derivatives are uniformly bounded; they are uniformly Lipschitz in v and are bounded by $C(1+|x|+|y|+|z|+|v|)$;
(iii) h is continuously differentiable in (x,y) and continuous in v, its derivatives and h are all uniformly bounded;
(iv) σ is continuously differentiable in Γ^v, its partial derivatives are uniformly bounded, and σ is bounded by $C(1+|x|+|y|+|z|)$;
(v) g is continuously differentiable in x, g_x is uniformly bounded and g is bounded by $C(1+|x|)$;
(vi) For each $x \in \Re^n$, $g(x) \in L^2(\Omega, \mathcal{F}_T^W; \Re^m)$.

For any $v(\cdot) \in \mathcal{U}_{ad}$, in order to ensure the existence and uniqueness of the solution of the above fully-coupled FBSDE (1), we introduce the following G-monotonic conditions which were used by Peng and Wu [1]:

$$\text{(H2)}\quad \begin{aligned} \langle \mathcal{A}(t,\Gamma^v,v) - \mathcal{A}(t,\bar{\Gamma}^v,v), \Gamma^v - \bar{\Gamma}^v\rangle &\le -\beta_1|G\hat{x}^v|^2 \\ &\quad - \beta_2(|G^\top \hat{y}^v|^2 + |G^\top \hat{z}^v|^2), \\ \langle g(x^v) - g(\bar{x}^v), G(x^v - \bar{x}^v)\rangle &\ge \mu_1|G\hat{x}^v|^2, \end{aligned}$$

or

$$\text{(H2)}'\quad \begin{aligned} &\langle \mathcal{A}(t,\Gamma^v,v) - \mathcal{A}(t,\bar{\Gamma}^v,v), \Gamma^v - \bar{\Gamma}^v\rangle \ge \beta_1|G\hat{x}^v|^2 + \beta_2(|G^\top \hat{y}^v|^2 + |G^\top \hat{z}|^2), \\ &\langle g(x^v) - g(\bar{x}^v), G(x^v - \bar{x}^v)\rangle \le -\mu_1|G\hat{x}^v|^2, \end{aligned}$$

$$\begin{aligned} &\forall \Gamma^v = (x^v, y^v, z^v), \quad \bar{\Gamma}^v = (\bar{x}^v, \bar{y}^v, \bar{z}^v), \quad \hat{x}^v = x^v - \bar{x}^v, \\ &\hat{y}^v = y^v - \bar{y}^v, \quad \hat{z}^v = z^v - \bar{z}^v, \end{aligned}$$

where β_1, β_2 and μ_1 are given nonnegative constants with $\beta_1 + \beta_2 > 0$, $\beta_2 + \mu_1 > 0$. Moreover we have $\beta_1 > 0$, $\mu_1 > 0$ (resp., $\beta_2 > 0$), when $m > n$ (resp., $m < n$).

For any $v(\cdot) \in \mathcal{U}_{ad}$, we suppose that (H1) and (H2) hold. Then we know FBSDE (1) has a unique solution $\Gamma^v(\cdot) \equiv (x^v(\cdot), y^v(\cdot), z^v(\cdot))$ by Theorem 2.6 of Peng and Wu [1], which is called the corresponding trajectory. Define $d\mathbb{P}^v := Z^v(t)d\mathbb{P}$, where

$$\begin{aligned} Z^v(t) := \exp\bigg\{ &\int_0^t \langle h(s, x^v(s), y^v(s), v(s)), dY(s)\rangle \\ &- \frac{1}{2}\int_0^t |h(s, x^v(s), y^v(s), v(s))|^2 ds \bigg\}. \end{aligned}$$

Obviously, $Z^v(\cdot)$ is the unique $\mathcal{F}_t^Y$-adapted solution of

$$dZ^v(t) = Z^v(t)\langle h(s, x^v(s), y^v(s), v(s)), dY(t)\rangle, \quad Z^v(0) = 1. \tag{3}$$

By virtue of Itô's formula, we can prove that $\sup_{0\le t\le T}\mathbb{E}|Z^v(t)|^i<\infty, i=1,2,\cdots$. Hence by Girsanov's theorem and (H1), $\mathbb{P}^v$ is a new probability measure and $(W(\cdot),\tilde W(\cdot))$ is an $\mathfrak{R}^{d+r}$-valued standard Brownian motion defined on the new filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P}^v)$.

We introduce the following cost functional:

$$J(v(\cdot)):=\mathbb{E}^v\left[\int_0^T l(t,x^v(t),y^v(t),v(t))dt+\Phi(x^v(T))+\gamma(y^v(0))\right], \tag{4}$$

here $\mathbb{E}^v$ denotes expectation on $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P}^v)$ and

$$l:[0,T]\times\mathfrak{R}^n\times\mathfrak{R}^m\times U\to\mathfrak{R},\qquad \Phi:\mathfrak{R}^n\to\mathfrak{R},\qquad \gamma:\mathfrak{R}^n\to\mathfrak{R}.$$

We need the following hypothesis.

(H3)

(i) l is continuous in v, continuously differentiable in (x,y), its partial derivatives are continuous in (x,y,v) and bounded by $C(1+|x|+|y|+|v|)$;
(ii) Φ is continuously differentiable and Φ_x is bounded by $C(1+|x|)$;
(iii) γ is continuously differentiable and γ_y is bounded by $C(1+|y|)$.

Our partially-observed optimal control problem is to minimize the cost functional (4) over $v(\cdot)\in\mathcal{U}_{ad}$ subject to (1) and (2), i.e., to find $u(\cdot)\in\mathcal{U}_{ad}$ satisfying

$$J(u(\cdot))=\inf_{v(\cdot)\in\mathcal{U}_{ad}}J(v(\cdot)). \tag{5}$$

Obviously, cost functional (4) can be rewritten as the following

$$J(v(\cdot))=\mathbb{E}\left[\int_0^T Z^v(t)l(t,x^v(t),y^v(t),v(t))dt+Z^v(T)\Phi(x^v(T))+\gamma(y^v(0))\right]. \tag{6}$$

Then the original problem (5) is equivalent to minimize (6) over $v(\cdot)\in\mathcal{U}_{ad}$ subject to (1) and (3). Our main target is to find the necessary condition of the partially-observed optimal control $u(\cdot)$ in the form of Pontryagin stochastic maximum principle.

The subject of stochastic maximum principles for partially-observed forward and forward-backward stochastic optimal control problems has been discussed by many authors, such as Bensoussan [2], Haussmann [3], Baras et al. [4], Li and Tang [5], Tang [6], Wu [7], Wang and Wu [8], etc. Li and Tang [5] obtained the maximum principle for the partially-observed forward stochastic optimal control problem in which the control variable is allowed to enter the diffusion and the observation coefficients. Moreover, the control domain is not necessarily convex. Tang [6] extended this result to the case with correlated noises between the system and the observation. A general maximum principle is proved for the partially-observed optimal control and the relations among the adjoint processes are established. Adjoint vector fields are introduced as the solutions to some backward stochastic partial differential equations (BSPDEs for short) and their relations are established. Under suitable conditions, the adjoint processes are characterized in terms of the adjoint vector fields, their differentials and Hessians, along the optimal state process. Wu [7] studied the optimal control problem for partially-observed forward-backward stochastic system when the control domain is convex. He obtained the maximum principle by the convex variational method. Wang and Wu [8] also researched this problem, on the assumption that the control domain is not necessarily convex while the forward diffusion coefficient does not contain the control variable. The maximum principle was obtained by means of the spike variational technique.

However, the forward-backward stochastic control systems in [7, 8] are "not fully-coupled", which means that the coefficients in the forward equations do not contain the backward state processes. For example, Wang and Wu [8] considered the following forward-backward stochastic control system

$$\begin{aligned} dx^v(t) &= b(t, x^v(t), v(t))dt + \sigma(t, x^v(t))dW(t), \\ -dy^v(t) &= f(t, x^v(t), y^v(t), z^v(t), v(t))dt - z^v(t)dW(t), \quad t \in [0, T], \\ x^v(0) &= x_0, \quad y^v(T) = g(x^v(T)), \end{aligned} \tag{7}$$

with observation

$$dY(t) = h(t, x^v(t), v(t))dt + d\tilde{W}(t), \quad Y(0) = 0. \tag{8}$$

Such kind of systems can be known as a special case of (1), (2). Recently, more researching attention has been attracted by optimal control problems for fully-coupled forward-backward stochastic systems. One reason is that the theory in itself is interesting and challenging. Another is that these kinds of systems are usually encountered when we study some financial optimization problems for some "large investors" (see Cvitanic and Ma [9], Ma and Yong [10]). In this case, state processes are described as fully-coupled forward-backward stochastic differential equations (FBSDEs for short). There have been many results on the solvability of fully-coupled FBSDEs. Making use of the fixed point theorem, Antonelli [11] obtained the first result of the solvability of an FBSDE on a "small" time duration. Among others, as far as we know, there are three main methods to investigate the solvability of an FBSDE on an arbitrarily prescribed time duration. The first one is the "four step scheme" by Ma et al. [12] which can be regarded as combination of methods of PDE and probability. They provided the explicit relations among the forward and backward components of the adapted solution via a quasilinear PDE, while they needed the forward equation to be nondegenerate and required that the coefficients can not be randomly disturbed. The second one is the "method of continuation" by Hu and Peng [13], Peng and Wu [1], Yong [14] and Pardoux and Tang [15]. They relaxed the above conditions, but required the "monotonicity" condition on the coefficients. The third one is motivated by the study of numerical methods for some linear FBSDEs (see Delarue [16] and Zhang [17]). Delarue [16] relied on PDE arguments, so its coefficients have to be deterministic while Zhang [17] imposed some assumptions on the derivatives of the coefficients instead of the monotonicity condition. For systemic theory of FBSDEs, we refer to the book of Ma and Yong [10] and the references therein.

In this paper, we study the partially-observed optimal control problem for fully-coupled forward-backward stochastic systems. We assume that the forward diffusion coefficient does not contain the control variable and the control domain is not necessarily convex. As far as we know, the general maximum principle for forward-backward stochastic control systems is still an open problem even if the system is completely observed.

One difficulty we meet to get the maximum principle for our problem is the fully-coupling. The core of our approach is that how to use the classical spike variational method to get the estimations of the variational processes with high enough ε-order. We use the FBSDE method of Peng and Wu [1] and the iteration technique of Xu [18] to overcome the difficulty of fully-coupling. We also employ the technique of FBSDE to obtain the estimations of the difference between the perturbed state processes and the sum of the optimal state processes and the solution of the variational equations with high enough ε-order. This approach derives from Shi and Wu [19].

Another difficulty we encounter is the partial observation. We use a pure probabilistic approach of Li and Tang [5]. Firstly, by Girsanov's theorem we reformulate our original partially-observed optimal control problem (1), (2), (4) to (1), (3), (6), which is quite similar to a completely observed one. The only difference lies in the admissible control set. One advantage of this approach is that it needs neither the Zakaï equation as adopted in Haussmann [3] nor the theory of stochastic flows as adopted in Baras et al. [4]. So we get around the complicated stochastic calculus in infinite-dimensional spaces. Secondly, we obtain some estimations of the variational observed process and the difference between the perturbed observed process and the sum of the optimal and variational observed processes. Then we give the variational inequality. Finally, by the classical duality technique we prove the maximum principle.

To illustrate our theoretical result, we give an example for a partially-observed linear-quadratic (LQ for short) fully-coupled forward-backward stochastic optimal control problem. Combining the classical linear filtering theory (see Liptser and Shiryayev [20]) with the technique of solving linear FBSDEs (see Ma and Yong [10]), we find an explicit observable control variable satisfying the necessary condition of optimality. Moreover, we can prove that the observable control variable is really optimal. In addition, we obtain the filtering estimates of the optimal trajectories which are given by the solution of some forward-backward ordinary differential equation with double dimensions (DFBODE for short) and Riccati equations.

We emphasize that our problem should be distinguished from the optimal control problems with partial information, where a subfiltration is given to represent the information available to the controller instead of an observation process. For problems with partial information, there is a rich literature. See Baghery and Øksendal [21], Meyer-Brandis et al. [22], Øksendal and Sulem [23], Meng [24], etc.

The rest of this paper is organized as follows. In Sect. 2, we obtain the partially-observed stochastic maximum principle. In Sect. 3, we give an LQ example to show the applications of our theoretical result.

2 Partially-Observed Stochastic Maximum Principle

In this section, combining the classical spike variational method with the filtering technique we obtain the maximum principle for the aforementioned partially-observed optimal control problem. First of all, we give some prior estimations and the variational inequality. Then adjoint equations and Hamiltonian function are introduced and the maximum principle is obtained.

2.1 Spike Variation and Prior Estimations

Let $u(\cdot)\in\mathcal{U}_{ad}$ be optimal, $\Gamma(\cdot)\equiv(x(\cdot),y(\cdot),z(\cdot))$ be the corresponding optimal trajectory of (1) and $Z(\cdot)$ be the solution of (3). We introduce the spike variation

$$u^{\varepsilon}(t):=\begin{cases} v, & \text{if } \tau\le t\le\tau+\varepsilon,\\ u(t), & \text{otherwise,}\end{cases}$$

where $0\le\tau<T$ is fixed, $\varepsilon>0$ is sufficiently small, and $v\in U$ is an arbitrary $\mathcal{F}_{\tau}^{Y}$-measurable random variable such that $\sup_{\omega\in\Omega}|v(\omega)|<+\infty$. Obviously, $u^{\varepsilon}(\cdot)$ is admissible. Let $\Gamma^{\varepsilon}(\cdot)\equiv(x^{\varepsilon}(\cdot),y^{\varepsilon}(\cdot),z^{\varepsilon}(\cdot))$ be the perturbed trajectory of the control system (1) and $Z^{\varepsilon}(\cdot)$ be the solution of (3) corresponding to $u^{\varepsilon}(\cdot)$.

