J Optim Theory Appl (2010) 146: 795–809
DOI 10.1007/s10957-010-9683-4

A Parameter-Uniform B-Spline Collocation Method for Singularly Perturbed Semilinear Reaction-Diffusion Problems

S.C.S. Rao · S. Kumar · M. Kumar

Published online: 16 April 2010

Abstract We consider a Dirichlet boundary value problem for a class of singularly perturbed semilinear reaction-diffusion equations. A B-spline collocation method on a piecewise-uniform Shishkin mesh is developed to solve such problems numerically. The convergence analysis is given and the method is shown to be almost second-order convergent, uniformly with respect to the perturbation parameter ε in the maximum norm. Numerical results are presented to validate the theoretical results.

Keywords Singular perturbation · Reaction-diffusion problems · B-spline collocation method · Shishkin mesh

1 Introduction

Consider a class of singularly perturbed semilinear reaction-diffusion problems

$$Lu := -\varepsilon^2 u'' + f(x, u) = 0, \quad x \in \ \Omega = [0, 1],\ u(0) = 0,\ u(1) = 0, \tag{1}$$

where ε is a small parameter, such that $0 < \varepsilon \ll 1$, and f is a sufficiently smooth function. In general, as ε tends to zero, the solution u of (1) may exhibit boundary or internal layers of various types. The location of the layers and the behavior of the solution u depends on the character of $f(x, u)$ [1, 2].

We consider the problem (1) with the assumption that

$$f_u(x, u) \geq f_*^2 > 0, \quad \text{for } (x, u) \in \Omega \times \mathbb{R}. \tag{2}$$

Communicated by I. Galligani.

S.C.S. Rao (✉) · S. Kumar · M. Kumar
Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi, India
e-mail: scsr@maths.iitd.ac.in

This is a standard stability condition which implies that the problem (1) and the reduced problem $f(x,u_0(x))=0$, for $x \in \Omega$, have a unique solution. Under the assumption (2) the problem (1) exhibits two exponential boundary layers, when the reduced solution u_0 does not satisfy the boundary conditions in (1).

Singular perturbation problems often arise in many areas of science and engineering such as heat transfer problem with high Peclet numbers [3], drift-diffusion equations of semiconductor device physics [4], Navier-Stokes equations of fluid flow with high Renoylds number [5], Michaelis-Menten theory for enzyme reactions [6] and the mathematical models of liquid crystal materials and chemical reactions [7]. The classical numerical methods on uniform mesh for solving such problems may give rise to difficulties when the singular perturbation parameter ε becomes small. Then the mesh needs to be refined substantially to grasp the solution within the boundary layers. To avoid this, a class of non-equidistant meshes, dense in layers, are available in the literature [8–11]. Among these meshes, the piecewise-uniform Shishkin ones gained more popularity, not only because they are simple in nature, but also because they can be easily extended to higher dimensions. A wide class of fitted numerical methods on piecewise-uniform Shishkin meshes are discussed in Miller et al. [12].

For the numerical solution of singularly perturbed semilinear reaction-diffusion problem (1), Vulanović [13] considered a central difference scheme and achieved the second order parameter-uniform convergence result on a special graded mesh of Bakhvalov type. Surla and Uzelac [14] considered the collocation method with quadratic splines on a slightly modified piecewise-uniform Shishkin type mesh and proved almost second order parameter-uniform convergence of the method. Surla [15] proposed a cubic spline difference scheme on a simple piecewise-uniform Shishkin type mesh and proved that the scheme is second order uniformly convergent with respect to small parameter ε in the discrete maximum norm. Rao and Kumar [16] derived an exponential spline difference scheme based on spline in tension on a piecewise-uniform Shishkin mesh and proved almost second order parameter-uniform convergence of the scheme.

Some spline approximation methods for the numerical solution of nonlinear singularly perturbed boundary problems are given in [17–19], and the references therein. Among the various classes of splines, the polynomial one has received a greater attention primarily because it admits a B-splines basis which can be computed efficiently. Kadalbajoo and Gupta [20] gave a B-spline collocation method on a piecewise-uniform Shishkin mesh for singularly perturbed linear convection-diffusion problem and proved the second order parameter-uniform convergence of the method. Also, they extended this method for singularly perturbed one-dimensional time dependent linear convection-diffusion problem [21]. Rao and Kumar [19] considered a B-spline collocation method for solving nonlinear singularly perturbed boundary value problem. The quasilinearization technique is used to linearize the original nonlinear singular perturbation problem into a sequence of linear singular perturbation problems and a B-spline collocation method on a piecewise uniform Shishkin mesh is developed to solve the set of linear singular perturbation problems obtained through quasilinearization.