For simplification, we introduce the notations

$$\theta(u^{\varepsilon}(t)):=\theta(t,x(t),y(t),z(t),u^{\varepsilon}(t)),\qquad \theta(u(t)):=\theta(t,x(t),y(t),z(t),u(t)),$$

where $\theta=b,\sigma,f,h,l$ as well as their partial derivatives in the optimal trajectory (x,y,z).

Now we introduce the following variational equations which is a linear FBSDE:

$$\begin{aligned} dx^1(t)&=[b_x(u(t))x^1(t)+b_y(u(t))y^1(t)+b_z(u(t))z^1(t)+b(u^{\varepsilon}(t))-b(u(t))]dt\\ &\qquad+[\sigma_x(t)x^1(t)+\sigma_y(t)y^1(t)+\sigma_z(t)z^1(t)]dW(t),\\ -dy^1(t)&=[f_x(u(t))x^1(t)+f_y(u(t))y^1(t)+f_z(u(t))z^1(t)+f(u^{\varepsilon}(t))\\ &\qquad-f(u(t))]dt-z^1(t)dW(t),\\ x^1(0)&=0,\quad y^1(T)=g_x(x(T))x^1(T), \end{aligned}\tag{9}$$

and a linear SDE:

$$\begin{aligned} dZ^1(t)&=\langle Z^1(t)h(u(t))+Z(t)h_x(u(t))x^1(t)+Z(t)h_y(u(t))y^1(t)\\ &\qquad+Z(t)(h(u^{\varepsilon}(t))-h(u(t))),dY(t)\rangle,\\ Z^1(0)&=0. \end{aligned}\tag{10}$$

By (H1) and (H2), it is easy to know that (9) and (10) admit unique adapted solutions $(x^1(\cdot),y^1(\cdot),z^1(\cdot))$ and $Z^1(\cdot)$, respectively.

The following lemmas in this subsection are all preparations to derive the variational inequality in next subsection. At first, we have the following elementary lemma.

Lemma 2.1 *Given bounded $\Re^{n\times n}$-valued $\mathcal{F}_t^W$-adapted processes $a_1(\cdot),b_{11}(\cdot),\cdots,b_{1d}(\cdot)$ and processes $a_0(\cdot),b_{01}(\cdot),\cdots,b_{0d}(\cdot)\in L^2_{\mathcal{F}^W}([0,T];\Re^n)$. Then for the fol-*

lowing linear SDE

$$\tilde{x}(t)=\int_0^t [a_1(s)\tilde{x}(s)+a_0(s)]ds+\sum_{j=1}^d \int_0^t [b_{1j}(s)\tilde{x}(s)+b_{0j}(s)]dW_j(s), \tag{11}$$

there exists a constant $K_1>0$, *such that the unique solution* $\tilde{x}(\cdot)\in L^2_{\mathcal{F}^W}([0,T];\Re^n)$ *satisfies*

$$\mathbb{E}\Big[\sup_{0\le t\le T}|\tilde{x}(t)|^2\Big]\le K_1\Big\{\mathbb{E}\int_0^T |a_0(s)|^2ds+\mathbb{E}\int_0^T |b_0(s)|^2ds\Big\}, \tag{12}$$

where $b_0(\cdot)\equiv(b_{01}(\cdot),\cdots,b_{0d}(\cdot))$.

Given random variable $\xi\in L^2(\Omega,\mathcal{F}_T^W;\Re^m)$, *bounded* $\Re^{m\times m}$*-valued* $\mathcal{F}_t^W$*-adapted processes* $\alpha_0(\cdot),\alpha_{11}(\cdot),\cdots,\alpha_{1d}(\cdot)$ *and process* $\alpha_2(\cdot)\in L^2_{\mathcal{F}^W}([0,T];\Re^m)$. *Then for the following linear BSDE*

$$\tilde{y}(t)=\xi+\int_t^T \Big[\alpha_0(s)\tilde{y}(s)+\sum_{j=1}^d \alpha_{1j}(s)\tilde{z}_j(s)+\alpha_2(s)\Big]ds-\int_t^T \tilde{z}(s)dW(s), \tag{13}$$

there exists a constant $K_2>0$, *such that the unique solution* $(\tilde{y}(\cdot),\tilde{z}(\cdot))\in L^2_{\mathcal{F}^W}([0,T];\Re^m)\times L^2_{\mathcal{F}^W}([0,T];\Re^{m\times d})$ *satisfies*

$$\mathbb{E}\Big[\sup_{0\le t\le T}|\tilde{y}(t)|^2\Big]\le K_2\Big\{\mathbb{E}\int_0^T |\tilde{z}(s)|^2ds+\mathbb{E}\int_0^T |\alpha_2(s)|^2ds+\mathbb{E}|\xi|^2\Big\}, \tag{14}$$

where $\tilde{z}(\cdot)\equiv(\tilde{z}_1(\cdot),\cdots,\tilde{z}_d(\cdot))$.

Proof By (11) we can get

$$\begin{aligned}\sup_{0\le t\le T}|\tilde{x}(t)|^2&\le C\Big(\int_0^T |\tilde{x}(s)|^2ds+\int_0^T |a_0(s)|^2ds\Big)\\&\quad+C\sup_{0\le t\le T}\Big\{\sum_{j=1}^d\int_0^t [b_{1j}(s)\tilde{x}(s)+b_{0j}(s)]dW_j(s)\Big\}^2.\end{aligned}$$

Taking expectation on both sides and by the Davis-Burkholder-Gundy inequality, we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le t\le T}|\tilde{x}(t)|^2\Big]&\le C\bigg\{\int_0^T\mathbb{E}|\tilde{x}(s)|^2ds+\mathbb{E}\int_0^T|a_0(s)|^2ds\\&\quad+\sum_{j=1}^d\mathbb{E}\int_0^T|b_{1j}(s)\tilde{x}(s)+b_{0j}(s)|^2ds\bigg\}\\&\le C\int_0^T\mathbb{E}\Big[\sup_{0\le s\le t}|\tilde{x}(s)|^2\Big]dt\\&\quad+C\bigg\{\mathbb{E}\int_0^T|a_0(s)|^2ds+\sum_{j=1}^d\mathbb{E}\int_0^T|b_{0j}(s)|^2ds\bigg\}.\end{aligned}$$

By the Gronwall inequality, we obtain (12). And, by (13), we can get

$$\begin{aligned}\sup_{0\le t\le T}|\tilde{y}(t)|^2&\le C|\xi|^2+C\bigg\{\int_0^T(|\tilde{y}(t)|^2+\sum_{j=1}^d|\tilde{z}_j(t)|^2+|\alpha_2(t)|^2)dt\bigg\}\\&\quad+C\Big(\int_0^T\tilde{z}(s)dW(s)\Big)^2+C\sup_{0\le t\le T}\Big(\int_0^t\tilde{z}(s)dW(s)\Big)^2.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le t\le T}|\tilde{y}(t)|^2\Big]&\le C\mathbb{E}|\xi|^2+C\bigg\{\int_0^T\mathbb{E}|\tilde{y}(t)|^2dt+\sum_{j=1}^d\mathbb{E}\int_0^T\big(|\tilde{z}_j(t)|^2+|\alpha_2(t)|^2\big)dt\bigg\}\\&\le C\mathbb{E}|\xi|^2+C\bigg\{\int_0^T\mathbb{E}\Big[\sup_{0\le s\le t}|\tilde{y}(s)|^2\Big]dt+\mathbb{E}\int_0^T|\tilde{z}(s)|^2ds\\&\quad+\mathbb{E}\int_0^T|\alpha_2(s)|^2ds\bigg\}.\end{aligned}$$

By the Gronwall inequality, we obtain (14). The proof is complete. □

We need the following lemma.

Lemma 2.2 *Let* (H1) *and* (H2) *hold. Then*, *we have the following estimations*:

$$\sup_{0\le t\le T}\mathbb{E}|x^1(t)|^2\le C\varepsilon, \tag{15}$$

$$\sup_{0\le t\le T}\mathbb{E}|y^1(t)|^2\le C\varepsilon, \tag{16}$$

$$\mathbb{E}\int_0^T|z^1(t)|^2dt\le C\varepsilon. \tag{17}$$

Proof Applying Itô's Formula to $\langle Gx^1(\cdot), y^1(\cdot)\rangle$, we can get

$$\begin{aligned}
&\mathbb{E}\langle g_x(x(T))x^1(T), Gx^1(T)\rangle \\
&\quad = \mathbb{E}\int_0^T \big[-\langle f_x(u(t))x^1(t) + f_y(u(t))y^1(t) + f_z(u(t))z^1(t), Gx^1(t)\rangle \\
&\qquad + \langle b_x(u(t))x^1(t) + b_y(u(t))y^1(t) + b_z(u(t))z^1(t), G^\top y^1(t)\rangle \\
&\qquad + \langle \sigma_x(t)x^1(t) + \sigma_y(t)y^1(t) + \sigma_z(t)z^1(t), G^\top z^1(t)\rangle\big]dt \\
&\qquad - \mathbb{E}\int_0^T \langle f(u^\varepsilon(t)) - f(u(t)), Gx^1(t)\rangle dt \\
&\qquad + \mathbb{E}\int_0^T \langle b(u^\varepsilon(t)) - b(u(t)), G^\top y^1(t)\rangle dt.
\end{aligned}$$

Then by the G-monotonic condition (H2), we can obtain

$$\begin{aligned}
&\mu_1\mathbb{E}|Gx^1(T)|^2 + \beta_1\mathbb{E}\int_0^T |Gx^1(t)|^2 dt + \beta_2\mathbb{E}\int_0^T (|G^\top y^1(t)|^2 + |G^\top z^1(t)|^2)dt \\
&\quad \le \mathbb{E}\int_0^T \langle b(u^\varepsilon(t)) - b(u(t)), G^\top y^1(t)\rangle dt \\
&\qquad - \mathbb{E}\int_0^T \langle f(u^\varepsilon(t)) - f(u(t)), Gx^1(t)\rangle dt. \qquad (18)
\end{aligned}$$

Case 1. When $m > n$, we assume $\beta_1 > 0$, $\beta_2 \ge 0$, $\mu_1 > 0$. From (18) we have

$$\begin{aligned}
&\mu_1\mathbb{E}|Gx^1(T)|^2 + \beta_1\mathbb{E}\int_0^T |Gx^1(t)|^2 dt \\
&\quad \le \mathbb{E}\int_0^T \langle b(u^\varepsilon(t)) - b(u(t)), G^\top y^1(t)\rangle dt \\
&\qquad - \mathbb{E}\int_0^T \langle f(u^\varepsilon(t)) - f(u(t)), Gx^1(t)\rangle dt \\
&\quad \le \mathbb{E}\int_0^T |G^\top y^1(t)|^2 dt + \frac{1}{4}\mathbb{E}\int_0^T |b(u^\varepsilon(t) - b(u(t))|^2 dt \\
&\qquad + \frac{\beta_1}{2}\mathbb{E}\int_0^T |Gx^1(t)|^2 dt + \frac{1}{2\beta_1}\mathbb{E}\int_0^T |f(u^\varepsilon(t)) - f(u(t)|^2 dt,
\end{aligned}$$

i.e.,

$$\mu_1\mathbb{E}|Gx^1(T)|^2 + \frac{\beta_1}{2}\mathbb{E}\int_0^T |Gx^1(t)|^2 dt \le \mathbb{E}\int_0^T |G^\top y^1(t)|^2 dt + C\varepsilon, \qquad (19)$$

where C is a constant depending on Lipschitz constants. From the second equation of variational equation (9), we get

$$\begin{aligned}
&\mathbb{E}|y^1(t)|^2 + \mathbb{E}\int_t^T |z^1(s)|^2 ds \\
&= \mathbb{E}\bigg\{h_x(x(T))x^1(T) + \int_t^T [f_x(u(s))x^1(s) + f_y(u(s))y^1(s) \\
&\quad + f_z(u(s))z^1(s) + f(u^\varepsilon(s)) - f(u(s))]ds\bigg\}^2 \\
&\le 5C_1\bigg(\mathbb{E}|x^1(T)|^2 + T\mathbb{E}\int_t^T |x^1(s)|^2 ds + T\mathbb{E}\int_t^T |y^1(s)|^2 ds \\
&\quad + (T-t)\mathbb{E}\int_t^T |z^1(s)|^2 ds\bigg) + 5\mathbb{E}\bigg(\int_t^T (f(u^\varepsilon(s)) - f(u(s))ds\bigg)^2,
\end{aligned}$$

where C_1 is a constant. So for $t \in [T-\delta, T]$, $\delta = \frac{1}{10C_1}$, we have

$$\begin{aligned}
&\mathbb{E}|y^1(t)|^2 + \frac{1}{2}\mathbb{E}\int_t^T |z^1(s)|^2 ds \\
&\quad \le 5C_1\bigg(\mathbb{E}|x^1(T)|^2 + T\mathbb{E}\int_t^T |x^1(s)|^2 ds + T\mathbb{E}\int_t^T |y^1(s)|^2 ds\bigg) \\
&\qquad + 5\mathbb{E}\bigg(\int_t^T (f(u^\varepsilon(s)) - f(u(s))ds\bigg)^2.
\end{aligned}$$

From (19), we get

$$\mathbb{E}|y^1(t)|^2 + \frac{1}{2}\mathbb{E}\int_t^T |z^1(s)|^2 ds \le C_2\int_t^T \mathbb{E}|y^1(s)|^2 ds + C\varepsilon,$$

where C_2 is a constant depending on C_1, μ_1, β_1 and T. By the Gronwall inequality, we have

$$\mathbb{E}|y^1(t)|^2 \le C\varepsilon, \qquad \mathbb{E}\int_t^T |z^1(s)|^2 ds \le C\varepsilon, \quad t \in [T-\delta, T].$$

Repeating this process, the above estimates hold for $t \in [T-2\delta, T-\delta]$. Obviously, after a finite number of iterations, we obtain

$$\mathbb{E}|y^1(t)|^2 \le C\varepsilon, \qquad \mathbb{E}\int_t^T |z^1(s)|^2 ds \le C\varepsilon, \quad \forall t \in [0, T].$$

Then estimations (16) and (17) are obtained. By (9), (16), and (17), we get

$$\begin{aligned}
\mathbb{E}|x^1(t)|^2 &= \mathbb{E}\bigg\{\int_0^t [b_x(u(s))x^1(s) + b_y(u(s))y^1(s) \\
&\qquad + b_z(u(s))z^1(s) + b(u^\varepsilon(s)) - b(u(s))]ds \\
&\qquad + \int_0^t [\sigma_x(s)x^1(s) + \sigma_y(s)y^1(s) + \sigma_z(s)z^1(s)]dW(s)\bigg\}^2 \\
&\le 7C_3\bigg(\int_0^t \mathbb{E}|x^1(s)|^2 ds + \sup_{0\le t\le T}\mathbb{E}|y^1(t)|^2 + \mathbb{E}\int_0^t |z^1(s)|^2 ds\bigg) \\
&\qquad + 7\mathbb{E}\bigg(\int_0^t (b(u^\varepsilon(s)) - b(u(s))ds\bigg)^2 \\
&\le 7C_3\int_0^t \mathbb{E}|x^1(s)|^2 ds + C\varepsilon + C\varepsilon^2,
\end{aligned}$$

where C_3 is a constant. By the Gronwall inequality, we obtain (15).

Case 2. When $m < n$, we assume $\beta_1 \ge 0$, $\beta_2 > 0$, $\mu_1 \ge 0$. From (18) we have

$$\begin{aligned}
&\beta_2\mathbb{E}\int_0^T (|G^\top y^1(t)|^2 + |G^\top z^1(t)|^2)dt \\
&\quad \le \mathbb{E}\int_0^T \langle b(u^\varepsilon) - b(u), G^\top y^1(t)\rangle dt - \mathbb{E}\int_0^T \langle f(u^\varepsilon) - f(u), Gx^1(t)\rangle dt \\
&\quad \le \frac{\beta_2}{2}\mathbb{E}\int_0^T |G^\top y^1(t)|^2 dt + \frac{1}{2\beta_2}\mathbb{E}\int_0^T |b(u^\varepsilon(t) - b(u(t))|^2 dtr \\
&\qquad + \mathbb{E}\int_0^T |Gx^1(t)|^2 dt + \frac{1}{4}\mathbb{E}\int_0^T |f(u^\varepsilon(t)) - f(u(t)|^2 dt,
\end{aligned}$$

i.e.,

$$\frac{\beta_2}{2}\mathbb{E}\int_0^T |G^\top y^1(t)|^2 dt + \beta_2\mathbb{E}\int_0^T |G^\top z^1(t)|^2 dt \le \mathbb{E}\int_0^T |Gx^1(t)|^2 dt + C\varepsilon, \quad (20)$$

where C is a constant depending on Lipschitz constants, β_2. By (9) and (20), we get

$$\begin{aligned}
\mathbb{E}|x^1(t)|^2 &\le 7C_3\bigg(\mathbb{E}\int_0^t |x^1(s)|^2 ds + \mathbb{E}\int_0^t |y^1(s)|^2 ds + \mathbb{E}\int_0^t |z^1(s)|^2 ds\bigg) \\
&\qquad + 7\mathbb{E}\bigg(\int_0^t (b(u^\varepsilon(s)) - b(u(s))ds\bigg)^2 \\
&\le C_4\int_0^t \mathbb{E}|x^1(s)|^2 ds + C\varepsilon + C\varepsilon^2,
\end{aligned}$$

where C_4 is a constant depending on C_3, β_2. By the Gronwall inequality, estimations (15) are obtained. By (9) and (15), we get

$$\begin{aligned}&\mathbb{E}|y^1(t)|^2+\mathbb{E}\int_t^T|z^1(s)|^2ds\\&\quad\le C_5\bigg(T\int_t^T\mathbb{E}|y^1(s)|^2ds+(T-t)\mathbb{E}\int_t^T|z^1(s)|^2ds\bigg)+C\varepsilon+C\varepsilon^2,\end{aligned}$$

where C_5 is a constant. Using the above iteration process, (16) and (17) are obtained.
Case 3. When $m=n$, similarly to the above two cases, the result can be obtained easily. □

However, the ε-order estimations of the variational state processes $(x^1(\cdot), y^1(\cdot), z^1(\cdot))$ in Lemma 2.2 is too low to get the variational inequality for our desired maximum principle. We need to give some more elaborate estimations making use of Lemma 2.2 and the FBSDE technique again. Thus we have the following lemma.

Lemma 2.3 *Let* (H1) *and* (H2) *hold. Then we have*

$$\sup_{0\le t\le T}\mathbb{E}|x^1(t)|^2\le C\varepsilon^{\frac{3}{2}},\tag{21}$$

$$\sup_{0\le t\le T}\mathbb{E}|y^1(t)|^2\le C\varepsilon^{\frac{3}{2}},\qquad \mathbb{E}\int_0^T|z^1(t)|^2dt\le C\varepsilon^{\frac{3}{2}}.\tag{22}$$

Proof By Lemma 2.1 and 2.2, we can easily get

$$\mathbb{E}\Big[\sup_{0\le t\le T}|x^1(t)|^2\Big]\le C\varepsilon,\qquad \mathbb{E}\Big[\sup_{0\le t\le T}|y^1(t)|^2\Big]\le C\varepsilon.\tag{23}$$

By (18), the Hölder inequality and (23), we have

$$\begin{aligned}&\mu_1\mathbb{E}|Gx^1(T)|^2+\beta_1\mathbb{E}\int_0^T|Gx^1(t)|^2dt+\beta_2\mathbb{E}\int_0^T(|G^\top y^1(t)|^2+|G^\top z^1(t)|^2)dt\\&\quad\le\mathbb{E}\int_0^T\langle b(u^\varepsilon(t))-b(u(t)),G^\top y^1(t)\rangle dt\\&\qquad-\mathbb{E}\int_0^T\langle f(u^\varepsilon(t))-f(u(t)),Gx^1(t)\rangle dt\\&\quad\le\sqrt{\mathbb{E}\Big[\sup_{0\le t\le T}|y^1(t)|^2\Big]\mathbb{E}\bigg(\int_0^T|G(b(u^\varepsilon(t))-b(u(t)))|dt\bigg)^2}\\&\qquad+\sqrt{\mathbb{E}\Big[\sup_{0\le t\le T}|x^1(t)|^2\Big]\mathbb{E}\bigg(\int_0^T|G^\top(f(u^\varepsilon(t))-f(u(t)))|dt\bigg)^2}\le C\varepsilon^{\frac{3}{2}}.\end{aligned}$$

Case 1. When $m > n$, we have $\beta_1 > 0, \beta_2 \geq 0, \mu_1 > 0$. Then we obtain

$$\mu_1 \mathbb{E}|Gx^1(T)|^2 + \beta_1 \mathbb{E}\int_0^T |Gx^1(t)|^2 dt \leq C\varepsilon^{\frac{3}{2}}. \tag{24}$$

Using the same method of Lemma 2.2, we can get estimation (22). Then (21) holds.
Case 2. When $m < n$, we have $\beta_1 \geq 0, \beta_2 > 0, \mu_1 \geq 0$. Then we have

$$\beta_2 \mathbb{E}\int_0^T (|G^\top y^1(t)|^2 + |G^\top z^1(t)|^2) dt \leq C\varepsilon^{\frac{3}{2}}. \tag{25}$$

Using the method of Lemma 2.2 once more, we can get estimations (21). Then (22) holds.
Case 3. When $m = n$, the result can be obtained similarly. □

Different from the complete observation case (see Shi and Wu [19]), we need the following two lemmas. They give the higher ε-order estimations of variational processes $(x^1(\cdot), y^1(\cdot), z^1(\cdot))$ to deal with the related observation process.

Lemma 2.4 *Let* (H1) *and* (H2) *hold. Then we have*

$$\sup_{0\leq t\leq T} \mathbb{E}|x^1(t)|^4 \leq C\varepsilon^3, \tag{26}$$

$$\sup_{0\leq t\leq T} \mathbb{E}|y^1(t)|^4 \leq C\varepsilon^3, \qquad \mathbb{E}\left(\int_0^T |z^1(t)|^2 dt\right)^2 \leq C\varepsilon^3. \tag{27}$$

Proof Applying Itô's formula to $\langle Gx^1(\cdot)x^1(\cdot)^\top G^\top Gx^1(\cdot), y^1(\cdot)\rangle$, we get

$$\begin{aligned}
&\mathbb{E}\langle g_x(x(T))x^1(T), Gx^1(T)x^1(T)^\top G^\top Gx^1(T)\rangle \\
&= \mathbb{E}\int_0^T \Big\{\big[-\langle f_x(u(t))x^1(t) + f_y(u(t))y^1(t) + f_z(u(t))z^1(t), Gx^1(t)\rangle \\
&\qquad + 3\langle b_x(u(t))x^1(t) + b_y(u(t))y^1(t) + b_z(u(t))z^1(t), G^\top y^1(t)\rangle \\
&\qquad + 3\langle \sigma_x(t)x^1(t) + \sigma_y(t)y^1(t) + \sigma_z(t)z^1(t), G^\top z^1(t)\rangle\big] x^1(t)^\top G^\top Gx^1(t) \\
&\qquad - \langle f(u^\varepsilon(t)) - f(u(t)), Gx^1(t)\rangle x^1(t)^\top G^\top Gx^1(t) \\
&\qquad + 3\langle b(u^\varepsilon(t)) - b(u(t)), G^\top y^1(t)\rangle x^1(t)^\top G^\top Gx^1(t)\Big\} dt.
\end{aligned}$$

Case 1. When $m > n$, we have $\beta_1 > 0, \beta_2 \geq 0, \mu_1 > 0$. By the G-monotonic condition (H2), the Hölder inequality, Lemma 2.1 and 2.3, we have

$$\begin{aligned}
&\mu_1 \mathbb{E}|Gx^1(T)|^4 + \beta_1 \mathbb{E}\int_0^T |Gx^1(t)|^4 dt \\
&\leq \mathbb{E}\int_0^T \Big\{\big[2\langle b_x(u(t))x^1(t) + b_y(u(t))y^1(t) + b_z(u(t))z^1(t), G^\top y^1(t)\rangle
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\langle \sigma_x(t)x^1(t) + \sigma_y(t)y^1(t) + \sigma_z(t)z^1(t), G^\top z^1(t)\rangle\big]x^1(t)^\top G^\top G x^1(t) \\
&\quad - \langle f(u^\varepsilon(t)) - f(u(t)), Gx^1(t)\rangle x^1(t)^\top G^\top G x^1(t) \\
&\quad + 3\langle b(u^\varepsilon(t)) - b(u(t)), G^\top y^1(t)\rangle x^1(t)^\top G^\top G x^1(t)\Big\}dt \\
&\le C\Big(\sup_{0\le t\le T}\mathbb{E}|x^1(t)|^2\Big)\Big(\mathbb{E}\int_0^T \langle x^1(t), G^\top y^1(t)\rangle dt + \mathbb{E}\int_0^T \langle x^1(t), G^\top z^1(t)\rangle dt \\
&\quad + \mathbb{E}\int_0^T y^1(t)^\top z^1(t)dt + \mathbb{E}\int_0^T |y^1(t)|^2 dt + \mathbb{E}\int_0^T |z^1(t)|^2 dt\Big) \\
&\quad + C\Big(\sup_{0\le t\le T}\mathbb{E}|x^1(t)|^2\Big)\sqrt{\Big(\mathbb{E}\sup_{0\le t\le T}|y^1(t)|^2\Big)\mathbb{E}\Big(\int_0^T |G(b(u^\varepsilon(t)) - b(u(t)))|dt\Big)^2} \\
&\quad + C\Big(\sup_{0\le t\le T}\mathbb{E}|y^1(t)|^2\Big)\sqrt{\Big(\mathbb{E}\sup_{0\le t\le T}|x^1(t)|^2\Big)\mathbb{E}\Big(\int_0^T |G^\top(f(u^\varepsilon(t)) - f(u(t)))|dt\Big)^2} \\
&\le C\varepsilon^3.
\end{aligned}
\tag{28}
$$

By variational equation (9) and the Davis-Burkholder-Gundy inequality, we obtain

$$
\begin{aligned}
&\mathbb{E}|y^1(t)|^4 + \mathbb{E}\Big(\int_t^T |z^1(s)|^2 ds\Big)^2 \\
&\quad \le \mathbb{E}\Big\{h_x(x(T))x^1(T) + \int_t^T [f_x(u(s))x^1(s) + f_y(u(s))y^1(s) \\
&\qquad + f_z(u(s))z^1(s) + f(u^\varepsilon(s)) - f(u(s))]ds\Big\}^4 \\
&\quad \le C\Big\{\mathbb{E}|x^1(T)|^4 + T\mathbb{E}\int_t^T |x^1(s)|^4 ds + T\mathbb{E}\int_t^T |y^1(s)|^4 ds \\
&\qquad + (T-t)\mathbb{E}\Big(\int_t^T |z^1(s)|^2 ds\Big)^2\Big\} + C\mathbb{E}\Big(\int_t^T (f(u^\varepsilon(s)) - f(u(s))ds\Big)^4.
\end{aligned}
$$

Making use of the iteration process again, (28) and the Gronwall inequality, (27) holds. Then by (9) and (27), we have

$$
\begin{aligned}
\mathbb{E}|x^1(t)|^4 = \mathbb{E}\Big\{&\int_0^t [b_x(u(s))x^1(s) + b_y(u(s))y^1(s) + b_z(u(s))z^1(s) + b(u^\varepsilon(s)) \\
&- b(u(s))]ds + \int_0^t [\sigma_x(s)x^1(s) + \sigma_y(s)y^1(s) + \sigma_z(s)z^1(s)]dW(s)\Big\}^4
\end{aligned}
$$

$$\leq C\bigg(\int_0^t \mathbb{E}|x^1(s)|^4 ds + \sup_{0\leq t\leq T} \mathbb{E}|y^1(t)|^4 + \mathbb{E}\bigg(\int_0^t |z^1(s)|^2 ds\bigg)^2\bigg)$$
$$+ C\mathbb{E}\bigg(\int_0^t (b(u^\varepsilon(s)) - b(u(s))ds\bigg)^4$$
$$\leq C\int_0^t \mathbb{E}|x^1(s)|^4 ds + C\varepsilon^3 + C\varepsilon^4.$$

By the Gronwall inequality again, (26) holds.
Case 2. When $m < n$, we have $\beta_1 \geq 0, \beta_2 > 0, \mu_1 \geq 0$. By the G-monotonic condition (H2), the Hölder inequality, Lemma 2.1, 2.3 and (28), we have

$$3\beta_2\mathbb{E}\int_0^T (|G^\top y^1(t)|^4 + |G^\top z^1(t)|^4)dt$$
$$\leq \mathbb{E}\int_0^T \Big\{\big[\langle f_x(u(t))x^1(t) + f_y(u(t))y^1(t) + f_z(u(t))z^1(t), Gx^1(t)\rangle x^1(t)^\top G^\top Gx^1(t)$$
$$- \langle f(u^\varepsilon(t)) - f(u(t)), Gx^1(t)\rangle x^1(t)^\top G^\top Gx^1(t)$$
$$+ 3\langle b(u^\varepsilon(t)) - b(u(t)), G^\top y^1(t)\rangle x^1(t)^\top G^\top Gx^1(t)\Big\}dt$$
$$\leq C\Big(\sup_{0\leq t\leq T} \mathbb{E}|x^1(t)|^2\Big)\bigg(\mathbb{E}\int_0^T \langle x^1(t), G^\top y^1(t)\rangle dt + \mathbb{E}\int_0^T \langle x^1(t), G^\top z^1(t)\rangle dt$$
$$+ \mathbb{E}\int_0^T y^1(t)^\top z^1(t)dt + \mathbb{E}\int_0^T |x^1(t)|^2 dt + \mathbb{E}\int_0^T |y^1(t)|^2 dt + \mathbb{E}\int_0^T |z^1(t)|^2 dt\bigg)$$
$$+ C\Big(\sup_{0\leq t\leq T} \mathbb{E}|x^1(t)|^2\Big)\sqrt{\Big(\mathbb{E}\sup_{0\leq t\leq T} |y^1(t)|^2\Big)\mathbb{E}\bigg(\int_0^T |G(b(u^\varepsilon(t)) - b(u(t)))|dt\bigg)^2}$$
$$+ C\Big(\sup_{0\leq t\leq T} \mathbb{E}|y^1(t)|^2\Big)\sqrt{\Big(\mathbb{E}\sup_{0\leq t\leq T} |x^1(t)|^2\Big)\mathbb{E}\bigg(\int_0^T |G^\top(f(u^\varepsilon(t)) - f(u(t)))|dt\bigg)^2}$$
$$\leq C\varepsilon^3. \tag{29}$$

Then by (9), we get

$$\mathbb{E}|x^1(t)|^4 \leq C\bigg(\int_0^t \mathbb{E}|x^1(s)|^4 ds + \mathbb{E}\int_0^T |y^1(t)|^4 dt + \mathbb{E}\int_0^t |z^1(s)|^4 ds\bigg)$$
$$+ C\mathbb{E}\bigg(\int_0^t (b(u^\varepsilon(s)) - b(u(s))ds\bigg)^4$$
$$\leq C\int_0^t \mathbb{E}|x^1(s)|^4 ds + C\varepsilon^3 + C\varepsilon^4.$$

By the Gronwall inequality, estimation (26) are obtained. Then by (9), we have

$$\mathbb{E}|y^1(t)|^4+\mathbb{E}\bigg(\int_t^T |z^1(s)|^2 ds\bigg)^2 \\ \le C\bigg(T\int_t^T \mathbb{E}|y^1(s)|^4 ds+(T-t)\mathbb{E}\bigg(\int_t^T |z^1(s)|^2 ds\bigg)^2\bigg)+C\varepsilon^3+C\varepsilon^4.$$

Using the iteration process again, (27) are obtained.
Case 3. When $m=n$, similarly to the above two cases, the result can be obtained easily. □

Lemma 2.5 *Let* (H1) *and* (H2) *hold. Then we have*

$$\sup_{0\le t\le T} \mathbb{E}|x^1(t)|^8 \le C\varepsilon^6, \tag{30}$$

$$\sup_{0\le t\le T} \mathbb{E}|y^1(t)|^8 \le C\varepsilon^6, \qquad \mathbb{E}\bigg(\int_0^T |z^1(t)|^2 dt\bigg)^4 \le C\varepsilon^6. \tag{31}$$

Proof Applying Itô's formula to $\langle Gx^1(\cdot)x^1(\cdot)^\top G^\top Gx^1(\cdot)x^1(\cdot)^\top G^\top Gx^1(\cdot)x^1(\cdot)^\top G^\top Gx^1(\cdot), y^1(\cdot)\rangle$, using the same method as the proof of (26) and (27), by the G-monotonic condition (H2), the Hölder inequality, Lemma 2.1, 2.3 and 2.4, we can get estimates (30) and (31). We omit the detail. □

The following lemma plays an important role in deriving the variational inequality. It gives the ε-order estimations of the differences between the perturbed state processes $(x^\varepsilon(\cdot), y^\varepsilon(\cdot), z^\varepsilon(\cdot))$ and the sum of the optimal state processes $(x(\cdot), y(\cdot), z(\cdot))$ and the variational processes $(x^1(\cdot), y^1(\cdot), z^1(\cdot))$.

Lemma 2.6 *Let* (H1) *and* (H2) *hold. Then we have*

$$\sup_{0\le t\le T} \mathbb{E}|x^\varepsilon(t)-x(t)-x^1(t)|^2 \le C_\varepsilon\varepsilon^2, \tag{32}$$

$$\sup_{0\le t\le T} \mathbb{E}|y^\varepsilon(t)-y(t)-y^1(t)|^2 \le C_\varepsilon\varepsilon^2,$$

$$\mathbb{E}\int_0^T |z^\varepsilon(t)-z(t)-z^1(t)|^2 dt \le C_\varepsilon\varepsilon^2, \tag{33}$$

$$\sup_{0\le t\le T} \mathbb{E}|x^\varepsilon(t)-x(t)-x^1(t)|^4 \le C_\varepsilon\varepsilon^4, \tag{34}$$

$$\sup_{0\le t\le T} \mathbb{E}|y^\varepsilon(t)-y(t)-y^1(t)|^4 \le C_\varepsilon\varepsilon^4,$$

$$\mathbb{E}\bigg(\int_0^T |z^\varepsilon(t)-z(t)-z^1(t)|^2 dt\bigg)^2 \le C_\varepsilon\varepsilon^4. \tag{35}$$

Hereafter C_ε denotes some nonnegative constant such that $C_\varepsilon\to 0$ as $\varepsilon\to 0$.

Proof Because (34) and (35) imply (32) and (33), we only prove (34) and (35). It is easy to see that

$$
\begin{aligned}
&\int_0^t b(s,\Gamma(s)+\Gamma^1(s),u^\varepsilon(s))ds+\int_0^t \sigma(s,\Gamma(s)+\Gamma^1(s))dW(s)\\
&\quad=\int_0^t\bigg[b(u^\varepsilon(s))+\int_0^1 b_x(s,\Gamma(s)+\lambda\Gamma^1(s),u^\varepsilon(s))d\lambda x^1(s)\\
&\qquad+\int_0^1 b_y(s,\Gamma(s)+\lambda\Gamma^1(s),u^\varepsilon(s))d\lambda y^1(s)\\
&\qquad+\int_0^1 b_z(s,\Gamma(s)+\lambda\Gamma^1(s),u^\varepsilon(s))d\lambda z^1(s)\bigg]ds\\
&\qquad+\int_0^t\bigg[\sigma(s)+\int_0^1 \sigma_x(s,\Gamma(s)+\lambda\Gamma^1(s))d\lambda x^1(s)\\
&\qquad+\int_0^1 \sigma_y(s,\Gamma(s)+\lambda\Gamma^1(s))d\lambda y^1(s)\\
&\qquad+\int_0^1 \sigma_z(s,\Gamma(s)+\lambda\Gamma^1(s))d\lambda z^1(s)\bigg]dW(s)\\
&\quad=\int_0^t b(u(s))ds+\int_0^t\sigma(s)dW(s)+\int_0^t\big[b_x(u(s))x^1(s)+b_y(u(s))y^1(s)\\
&\qquad+b_z(u(s))z^1(s)+b(u^\varepsilon(s))-b(u(s))\big]ds+\int_0^t\big[\sigma_x(s)x^1(s)+\sigma_y(s)y^1(s)\\
&\qquad+\sigma_z(s)z^1(s)\big]dW(s)+\int_0^t A^\varepsilon(s)ds+\int_0^t B^\varepsilon(s)dW(s)\\
&\quad=x(t)-x_0+x^1(t)+\int_0^t A_1^\varepsilon(s)ds+\int_0^t B_1^\varepsilon(s)dW(s),
\end{aligned}
\tag{36}
$$

where (for simplification we omit the time subscript s)

$$
\begin{aligned}
A_1^\varepsilon &:= \int_0^1[b_x(\Gamma+\lambda\Gamma^1,u^\varepsilon)-b_x(u)]d\lambda x^1+\int_0^1[b_y(\Gamma+\lambda\Gamma^1,u^\varepsilon)-b_y(u)]d\lambda y^1\\
&\quad+\int_0^1[b_z(\Gamma+\lambda\Gamma^1,u^\varepsilon)-b_z(u)]d\lambda z^1,\\
B_1^\varepsilon &:= \int_0^1[\sigma_x(\Gamma+\lambda\Gamma^1)-\sigma_x]d\lambda x^1+\int_0^1[\sigma_y(\Gamma+\lambda\Gamma^1)-\sigma_y]d\lambda y^1\\
&\quad+\int_0^1[\sigma_z(\Gamma+\lambda\Gamma^1)-\sigma_z]d\lambda z^1,
\end{aligned}
$$

and $\Gamma + \lambda\Gamma^1 \equiv (x + \lambda x^1, y + \lambda y^1, z + \lambda z^1)$. By (26) and (27), we can easily get

$$\sup_{0\le t\le T} \mathbb{E}\left\{\left(\int_0^t A_1^\varepsilon(s)ds\right)^4 + \left(\int_0^t B_1^\varepsilon(s)dW(s)\right)^4\right\} \le C_\varepsilon \varepsilon^4. \tag{37}$$

Then by

$$x^\varepsilon(t) = x_0 + \int_0^t b(s, \Gamma^\varepsilon(s), u^\varepsilon(s))ds + \int_0^t \sigma(s, \Gamma^\varepsilon(s))dW(s),$$

we have

$$\begin{aligned}
&x^\varepsilon(t) - x(t) - x^1(t)\\
&\quad= \int_0^t \big[b(s, \Gamma^\varepsilon(s), u^\varepsilon(s)) - b(s, \Gamma(s) + \Gamma^1(s), u^\varepsilon(s))\big]ds + \int_0^t A_1^\varepsilon(s)ds\\
&\qquad+ \int_0^t \big[\sigma(s, \Gamma^\varepsilon(s)) - \sigma(s, \Gamma(s) + \Gamma^1(s))\big]dW(s) + \int_0^t B_1^\varepsilon(s)dW(s)\\
&\quad= \int_0^t \big[I_1^\varepsilon(s)(x^\varepsilon(s) - x(s) - x^1(s)) + I_2^\varepsilon(s)(y^\varepsilon(s) - y(s) - y^1(s))\\
&\qquad+ I_3^\varepsilon(s)(z^\varepsilon(s) - z(s) - z^1(s))\big]ds + \int_0^t \big[D_1^\varepsilon(s)(x^\varepsilon(s) - x(s) - x^1(s))\\
&\qquad+ D_2^\varepsilon(s)(y^\varepsilon(s) - y(s) - y^1(s)) + D_3^\varepsilon(s)(z^\varepsilon(s) - z(s) - z^1(s))\big]dW(s)\\
&\qquad+ \int_0^t A_1^\varepsilon(s)ds + \int_0^t B_1^\varepsilon(s)dW(s),
\end{aligned}$$

where

$$\begin{aligned}
I_1^\varepsilon &:= \int_0^1 b_x(\Gamma + \Gamma^1 + \lambda(\Gamma^\varepsilon - \Gamma - \Gamma^1), u^\varepsilon)d\lambda,\\
I_2^\varepsilon &:= \int_0^1 b_y(\Gamma + \Gamma^1 + \lambda(\Gamma^\varepsilon - \Gamma - \Gamma^1), u^\varepsilon)d\lambda,\\
I_3^\varepsilon &:= \int_0^1 b_z(\Gamma + \Gamma^1 + \lambda(\Gamma^\varepsilon - \Gamma - \Gamma^1), u^\varepsilon)]d\lambda,\\
D_1^\varepsilon &:= \int_0^1 [\sigma_x(\Gamma + \Gamma^1 + \lambda(\Gamma^\varepsilon - \Gamma - \Gamma^1))d\lambda,\\
D_2^\varepsilon &:= \int_0^1 [\sigma_y(\Gamma + \Gamma^1 + \lambda(\Gamma^\varepsilon - \Gamma - \Gamma^1))d\lambda,\\
D_3^\varepsilon &:= \int_0^1 [\sigma_z(\Gamma + \Gamma^1 + \lambda(\Gamma^\varepsilon - \Gamma - \Gamma^1))d\lambda
\end{aligned}$$

and

$$\Gamma + \Gamma^1 + \lambda(\Gamma^\varepsilon - \Gamma - \Gamma^1)$$
$$\equiv (x + x^1 + \lambda(x^\varepsilon - x - x^1), y + y^1 + \lambda(y^\varepsilon - y - y^1), z + z^1 + \lambda(z^\varepsilon - z - z^1)).$$

Then by (37), we get

$$\begin{aligned}
&\mathbb{E}|x^\varepsilon(t) - x(t) - x^1(t)|^4 \\
&\quad = \mathbb{E}\bigg\{ \int_0^t \big[I_1^\varepsilon(s)(x^\varepsilon(s) - x(s) - x^1(s)) + I_2^\varepsilon(s)(y^\varepsilon(s) - y(s) - y^1(s)) \\
&\qquad + I_3^\varepsilon(s)(z^\varepsilon(s) - z(s) - z^1(s))\big]ds + \int_0^t A_1^\varepsilon(s)ds \\
&\qquad + \int_0^t \big[D_1^\varepsilon(s)(x^\varepsilon(s) - x(s) - x^1(s)) + D_2^\varepsilon(s)(y^\varepsilon(s) - y(s) - y^1(s)) \\
&\qquad + D_3^\varepsilon(s)(z^\varepsilon(s) - z(s) - z^1(s))\big]dW(s) + \int_0^t B_1^\varepsilon(s)dW(s)\bigg\}^4 \\
&\quad \le C\mathbb{E}\int_0^t \Big(|x^\varepsilon(s) - x(s) - x^1(s)|^4 + |y^\varepsilon(s) - y(s) - y^1(s)|^4\Big)ds \\
&\qquad + C\mathbb{E}\bigg(\int_0^t |z^\varepsilon(s) - z(s) - z^1(s)|^2 ds\bigg)^2 + C_\varepsilon \varepsilon^4.
\end{aligned} \tag{38}$$

Similarly, we have

$$\begin{aligned}
&-\int_t^T f(s, \Gamma(s) + \Gamma^1(s), u^\varepsilon(s))ds + \int_t^T (z(s) + z^1(s))dW(s) \\
&\quad = g(x(T)) + g_x(x(T))x^1(T) - y(t) - y^1(t) - \int_t^T I_4^\varepsilon(s)ds,
\end{aligned}$$

where

$$\begin{aligned}
I_4^\varepsilon :=& \int_0^1 [f_x(\Gamma + \lambda\Gamma^1, u^\varepsilon) - f_x(u)]d\lambda x^1 + \int_0^1 [f_y(\Gamma + \lambda\Gamma^1, u^\varepsilon) - f_y(u)]d\lambda y^1 \\
&+ \int_0^1 [f_z(\Gamma + \lambda\Gamma^1, u^\varepsilon) - f_z(u)]d\lambda z^1.
\end{aligned}$$

By (26) and (27), we can easily find

$$\sup_{0\le t\le T} \mathbb{E}\bigg(\int_t^T I_4^\varepsilon(s)ds\bigg)^4 \le C_\varepsilon \varepsilon^4. \tag{39}$$

From

$$y^\varepsilon(t) = g(x^\varepsilon(T)) + \int_t^T f(s, \Gamma^\varepsilon(s), u^\varepsilon(s))ds - \int_t^T z^\varepsilon(s)dW(s),$$

we have

$$
\begin{aligned}
&\big[y^\varepsilon(t)-y(t)-y^1(t)\big]+\int_t^T\big[z^\varepsilon(s)-z(s)-z^1(s)\big]dW(s)\\
&\quad=\int_t^T\big[f(s,\Gamma^\varepsilon(s),u^\varepsilon(s))-f(s,\Gamma(s)+\Gamma^1(s),u^\varepsilon(s))\big]ds\\
&\qquad+\int_t^T I_4^\varepsilon(s)ds+g(x^\varepsilon(T))-g(x(T))-g_x(x(T))x^1(T)\\
&\quad=\int_t^T\big[H_1^\varepsilon(s)(x^\varepsilon(s)-x(s)-x^1(s))+H_2^\varepsilon(s)(y^\varepsilon(s)-y(s)-y^1(s))\\
&\qquad+H_3^\varepsilon(s)(z^\varepsilon(s)-z(s)-z^1(s))\big]ds+\int_t^T I_4^\varepsilon(s)ds+g(x^\varepsilon(T))\\
&\qquad-g(x(T)+x^1(T))+\int_0^1\big[g_x(x(T)+\lambda x^1(T))-g_x(x(T))\big]d\lambda x^1(T),
\end{aligned}
$$

where

$$
\begin{aligned}
H_1^\varepsilon&:=\int_0^1 f_x(\Gamma+\Gamma^1+\lambda(\Gamma^\varepsilon-\Gamma-\Gamma^1),u^\varepsilon)d\lambda,\\
H_2^\varepsilon&:=\int_0^1 f_y(\Gamma+\Gamma^1+\lambda(\Gamma^\varepsilon-\Gamma-\Gamma^1),u^\varepsilon)d\lambda,\\
H_3^\varepsilon&:=\int_0^1 f_z(\Gamma+\Gamma^1+\lambda(\Gamma^\varepsilon-\Gamma-\Gamma^1),u^\varepsilon)]d\lambda.
\end{aligned}
$$

By (39), (26), (27) and the same iteration method of Lemma 2.1, we have

$$
\begin{aligned}
&\mathbb{E}|y^\varepsilon(t)-y(t)-y^1(t)|^4+\mathbb{E}\left(\int_t^T|z^\varepsilon(s)-z(s)-z^1(s)|^2ds\right)^2\\
&\quad\le C\mathbb{E}\bigg\{\int_t^T\big[H_1^\varepsilon(s)\big(x^\varepsilon(s)-x(s)-x^1(s)\big)+H_2^\varepsilon(s)\big(y^\varepsilon(s)-y(s)-y^1(s)\big)\\
&\qquad+H_3^\varepsilon(s)\big(z^\varepsilon(s)-z(s)-z^1(s)\big)\big]ds+\int_t^T I_4^\varepsilon(s)ds+h(x^\varepsilon(T))\\
&\qquad-h(x(T)+x^1(T))+\int_0^1\big[h_x(x(T)+\lambda x^1(T))-h_x(x(T))\big]d\lambda x^1(T)\bigg\}^4\\
&\quad\le C\mathbb{E}\int_t^T\big(|x^\varepsilon(s)-x(s)-x^1(s)|^4+|y^\varepsilon(s)-y(s)-y^1(s)|^4\big)ds\\
&\qquad+\mathbb{E}|h(x^\varepsilon(T))-h(x(T)+x^1(T))|^4+C_\varepsilon\varepsilon^4. \qquad (40)
\end{aligned}
$$

From (38) and (40), using the method once more as the proof of (26) and (27) with $(x^1(\cdot), y^1(\cdot), z^1(\cdot))$ replaced by $(x^\varepsilon(\cdot) - x(\cdot) - x^1(\cdot), y^\varepsilon(\cdot) - y(\cdot) - y^1(\cdot), z^\varepsilon(\cdot) - z(\cdot) - z^1(\cdot))$, we can get (34) and (35). We omit the detail. □

2.2 Variational Inequality

In this subsection, firstly we obtain some ε-order estimations of the variational observed process $Z^1(\cdot)$ and the difference between the perturbed observed process $Z^\varepsilon(\cdot)$ with the sum of the optimal observed process $Z(\cdot)$ and the variational observed process $Z^1(\cdot)$. Then we give the variational inequality which has two equivalent forms and is important to obtain the maximum principle.

Lemma 2.7 *Let* (H1) *and* (H2) *hold. Then we have*

$$\sup_{0\le t\le T} \mathbb{E}|Z^1(t)|^2 \le C\varepsilon, \qquad \sup_{0\le t\le T} \mathbb{E}|Z^1(t)|^4 \le C\varepsilon^2, \tag{41}$$

$$\sup_{0\le t\le T} \mathbb{E}|Z^\varepsilon(t) - Z(t) - Z^1(t)|^2 \le C_\varepsilon \varepsilon^2. \tag{42}$$

Proof Because the second estimate in (41) implies the first one, we only prove the second one. From (10), by the Hölder and Davis-Burkholder-Gundy inequalities, we can get

$$\begin{aligned}
&\mathbb{E}|Z^1(t)|^4 \\
&\quad\le C\bigg(\mathbb{E}\bigg|\int_0^t Z^1(s)\langle h(u(s)), dY(s)\rangle\bigg|^4 + \mathbb{E}\bigg|\int_0^t Z(s)\langle h_x(u(s))x^1(s), dY(s)\rangle\bigg|^4 \\
&\qquad + \mathbb{E}\bigg|\int_0^t Z(s)\langle h_y(u(s))y^1(s), dY(s)\rangle\bigg|^4 \\
&\qquad + \mathbb{E}\bigg|\int_0^t Z(s)\langle h(u^\varepsilon(s)) - h(u(s)), dY(s)\rangle\bigg|^4\bigg) \\
&\quad\le C\bigg(\int_0^t \mathbb{E}|Z^1(s)|^4 ds + \mathbb{E}\bigg(\int_0^t |Z(s)x^1(s)|^2 ds\bigg)^2 + \mathbb{E}\bigg(\int_0^t |Z(s)y^1(s)|^2 ds\bigg)^2 \\
&\qquad + \varepsilon\int_\tau^{\tau+\varepsilon} \mathbb{E}|Z(s)(h(u^\varepsilon(s)) - h(u(s)))|^4 ds\bigg) \\
&\quad\le C\bigg(\int_0^t \mathbb{E}|Z^1(s)|^4 ds + \mathbb{E}\bigg(\int_0^t |Z(s)|^4 ds \cdot \int_0^t |x^1(s)|^4 ds\bigg) \\
&\qquad + \mathbb{E}\bigg(\int_0^t |Z(s)|^4 ds \cdot \int_0^t |y^1(s)|^4 ds\bigg)\bigg) + C\varepsilon^2 \sup_{0\le t\le T} \mathbb{E}|Z(s)|^4 \\
&\quad\le C\bigg(\int_0^t \mathbb{E}|Z^1(s)|^4 ds + \sqrt{\mathbb{E}\int_0^t |Z(s)|^8 ds \cdot \mathbb{E}\int_0^t |x^1(s)|^8 ds}
\end{aligned}$$

$$
\begin{aligned}
&+\sqrt{\mathbb{E}\int_0^t |Z(s)|^8 ds\cdot \mathbb{E}\int_0^t |y^1(s)|^8 ds}\Bigg)+C\varepsilon^2\\
&\le C\Bigg(\int_0^t \mathbb{E}|Z^1(s)|^4 ds+\sqrt{T\sup_{0\le t\le T}\mathbb{E}|Z(s)|^8}\sqrt{\mathbb{E}\int_0^t |x^1(s)|^8 ds}\\
&+\sqrt{T\sup_{0\le t\le T}\mathbb{E}|Z(s)|^8}\sqrt{\mathbb{E}\int_0^t |y^1(s)|^8 ds}\Bigg)+C\varepsilon^2\\
&\le C\left(\int_0^t \mathbb{E}|Z^1(s)|^4 ds+\sqrt{T\sup_{0\le t\le T}\mathbb{E}|x^1(s)|^8}+\sqrt{T\sup_{0\le t\le T}\mathbb{E}|y^1(s)|^8}\right)+C\varepsilon^2.
\end{aligned}
$$

By (30), (31) and the Gronwall inequality we get (41). We now prove (42). At first, we have

$$
\begin{aligned}
&\int_0^t Z^1(s)\langle h(u(s)), dY(s)\rangle+\int_0^t Z(s)\langle h(s,x(s)+x^1(s),y(s)+y^1(s),u^\varepsilon(s)),dY(s)\rangle\\
&\quad=\int_0^t Z^1(s)\langle h(u(s)), dY(s)\rangle+\int_0^t Z(s)\langle h(u^\varepsilon(s)),dY(s)\rangle\\
&\qquad+\int_0^t Z(s)\left\langle\int_0^1 h_x(s,x(s)+\lambda x^1(s),y(s)+\lambda y^1(s),u^\varepsilon(s))x^1(s)d\lambda, dY(s)\right\rangle\\
&\qquad+\int_0^t Z(s)\left\langle\int_0^1 h_y(s,x(s)+\lambda x^1(s),y(s)+\lambda y^1(s),u^\varepsilon(s))y^1(s)d\lambda, dY(s)\right\rangle\\
&\quad=\int_0^t \big\langle Z^1(s)h(u(s))+Z(s)h_x(u(s))x^1(s)+Z(s)h_y(u(s))y^1(s)\\
&\qquad+Z(s)(h(u^\varepsilon(s))-h(u(s))),dY(s)\big\rangle+\int_0^t Z(s)\langle h(u(s)),dY(s)\rangle\\
&\qquad+\int_0^t Z(s)\langle A_2^\varepsilon(s),dY(s)\rangle,\\
&\quad=Z(t)-1+Z^1(t)+\int_0^t Z(s)\langle A_2^\varepsilon(s),dY(s)\rangle,
\end{aligned}
$$

where

$$
\begin{aligned}
A_2^\varepsilon(s):=&\int_0^1 (h_x(s,x(s)+\lambda x^1(s),y(s)+\lambda y^1(s),u^\varepsilon(s))-h_x(u(s)))d\lambda x^1(s)\\
&+\int_0^1 (h_y(s,x(s)+\lambda x^1(s),y(s)+\lambda y^1(s),u^\varepsilon(s))-h_y(u(s))d\lambda y^1(s).
\end{aligned}
$$

By (30) and (31), it is easy to know that

$$\sup_{0\le t\le T} \mathbb{E}\bigg(\int_0^t Z(s)A_2^\varepsilon(s)dY(s)\bigg)^2 \le C_\varepsilon \varepsilon^2. \tag{43}$$

Then by

$$Z^\varepsilon(t) = 1 + \int_0^t Z^\varepsilon(s)\langle h(s, x^\varepsilon(s), y^\varepsilon(s), u^\varepsilon(s)), dY(s)\rangle,$$

we can get

$$\begin{aligned}
&Z^\varepsilon(t) - Z(t) - Z^1(t)\\
&\quad= \int_0^t Z^\varepsilon(s)\langle h(s, x^\varepsilon(s), y^\varepsilon(s), u^\varepsilon(s)), dY(s)\rangle - \int_0^t Z^1(s)\langle h(u(s)), dY(s)\rangle\\
&\qquad - \int_0^t Z(s)\langle h(s, x(s)+x^1(s), y(s)+y^1(s), u^\varepsilon(s)), dY(s)\rangle\\
&\qquad + \int_0^t Z(s)\langle A^\varepsilon(s), dY(s)\rangle\\
&\quad= \int_0^t (Z^\varepsilon(s) - Z(s) - Z^1(s))\langle h(s, x^\varepsilon(s), y^\varepsilon(s), u^\varepsilon(s)), dY(s)\rangle\\
&\qquad + \int_0^t (Z(s)+Z^1(s))\langle h(s, x^\varepsilon(s), y^\varepsilon(s), u^\varepsilon(s))\\
&\qquad - h(s, x(s)+x^1(s), y(s)+y^1(s), u^\varepsilon(s)), dY(s)\rangle\\
&\qquad + \int_0^t Z^1(s)\langle h(s, x(s)+x^1(s), y(s)+y^1(s), u^\varepsilon(s)) - h(u^\varepsilon(s)), dY(s)\rangle\\
&\qquad + \int_0^t Z^1(s)\langle h(u^\varepsilon(s)) - h(u(s)), dY(s)\rangle + \int_0^t Z(s)\langle A^\varepsilon(s), dY(s)\rangle\\
&\quad= \int_0^t (Z^\varepsilon(s) - Z(s) - Z^1(s))\langle h(s, x^\varepsilon(s), y^\varepsilon(s), u^\varepsilon(s)), dY(s)\rangle\\
&\qquad + \int_0^t (Z(s)+Z^1(s))\langle B_2^\varepsilon(s), dY(s)\rangle + \int_0^t Z^1(s)\langle B_3^\varepsilon(s), dY(s)\rangle\\
&\qquad + \int_0^t Z^1(s)\langle h(u^\varepsilon(s)) - h(u(s)), dY(s)\rangle + \int_0^t Z(s)\langle A_2^\varepsilon(s), dY(s)\rangle,
\end{aligned}$$

where

$$\begin{aligned}
B_2^\varepsilon := \int_0^1 & h_x(x + x^1 + \lambda(x^\varepsilon - x - x^1), y + y^1\\
&+ \lambda(y^\varepsilon - y - y^1), u^\varepsilon)d\lambda(x^\varepsilon - x - x^1)
\end{aligned}$$

$$+\int_0^1 h_y(x+x^1+\lambda(x^\varepsilon-x-x^1), y+y^1$$
$$+\lambda(y^\varepsilon-y-y^1), u^\varepsilon)d\lambda(y^\varepsilon-y-y^1),$$
$$B_3^\varepsilon := \int_0^1 h_x(x+\lambda x^1, y+\lambda y^1, u^\varepsilon)d\lambda x^1 + \int_0^1 h_y(x+\lambda x^1, y+\lambda y^1, u^\varepsilon)d\lambda y^1.$$

By (30) and (31), it is easy to see that

$$\sup_{0\le t\le T} \mathbb{E}\bigg(\int_0^t Z(s)B_2^\varepsilon(s)dY(s)\bigg)^2 \le C_\varepsilon \varepsilon^2. \tag{44}$$

By (41), (43) and (44), we have

$$\begin{aligned}
&\mathbb{E}|Z^\varepsilon(t)-Z(t)-Z^1(t)|^2\\
&\quad\le C\bigg\{\int_0^t \mathbb{E}|Z^\varepsilon(s)-Z(s)-Z^1(s)|^2ds + \mathbb{E}\int_0^t |Z^1(s)B-2^\varepsilon(s)|^2ds\\
&\qquad + \mathbb{E}\int_0^t |Z^1(s)B_3^\varepsilon(s)|^2ds + \mathbb{E}\int_0^t |Z^1(s)(h(u^\varepsilon(s))-h(u(s)))|^2ds\\
&\qquad + \sup_{0\le t\le T}\mathbb{E}\Big(\int_0^t Z(s)A_2^\varepsilon(s)dY(s)\Big)^2 + \sup_{0\le t\le T}\mathbb{E}\bigg(\int_0^t Z(s)B_2^\varepsilon(s)dY(s)\bigg)^2\bigg\}\\
&\quad\le C\int_0^t \mathbb{E}|Z^\varepsilon(s)-Z(s)-Z^1(s)|^2ds + C_\varepsilon\varepsilon^2.
\end{aligned}$$

By the Gronwall inequality, (42) holds. The proof is complete. □

Now we can present the following variational inequality.

Lemma 2.8 *Let* (H1)∼(H3) *hold. Then we have*

$$\begin{aligned}
&\mathbb{E}^u\bigg[\int_0^T [\zeta(t)l(u(t)) + l_x^\top(u(t))x^1(t) + l_y^\top(u(t))y^1(t) + l(u^\varepsilon(t)) - l(u(t))]dt\\
&\quad + \zeta(T)\Phi(x(T)) + \Phi_x^\top(x(T))x^1(T) + \gamma_y^\top(y(0))y^1(0)\bigg] \ge o(\varepsilon),
\end{aligned} \tag{45}$$

where $\zeta(\cdot)$ *is the solution of the SDE*:

$$\begin{aligned}
&d\zeta(t) = \langle h_x(u(t))x^1(t) + h_y(u(t))y^1(t) + h(u^\varepsilon(t)) - h(u(t)), d\tilde{W}(t)\rangle,\\
&\zeta(0) = 0.
\end{aligned} \tag{46}$$

Proof Since $u(\cdot)$ is optimal, we have

$$\begin{aligned}0 &\le J(u^\varepsilon(\cdot)) - J(u(\cdot))\\ &= \mathbb{E}\int_0^T [Z^\varepsilon(t)l(t,x^\varepsilon(t),y^\varepsilon(t),u^\varepsilon(t)) - Z(t)l(u(t))]dt\\ &\quad + \mathbb{E}[Z^\varepsilon(T)\Phi(x^\varepsilon(T)) - Z(T)\Phi(x(T))] + \mathbb{E}[\gamma(y^\varepsilon(0)) - \gamma(y(0))].\end{aligned}$$

By (33), it is clear that $\mathbb{E}[\gamma(y^\varepsilon(0)) - \gamma(y(0)+y^1(0))] = o(\varepsilon)$. Hence

$$\begin{aligned}\mathbb{E}[\gamma(y^\varepsilon(0)) - \gamma(y(0))] &= \mathbb{E}[\gamma(y(0)+y^1(0)) - \gamma(y(0))] + o(\varepsilon)\\ &= \mathbb{E}[\gamma_y^\top(y(0))y^1(0)] + o(\varepsilon).\end{aligned}$$

Noting (32) and (42), we get

$$\begin{aligned}&\mathbb{E}[Z^\varepsilon(T)\Phi(x^\varepsilon(T)) - Z(T)\Phi(x(T))]\\ &\quad= \mathbb{E}[Z^1(T)\Phi(x(T))] + \mathbb{E}[Z(T)(\Phi(x(T)+x^1(T)) - \Phi(x(T)))]\\ &\qquad + \mathbb{E}[(Z(T)+Z^1(T))(\Phi(x^\varepsilon(T)) - \Phi(x(T)+x^1(T)))]\\ &\qquad + \mathbb{E}[Z^1(T)(\Phi(x(T)+x^1(T)) - \Phi(x(T)))]\\ &\qquad + \mathbb{E}[(Z^\varepsilon(T) - Z(T) - Z^1(T))\Phi(x^\varepsilon(T))]\\ &\quad= \mathbb{E}[Z^1(T)\Phi(x(T))] + \mathbb{E}[Z(T)\Phi_x^\top(x(T))x^1(T)]\\ &\qquad + \mathbb{E}[(Z(T)+Z^1(T))\Phi_x^\top(x(T)+x^1(T))(x^\varepsilon(T) - x(T) - x^1(T))]r\\ &\qquad + \mathbb{E}[Z^1(T)\Phi_x^\top(x(T))x^1(T)] + \mathbb{E}[(Z^\varepsilon(T) - Z(T) - Z^1(T))\Phi(x^\varepsilon(T))] + o(\varepsilon)\\ &\quad= \mathbb{E}[Z^1(T)\Phi(x(T))] + \mathbb{E}[Z(T)\Phi_x^\top(x(T))x^1(T)] + o(\varepsilon).\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\mathbb{E}\int_0^T \big[Z^\varepsilon(t)l(t,x^\varepsilon(t),y^\varepsilon(t),u^\varepsilon(t)) - Z(t)l(u(t))\big]dt\\ &\quad= \mathbb{E}\int_0^T Z^1(t)l(u(t))dt\\ &\qquad + \mathbb{E}\int_0^T (Z^\varepsilon(t) - Z(t) - Z^1(t))l(t,x(t)+x^1(t),y(t)+y^1(t),u^\varepsilon(t))dt\\ &\qquad + \mathbb{E}\int_0^T (Z^\varepsilon(t) - Z(t) - Z^1(t))\bullet\\ &\qquad\qquad \big[l(t,x^\varepsilon(t),y^\varepsilon(t),u^\varepsilon(t)) - l(t,x(t)+x^1(t),y(t)+y^1(t),u^\varepsilon(t))\big]dt\\ &\qquad + \mathbb{E}\int_0^T (Z(t)+Z^1(t))\bullet\\ &\qquad\qquad \big[l(t,x^\varepsilon(t),y^\varepsilon(t),u^\varepsilon(t)) - l(t,x(t)+x^1(t),y(t)+y^1(t),u^\varepsilon(t))\big]dt\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\int_0^T (Z(t)+Z^1(t))\big[l(t,x(t)+x^1(t),y(t)+y^1(t),u^\varepsilon(t))-l(u^\varepsilon(t))\big]dt\\
&-\mathbb{E}\int_0^T (Z(t)+Z^1(t))\big[l(t,x(t)+x^1(t),y(t)+y^1(t),u(t))-l(u(t))\big]dt\\
&+\mathbb{E}\int_0^T (Z(t)+Z^1(t))\big[l(u^\varepsilon(t))-l(u(t))\big]dt\\
&+\mathbb{E}\int_0^T (Z(t)+Z^1(t))\big[l(t,x(t)+x^1(t),y(t)+y^1(t),u(t))-l(u(t))\big]dt\\
=\,&\mathbb{E}\int_0^T Z^1(t)l(u(t))dt\\
&+\mathbb{E}\int_0^T (Z^\varepsilon(t)-Z(t)-Z^1(t))l(t,x(t)+x^1(t),y(t)+y^1(t),u^\varepsilon(t))dt\\
&+\mathbb{E}\int_0^T (Z^\varepsilon(t)-Z(t)-Z^1(t))\bullet\\
&\qquad l_x^\top(t,x(t)+x^1(t),y(t)+y^1(t),u^\varepsilon(t))(x^\varepsilon(t)-x(t)-x^1(t))dt\\
&+\mathbb{E}\int_0^T (Z(t)+Z^1(t))\bullet\\
&\qquad \big[l_x^\top(t,x(t)+x^1(t),y(t)+y^1(t),u^\varepsilon(t))(x^\varepsilon(t)-x(t)-x^1(t))\\
&\qquad +l_y^\top(t,x(t)+x^1(t),y(t)+y^1(t),u^\varepsilon(t))(y^\varepsilon(t)-y(t)-y^1(t))\big]dt\\
&+\mathbb{E}\int_0^T (Z(t)+Z^1(t))\big[(l_x^\top(u^\varepsilon(t)-l_x^\top(u(t))x^1(t)\\
&\qquad +(l_y^\top(u^\varepsilon(t)-l_y^\top(u(t))y^1(t)+l(u^\varepsilon(t))-l(u(t))\big]dt\\
&+\mathbb{E}\int_0^T (Z(t)+Z^1(t))\big[l_x^\top(u(t))x^1(t)+l_y^\top(u(t))y^1(t)\big]dt\\
=\,&\mathbb{E}\int_0^T \big[Z^1(t)l(u(t))+Z(t)(l_x^\top(u(t))x^1(t)+l_y^\top(u(t))y^1(t)\\
&\qquad +l(u^\varepsilon(t))-l(u(t)))\big]dt+o(\varepsilon).
\end{aligned}
$$

So we obtain (in fact a variational inequality of another form)

$$
\begin{aligned}
\mathbb{E}\bigg[\int_0^T &\big[Z^1(t)l(u(t))+Z(t)l_x^\top(u(t))x^1(t)+Z(t)l_y^\top(u(t))y^1(t)+Z(t)(l(u^\varepsilon(t))\\
&-l(u(t)))\big]dt+Z^1(T)\Phi(x(T))+Z(T)\Phi_x^\top(x(T))x^1(T)\\
&+\gamma_y^\top(y(0))y^1(0)\bigg]\ge o(\varepsilon). \qquad (47)
\end{aligned}
$$

Applying Itô's formula, we can get (see Baras et al. [4])

$$Z^1(t) = Z(t) \int_0^t \langle h_x(u(s))x^1(s) + h_y(u(s))y^1(s) + h(u^\varepsilon(s)) - h(u(s)), d\tilde{W}(s)\rangle.$$

While by (3), we can get

$$dZ^{-1}(t) = -Z^{-1}(t)\langle h(u(t)), dY(t)\rangle + Z^{-1}(t)|h(u(t))|^2 dt.$$

Applying Itô's formula to $\zeta(\cdot) := Z^{-1}(\cdot)Z^1(\cdot)$, we can easily confirm (46) holds. Then from (47), the variational inequality (45) holds. The proof is complete. □

2.3 Adjoint Equations and Partially-Observed Stochastic Maximum Principle

In order to obtain the partially-observed stochastic maximum principle, in this subsection we first eliminate the variational processes $(x^1(\cdot), y^1(\cdot), z^1(\cdot))$ and the auxiliary process $\zeta(\cdot)$ from the variational inequality (45), then we employ Itô's formula to obtain the necessary condition of optimality.

To deal with the stochastic process $\zeta(\cdot) \in \mathfrak{R}$, we introduce the following auxiliary BSDE:

$$\begin{aligned} -dP(t) &= l(u(t))dt - \langle Q(t), d\tilde{W}(t)\rangle, \\ P(T) &= \Phi(x(T)). \end{aligned} \tag{48}$$

By (H1) and (H3), we can easily verify that (48) admits a unique solution $(P(\cdot), Q(\cdot))$.

We introduce the following adjoint equations:

$$\begin{aligned} dp(t) &= [f_y^\top(u(t))p(t) - b_y^\top(u(t))q(t) - \sigma_y^\top(t)k(t) \\ &\quad - h_y^\top(u(t))Q(t) - l_y(u(t))]dt \\ &\quad + [f_z^\top(u(t))p(t) - b_z^\top(u(t))q(t) - \sigma_z^\top(t)k(t))Q(t)]dW(t), \\ -dq(t) &= [-f_x^\top(u(t))p(t) + b_x^\top(u(t))q(t) + \sigma_x^\top(t)k(t) \\ &\quad + h_x^\top(u(t))Q(t) + l_x(u(t))]dt - k(t)dW(t), \\ p(0) &= -\gamma_y(y(0)), \quad q(T) = -g_x^\top(x(T))p(T) + \Phi_x(x(T)). \end{aligned} \tag{49}$$

Similarly, by (H1), (H2′), (H3), we can easily verify that (49) admits a unique solution $(p(\cdot), q(\cdot), k(\cdot))$.

We define the Hamiltonian function $H : [0,T] \times \mathfrak{R}^n \times \mathfrak{R}^m \times \mathfrak{R}^{m\times d} \times U \times \mathfrak{R}^m \times \mathfrak{R}^n \times \mathfrak{R}^{n\times d} \times \mathfrak{R}^r \to \mathfrak{R}$ as

$$\begin{aligned} H(t,x,y,z,v,p,q,k,Q) := {}& \langle q, b(t,x,y,z,v)\rangle - \langle p, f(t,x,y,z,v)\rangle \\ & + \mathrm{tr}\{k^\top \sigma(t,x,y,z)\} + \langle Q, h(t,x,y,v)\rangle \\ & + l(t,x,y,v). \end{aligned} \tag{50}$$

Then (49) can be written as the following stochastic Hamiltonian system's type:

$$
\begin{aligned}
dp(t) &= -H_y(u(t))dt - H_z(u(t))dW(t), \\
-dq(t) &= H_x(u(t))dt - k(t)dW(t), \\
p(0) &= -\gamma_y(y(0)), \qquad q(T) = -g_x^\top(x(T))p(T) + \Phi_x(x(T)),
\end{aligned} \tag{51}
$$

where $H_x(u(t)) := H_x(t, x(t), y(t), z(t), u(t), p(t), q(t), k(t), Q(t))$, etc.

The main result of this paper is the following theorem.

Theorem 2.1 (Partially-Observed Stochastic Maximum Principle) *We suppose* (H1)$\sim$(H3) *hold. Let* $u(\cdot)$ *be an optimal control for our partially-observed optimal control problem* (5), $(x(\cdot), y(\cdot), z(\cdot))$ *be the optimal trajectory and* $Z(\cdot)$ *be the corresponding solution of* (3). *Let* $(P(\cdot), Q(\cdot))$ *be the solution of* (48) *and* $(p(\cdot), q(\cdot), k(\cdot))$ *be the solution of adjoint equation* (49). *Then we have*

$$
\begin{aligned}
&\mathbb{E}^u\big[H(t, x(t), y(t), z(t), v, p(t), q(t), k(t), Q(t)) \\
&\quad - H(t, x(t), y(t), z(t), u(t), p(t), q(t), k(t), Q(t))\big|\mathcal{F}_t^Y\big] \geq 0, \\
&\quad \forall v \in U, \quad a.e. t \in [0, T], a.s.,
\end{aligned} \tag{52}
$$

where the Hamiltonian function H *is defined by* (50).

Proof Applying Itô's formula to $\langle x^1(\cdot), q(\cdot)\rangle + \langle y^1(\cdot), p(\cdot)\rangle + \zeta(\cdot)P(\cdot)$. By the variational equations (9), (10), variational inequality (45), auxiliary BSDE (48) and adjoint equation (49), we get

$$
\begin{aligned}
&\mathbb{E}^u\bigg[\int_0^T \zeta(t)l(u(t)) + l_x^\top(u(t))x^1(t) + l_y^\top(u(t))y^1(t) + l(u^\varepsilon(t)) - l(u(t)) \\
&\qquad + \zeta(T)\Phi(x(T)) + \Phi_x^\top(x(T))x^1(T) + \gamma_y^\top(y(0))y^1(0)\bigg] \\
&= \mathbb{E}^u \int_0^T \big[\langle q(t), b(u^\varepsilon(t)) - b(u(t))\rangle - \langle p(t), f(u^\varepsilon(t)) - f(u(t))\rangle \\
&\qquad + \langle Q(t), h(u^\varepsilon(t)) - h(u(t))\rangle + l(u^\varepsilon(t)) - l(u(t))\big]dt \\
&= \mathbb{E}^u \int_0^T \big[H(t, x(t), y(t), z(t), u^\varepsilon(t), p(t), q(t), k(t), Q(t)) \\
&\qquad - H(t, x(t), y(t), z(t), u(t), p(t), q(t), k(t), Q(t))\big]dt \geq o(\varepsilon).
\end{aligned}
$$

From the definition of $u^\varepsilon(\cdot)$, we have

$$
\begin{aligned}
&\mathbb{E}^u \int_\tau^{\tau+\varepsilon} \big[H(t, x(t), y(t), z(t), v, p(t), q(t), k(t), Q(t)) \\
&\qquad - H(t, x(t), y(t), z(t), u(t), p(t), q(t), k(t), Q(t))\big]dt \geq o(\varepsilon).
\end{aligned}
$$

Dividing by ε, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}^u\int_\tau^{\tau+\varepsilon}&\big[H(t,x(t),y(t),z(t),v,p(t),q(t),k(t),Q(t))\\&-H(t,x(t),y(t),z(t),u(t),p(t),q(t),k(t),Q(t))\big]dt\geq 0.\end{aligned}$$

This implies that

$$\begin{aligned}&\mathbb{E}^u\big[H(\tau,x(\tau),y(\tau),z(\tau),v,p(\tau),q(\tau),k(\tau),Q(\tau))\\&\quad-H(\tau,x(\tau),y(\tau),z(\tau),u(\tau),p(\tau),q(\tau),k(\tau),Q(\tau))\big]\geq 0,\quad a.e. \tau\in[0,T].\end{aligned}$$

Now, let $a\in U$ be a deterministic element and F be an arbitrary element of the σ-algebra $\mathcal{F}_t^Y$. And set

$$w(t)=a\mathbf{1}_F+u(t)\mathbf{1}_{\Omega-F}.$$

It is obvious that $w(\cdot)$ is an admissible control.

Since $0\leq\tau\leq T$, then for every bounded U-valued, $\mathcal{F}_t^Y$-measurable random variable v such that $\sup_{\omega\in\Omega}|v(\omega)|<+\infty$, we get

$$\begin{aligned}&\mathbb{E}^u\big[H(t,x(t),y(t),z(t),v,p(t),q(t),k(t),Q(t))\\&\quad-H(t,x(t),y(t),z(t),u(t),p(t),q(t),k(t),Q(t))\big]\geq 0,\quad a.e. t\in[0,T].\end{aligned}$$

Applying the above inequality to $w(\cdot)$, we obtain

$$\begin{aligned}&\mathbb{E}^u\big[\mathbf{1}_F\big(H(t,x(t),y(t),z(t),a,p(t),q(t),k(t),Q(t))\\&\quad-H(t,x(t),y(t),z(t),u(t),p(t),q(t),k(t),Q(t))\big)\big]\geq 0,\\&\forall F\in\mathcal{F}_t^Y,\quad a.e. t\in[0,T],\end{aligned}$$

which implies

$$\begin{aligned}&\mathbb{E}^u\big[H(t,x(t),y(t),z(t),v,p(t),q(t),k(t),Q(t))\\&\quad-H(t,x(t),y(t),z(t),u(t),p(t),q(t),k(t),Q(t))\big|\mathcal{F}_t^Y\big]\geq 0,\\&a.e. t\in[0,T], a.s.\end{aligned}$$

Then (52) holds. The proof is complete. □

3 Applications: Linear-Quadratic Case

In this section, we give an LQ example to illustrate our theoretical result in Sect. 2. Let us consider the following stochastic control system ($n=m=1$):

$$\begin{aligned}dx^v(t)&=[A(t)x^v(t)-by^v(t)+C(t)v(t)]dt+D(t)dW(t),\\-dy^v(t)&=[ax^v(t)+A(t)y^v(t)+E(t)v(t)]dt-z^v(t)dW(t),\quad t\in[0,T],\end{aligned}$$

$$x^v(0)=x_0,\quad y^v(T)=cx^v(T),\tag{53}$$

and observation

$$dY(t)=F(t)dt+d\tilde{W}(t),\quad t\in[0,T],\quad Y(0)=0.\tag{54}$$

The cost functional is

$$J(v(\cdot))=\frac{1}{2}\mathbb{E}^v\left[\int_0^T L(t)v^2(t)dt+M_T(x^v(T))^2+N_0(y^v(0))^2\right],\tag{55}$$

where constants $a>0$, $b\ge 0$, $c>0$, $L(\cdot)>0$, $M_T\ge 0$, $N_0\ge 0$. Functions $A(\cdot)$, $C(\cdot)$, $E(\cdot)$, $F(\cdot)$ are bounded and deterministic, $L^{-1}(\cdot)$ is also bounded. $(W(\cdot),Y(\cdot))$ is a two-dimensional standard Brownian motion on $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$. By (54), we know $(W(\cdot),\tilde{W}(\cdot))$ is also a two-dimensional standard Brownian motion defined on another new probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P}^v)$, here $\mathbb{P}^v$ is a new probability measure just as we have defined in Sect. 2.

For any given $v(\cdot)$, it is easy to show that the G-monotonic condition (H2) holds. Then FBSDE (53) admits a unique solution $(x^v(\cdot),y^v(\cdot),z^v(\cdot))$. By (50), the Hamiltonian function is given by

$$\begin{aligned}H(t,x,y,z,v,p,k,Q)&=q\big(A(t)x-by+C(t)v\big)-p\big(ax+A(t)y+E(t)v\big)\\&\quad+kD(t)+QF(t)+\frac{1}{2}L(t)v^2.\end{aligned}\tag{56}$$

According to Theorem 2.1, if $u(\cdot)$ is optimal, then

$$u(t)=-L^{-1}(t)\Big(C(t)\mathbb{E}^u\big[q(t)|\mathcal{F}_t^Y\big]-E(t)\mathbb{E}^u\big[p(t)|\mathcal{F}_t^Y\big]\Big),\quad 0\le t\le T,\tag{57}$$

where $(p(\cdot),q(\cdot))$ is the solution of the following FBSDE:

$$\begin{aligned}dp(t)&=[A(t)p(t)+bq(t)]dt,\\-dq(t)&=[-ap(t)+A(t)q(t)]dt-k(t)dW(t),\quad 0\le t\le T,\\p(0)&=-N_0y^u(0),\quad q(T)=-cp(T)+M_Tx^u(T).\end{aligned}\tag{58}$$

Similarly, it is easy to verify that the G-monotonic condition (H2)' holds, then FBSDE (58) admits a unique solution $(p(\cdot),q(\cdot),k(\cdot))$.

Moreover, we can prove that the admissible control (57) which satisfying the necessary condition of optimality is really optimal. Noting (54), $\mathbb{E}^v$ and $\mathbb{E}^u$ are equivalent. Then for any admissible control $v(\cdot)$, we have

$$\begin{aligned}J(v(\cdot))-J(u(\cdot))&=\frac{1}{2}\mathbb{E}^u\bigg\{\int_0^T\big[L(t)(v(t)-u(t))^2+2L(t)u(t)(v(t)-u(t))\big]dt\\&\quad+M_T(x^v(T)-x^u(T))^2+2M_Tx^u(T)(x^v(T)-x^u(T))\\&\quad+N_0(y^v(0)-y^u(0))^2+2N_0y^u(0)(y^v(0)-y^u(0))\bigg\}.\end{aligned}$$

Applying Ito's formula to $(x^v(t)-x^u(t))q(t)+(y^v(t)-y^u(t))p(t)$, noting that (53), (58), we have

$$\begin{aligned}&\mathbb{E}^u[M_T x^u(T)(x^v(T)-x^u(T))+N_0 y^u(0)(y^v(0)-y^u(0))]\\&\quad=\mathbb{E}^u\int_0^T\big[(C(t)q(t)-E(t)p(t))(v(t)-u(t))\big]dt.\end{aligned}$$

As $L(t)>0$, $\forall t\in[0,T]$, $M_T\geq 0$, $N_0\geq 0$, noting that $(p(\cdot),q(\cdot))$ are not observable, we have

$$\begin{aligned}&J(v(\cdot))-J(u(\cdot))\\&\quad\geq\mathbb{E}^u\bigg\{\int_0^T L(t)u(t)(v(t)-u(t))dt+M_T x^u(T)(x^v(T)-x^u(T))\\&\qquad\qquad+N_0 y^u(0)(y^v(0)-y^u(0))\bigg\}\\&\quad=\mathbb{E}^u\int_0^T\big[\big(L(t)u(t)+C(t)q(t)-E(t)p(t)\big)(v(t)-u(t))\big]dt\\&\quad=\mathbb{E}^u\int_0^T\big[\big(C(t)q(t)-E(t)p(t)\big)(v(t)-u(t))\\&\qquad\qquad-L(t)L^{-1}(t)(C(t)\mathbb{E}^u\big[q(t)|\mathcal{F}_t^Y\big]-E(t)\mathbb{E}^u\big[p(t)|\mathcal{F}_t^Y\big])(v(t)-u(t))\big]dt\\&\quad=0.\end{aligned}$$

So (57) is an optimal control.

We want to obtain an explicit observable optimal control from (57) and the filtering estimates for corresponding optimal trajectories. For this purpose, noting the terminal condition of (58), we can get

$$q(t)=\Pi(t)p(t)-\pi(t),\quad \forall t\in[0,T],\tag{59}$$

where $\Pi(\cdot)$ satisfies the following Riccati type equation:

$$\begin{aligned}\dot{\Pi}(t)+2A(t)\Pi(t)+b\Pi^2(t)-a=0,&\quad 0\leq t\leq T,\\ \Pi(T)=-c,&\end{aligned}\tag{60}$$

which exists a unique solution from the classical Riccati equation theory and $(\pi(\cdot),\beta(\cdot))$ is the solution of the following BSDE:

$$\begin{aligned}&-d\pi(t)=[b\Pi(t)+A(t)]\pi(t)dt-\beta(t)dW(t),\quad 0\leq t\leq T,\\&\pi(T)=M_T x^u(T).\end{aligned}\tag{61}$$

Next, we know that

$$dp(t)=[(A(t)+b\Pi(t))p(t)-b\pi(t)]dt,\quad 0\leq t\leq T,$$

$$p(0) = -N_0 y^u(0), \tag{62}$$

and $k(t) = -\beta(t), a.e. t \in [0,T]$. The remaining task is to compute the filtering estimates of $p(\cdot)$ and $\pi(\cdot)$ under $\mathcal{F}_t^Y$:

$$\hat{p}(t) := \mathbb{E}^u[p(t)|\mathcal{F}_t^Y], \qquad \hat{\pi}(t) := \mathbb{E}^u[\pi(t)|\mathcal{F}_t^Y], \quad 0 \le t \le T.$$

Then by (57) and (59), we have the observable optimal control

$$u(t) = -L^{-1}(t)\big\{[C(t)\Pi(t) - E(t)]\hat{p}(t) - C(t)\hat{\pi}(t)\big\}, \quad 0 \le t \le T. \tag{63}$$

In fact, taking conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_t^Y]$ on both sides of (62), we have

$$\begin{aligned} &d\hat{p}(t) = [(A(t) + b\Pi(t))\hat{p}(t) - b\hat{\pi}(t)]dt, \quad 0 \le t \le T, \\ &\hat{p}(0) = -N_0 \hat{y}^u(0). \end{aligned} \tag{64}$$

Similarly, noting $W(\cdot)$ and $\tilde{W}(\cdot)$ are independent, from (61) we have

$$\begin{aligned} -d\hat{\pi}(t) &= [b\Pi(t) + A(t)]\hat{\pi}(t)dt, \quad 0 \le t \le T, \\ \hat{\pi}(T) &= M_T \hat{x}^u(T), \end{aligned} \tag{65}$$

where we define the filtering estimates of optimal trajectories under $\mathcal{F}_t^Y$ by

$$\hat{x}^u(t) := \mathbb{E}^u[x^u(t)|\mathcal{F}_t^Y], \hat{y}^u(t) := \mathbb{E}^u[y^u(t)|\mathcal{F}_t^Y], \hat{z}^u(t) := \mathbb{E}^u[z^u(t)|\mathcal{F}_t^Y], \quad \forall 0 \le t \le T.$$

Put (63) in (53) and taking $\mathbb{E}[\cdot|\mathcal{F}_t^Y]$ on both sides of it, we have

$$\begin{aligned} d\hat{x}^u(t) &= \big[A(t)\hat{x}^u(t) - b\hat{y}^u(t) + \Phi^1(t)\hat{p}(t) + \Psi^1(t)\hat{\pi}(t)\big]dt, \\ -d\hat{y}^u(t) &= \big[a\hat{x}^u(t) + A(t)\hat{y}^u(t) + \Phi^2(t)\hat{p}(t) + \Psi^2(t)\hat{\pi}(t)\big]dt, \quad 0 \le t \le T, \\ \hat{x}^u(0) &= x_0, \quad \hat{y}^u(T) = c\hat{x}^u(T), \end{aligned} \tag{66}$$

where for $t \in [0,T]$, we denote

$$\begin{aligned} \Phi^1(t) &\equiv -C(t)L^{-1}(t)[C(t)\Pi(t) - E(t)], \qquad \Psi^1(t) \equiv C(t)L^{-1}(t)C(t), \\ \Phi^2(t) &\equiv -E(t)L^{-1}(t)[C(t)\Pi(t) - E(t)], \qquad \Psi^2(t) \equiv E(t)L^{-1}(t)C(t). \end{aligned}$$

Combining (64), (65) with (66), we have the following

$$\begin{aligned} d\hat{p}(t) &= [(A(t) + b\Pi(t))\hat{p}(t) - b\hat{\pi}(t)]dt, \\ d\hat{x}^u(t) &= \big[A(t)\hat{x}^u(t) - b\hat{y}^u(t) + \Phi^1(t)\hat{p}(t) + \Psi^1(t)\hat{\pi}(t)\big]dt, \\ -d\hat{\pi}(t) &= [b\Pi(t) + A(t)]\hat{\pi}(t)dt, \\ -d\hat{y}^u(t) &= \big[a\hat{x}^u(t) + A(t)\hat{y}^u(t) + \Phi^2(t)\hat{p}(t) + \Psi^2(t)\hat{\pi}(t)\big]dt, \quad 0 \le t \le T, \\ \hat{p}(0) &= -N_0\hat{y}^u(0), \quad \hat{x}^u(0) = x_0, \\ \hat{\pi}(T) &= M_T\hat{x}^u(T), \quad \hat{y}^u(T) = c\hat{x}^u(T), \end{aligned} \tag{67}$$

or equivalently, the following forward-backward ordinary differential equation with double dimensions (DFBODE for short)

$$\begin{aligned} d\begin{pmatrix}\hat{p}\\ \hat{x}^u\end{pmatrix}(t) &= \left[\begin{pmatrix}A(t)+b\Pi(t) & 0\\ \Phi^1(t) & A(t)\end{pmatrix}\begin{pmatrix}\hat{p}\\ \hat{x}^u\end{pmatrix}(t)\right.\\ &\qquad \left. + \begin{pmatrix}-b & 0\\ \Psi^1(t) & -b\end{pmatrix}\begin{pmatrix}\hat{\pi}\\ \hat{y}^u\end{pmatrix}(t)\right]dt,\\ -d\begin{pmatrix}\hat{\pi}\\ \hat{y}^u\end{pmatrix}(t) &= \left[\begin{pmatrix}0 & 0\\ \Phi^2(t) & a\end{pmatrix}\begin{pmatrix}\hat{p}\\ \hat{x}^u\end{pmatrix}(t)\right.\\ &\qquad \left. + \begin{pmatrix}b\Pi(t)+A(t) & 0\\ \Psi^2(t) & A(t)\end{pmatrix}\begin{pmatrix}\hat{\pi}\\ \hat{y}^u\end{pmatrix}(t)\right]dt,\\ \begin{pmatrix}\hat{p}\\ \hat{x}^u\end{pmatrix}(0) &= \begin{pmatrix}0 & -N_0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\hat{\pi}\\ \hat{y}^u\end{pmatrix}(0) + \begin{pmatrix}0\\ x_0\end{pmatrix},\\ \begin{pmatrix}\hat{\pi}\\ \hat{y}^u\end{pmatrix}(T) &= \begin{pmatrix}0 & M_T\\ 0 & c\end{pmatrix}\begin{pmatrix}\hat{p}\\ \hat{x}^u\end{pmatrix}(T). \end{aligned} \tag{68}$$

We declare that if the DFBODE (68) admits a unique solution $(\hat{p}(\cdot), \hat{x}^u(\cdot), \hat{\pi}(\cdot), \hat{y}^u(\cdot))$, then (63) is an observable optimal control.

In fact, the above DFBODE (68) is the deterministic counterpart for forward-backward stochastic differential equation with double dimensions (DFBSDE for short). We refer to Yu [25] for general theory of this kind equations.

Finally, we can also obtain the filtering estimate $\hat{z}^u(\cdot)$. In fact, from the terminal condition of (53), we can get

$$y^u(t) = \Sigma(t)x^u(t) + \Delta(t), \quad a.e.t \in [0,T], \tag{69}$$

where $\Sigma(\cdot)$ is the solution of the following Riccati type equation

$$\begin{aligned} &\dot{\Sigma}(t) + 2A(t)\Sigma(t) - b\Sigma(t)^2 + a = 0,\\ &\Sigma(T) = c, \end{aligned} \tag{70}$$

and $\Delta(\cdot)$ satisfies

$$\begin{aligned} &\dot{\Delta}(t) + (A(t) - b\Sigma(t))\Delta(t) - (C(t)\Sigma(t) + E(t)\\ &\quad L^{-1}(t))[(C(t)\Pi(t) - E(t))\hat{p}(t) - C(t)\hat{\pi}(t)] = 0,\\ &\Delta(T) = 0. \end{aligned} \tag{71}$$

From the classical BSDE theory, it is easy to get

$$\hat{z}^u(t) \equiv z^u(t) = D(t)\Sigma(t), \quad a.e.t \in [0,T]. \tag{72}$$

To summarize, we have the following result.

Theorem 3.1 *For our partially-observed fully-coupled forward-backward LQ stochastic optimal control problem* (53)–(55), *an observable optimal control* $u(\cdot)$ *is given by* (63), *where* $\hat{p}(\cdot), \hat{\pi}(\cdot)$ *are solutions of DFBODE* (68) *and* $\Pi(\cdot)$ *is the solution of Riccati equation* (60). *Moreover, the filtering estimates of optimal trajectories* $(\hat{x}^u(\cdot), \hat{y}^u(\cdot), \hat{z}^u(\cdot))$ *are given by DFBODE* (68) *and* (72), *respectively, where* $\Sigma(\cdot)$ *is the solution of Riccati equation* (70).

Acknowledgements The authors thank the anonymous referees for helpful suggestions as well as Associate Editor F. Zirilli for efficient handling of this paper. The authors also thank Dr. G.C. Wang for helpful discussions.

References

1. Peng, S.G., Wu, Z.: Fully coupled forward-backward stochastic differential equations and applications to the optimal control. SIAM J. Control Optim. **37**(3), 825–843 (1999)
2. Bensoussan, A.: Stochastic Control of Partially Observable Systems. Cambridge University Press, Cambridge (1992)
3. Haussmann, U.G.: The maximum principle for optimal control of diffusions with partial information. SIAM J. Control Optim. **25**(2), 341–361 (1987)
4. Baras, J.S., Elliott, R.J., Kohlmann, M.: The partially-observed stochastic minimum principle. SIAM J. Control Optim. **27**(6), 1279–1292 (1989)
5. Li, X.J., Tang, S.J.: General necessary conditions for partially-observed optimal stochastic controls. J. Appl. Probab. **32**, 1118–1137 (1995)
6. Tang, S.J.: The maximum principle for partially-observed optimal control of stochastic differential equations. SIAM J. Control Optim. **36**(5), 1596–1617 (1998)
7. Wu, Z.: A maximum principle for partially observed optimal control of forward–backward stochastic control systems, accepted to be published in Sci. China Ser. F (2010)
8. Wang, G.C., Wu, Z.: The maximum principle for stochastic recursive optimal control problems under partial information. IEEE Trans. Autom. Control **54**, 1230–1242 (2009)
9. Cvitanic, J., Ma, J.: Hedging options for a large investor and forward-backward SDE's. Ann. Appl. Probab. **6**(2), 370–398 (1996)
10. Ma, J., Yong, J.M.: Forward-Backward Stochastic Differential Equations and Their Applications. Springer, Berlin (1999)
11. Antonelli, F.: Backward-Forward stochastic differential equations. Ann. Appl. Probab. **3**(3), 777–793 (1993)
12. Ma, J., Protter, P., Yong, J.M.: Solving forward-backward stochastic differential equations explicitly-a four step scheme. Probab. Theory Relat. Fields **98**, 339–359 (1994)
13. Hu, Y., Peng, S.G.: Solution of forward-backward stochastic differential equations. Probab. Theory Relat. Fields **103**, 273–283 (1995)
14. Yong, J.M.: Finding adapted solutions of forward-backward stochastic differential equations: method of continuation. Probab. Theory Relat. Fields **107**, 537–572 (1997)
15. Pardoux, E., Tang, S.J.: Forward-backward stochastic differential equations and quasilinear parabolic PDEs. Probab. Theory Relat. Fields **114**, 123–150 (1999)
16. Delarue, F.: On the Existence and Uniqueness of Solutions to FBSDEs in a Non-Degenerate Case. Stoch. Process. Appl. **99**, 209–286 (2002)
17. Zhang, J.F.: The wellposedness of FBSDEs. Discrete Contin. Dyn. Syst., Ser. B **6**, 927–940 (2006)
18. Xu, W.S.: Stochastic maximum principle for optimal control problem of forward and backward system. J. Aust. Math. Soc., Ser. B **37**, 172–185 (1995)
19. Shi, J.T., Wu, Z.: The maximum principle for fully coupled forward-backward stochastic control systems. Acta Autom. Sinica **32**, 161–169 (2006)
20. Liptser, R.S., Shiryayev, A.N.: Statistics of Random Process. Springer, New York (1977)
21. Baghery, F., Øksendal, B.: A maximum principle for stochastic control with partial information. Stoch. Anal. Appl. **25**, 705–717 (2007)
22. Meyer-Brandis, T., Øksendal, B., Zhou, X.Y.: A maximum principle via Malliavin calculus. Pure Math., Vol. 10 (2008). University of Oslo

23. Øksendal, B., Sulem, A.: Maximum principles for optimal control of forward-backward stochastic differential equations with jumps. Pure Math., Vol. 22 (2008). University of Oslo
24. Meng, Q.X.: A maximum principle for optimal control problem of fully coupled forward-backward stochastic systems with partial information. Sci. China Ser. A **52**(7), 1579–1588 (2009)
25. Yu, Z.Y.: Some backward problems in stochastic control and game theory. PhD Thesis, Shandong University (2008)