We reformulate the problem (1) to an equivalent form

$$L_\varepsilon u := -\varepsilon^2 u'' + bu = g(x,u), \quad x \in \Omega = [0,1],\ u(0)=0,\ u(1)=0, \tag{3}$$

where b is a sufficiently smooth function, with $b(x) \geq b_* > 0$ for $x \in \Omega$ and $g(x,u) = bu - f(x,u)$, that satisfies

$$|g(x,u_1) - g(x,u_2)| \leq M|u_1 - u_2|, \quad \forall x \in \Omega \text{ (Lipschitz condition)} \tag{4}$$

such that for $K = (\min_{\forall x} 2b(x))^{-1}$,

$$1 - 6KM > 0. \tag{5}$$

In this paper, we present a B-spline collocation method on a piecewise-uniform Shishkin mesh for the numerical solution of the modified problem (3). It is interesting to note that, in all the numerical experiments, there is no difficulty in solving (1) directly by the present B-spline collocation method, and they rendered exactly the same result as the modified form (3). The reformulation of the problem (1) to (3) is solely for the theoretical purpose. Three test problems are considered to demonstrate the efficiency of the proposed B-spline collocation method.

The paper is arranged as follows. The fitted mesh technique is employed to generate a piecewise-uniform Shishkin mesh in Sect. 2. In Sect. 3, the B-spline collocation method on a piecewise-uniform Shishkin mesh is developed for the numerical solution of the problem (3). In Sect. 4, the convergence analysis is given and the method is shown to be almost second order convergent, uniformly with respect to the small parameter ε in the maximum norm. In Sect. 5, numerical experiments are conducted to validate the theoretical results. The results of the experiments are discussed in Sect. 6. Finally, the conclusions are included in Sect. 7.

Notations: Throughout the paper, we use C to denote a generic positive constant independent of ε and the discretization parameter N. For a real valued function $g \in C(\Omega)$, define $\|g\|_\Omega = \max_{x\in\Omega} |g(x)|$. For a mesh function $g_N = (g_0, \ldots, g_N)$, define $\|g_N\| = \max_{0\leq i\leq N} |g_i|$, and denote the corresponding subordinate matrix norm in the same way.

2 The Mesh

In this section, a piecewise-uniform Shishkin mesh $\Omega^N = \{x_i : i = 0, \ldots, N\}$, is constructed in such a way that more mesh points are generated in the boundary layer region than outside of it. For this, let $N = 2^k$, $k \geq 2$ be a positive integer.

Define the mesh transition parameter

$$\sigma = \min\left\{\frac{1}{4}, \frac{2\varepsilon}{f_*} \ln N\right\}.$$

We divide the given interval $[0, 1]$ into three subintervals $[0, \sigma]$, $[\sigma, 1 - \sigma]$, $[1 - \sigma, 1]$, where the subintervals $[0, \sigma]$ and $[1 - \sigma, 1]$ represent the inner regions and the subinterval $[\sigma, 1 - \sigma]$ represents the outer region. The subintervals $[0, \sigma]$ and $[1 - \sigma, 1]$ are divided into $\frac{N}{4}$ equidistant elements, and the subinterval $[\sigma, 1 - \sigma]$ is divided into $\frac{N}{2}$ equidistant elements.

Set $i_0 = \frac{N}{4}$, then $x_{i_0} = \sigma$ and $x_{N-i_0} = 1 - \sigma$ are the transition points. Let $x_i = x_{i-1} + h_i$, $\forall i = 1, \dots, N$. Then the resulting piecewise-uniform Shishkin mesh is represented as

$$\tilde{h} := \begin{cases} h_i = \frac{4\sigma}{N}, & \text{for } i = 1, \dots, i_0; \\ h_i = \frac{2(1-2\sigma)}{N}, & \text{for } i = i_0 + 1, \dots, N - i_0; \\ h_i = \frac{4\sigma}{N}, & \text{for } i = N - i_0 + 1, \dots, N. \end{cases} \tag{6}$$

Note that, if $\sigma = 1/4$, then the mesh is uniform, N^{-1} is very small with respect to ε and therefore a classical analysis could be used to prove the uniform convergence of the scheme. So, in the convergence analysis of the present method we consider only the case $\sigma = 2\varepsilon \ln N / f_*$.

3 B-Spline Collocation Method

In this section, we derive a B-spline collocation method on a piecewise-uniform Shishkin mesh for the numerical solution of singularly perturbed problem (3). Let Ω^N be a partitioning of Ω defined by

$$\Omega^N : x_0 = 0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = 1.$$

We extend the partition Ω^N by introducing $x_{-3} < x_{-2} < x_{-1}$ mesh points on the left side and $x_{N+1} < x_{N+2} < x_{N+3}$ mesh points on the right side. Then, for $i = -1, 0, \dots, N+1$, the cubic B-splines are defined by [20, 21]

$$B_i(x) = \frac{1}{\tilde{h}^3} \begin{cases} (x - x_{i-2})^3, & \text{if } x \in [x_{i-2}, x_{i-1}]; \\ \tilde{h}^3 + 3\tilde{h}^2(x - x_{i-1}) + 3\tilde{h}(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & \text{if } x \in [x_{i-1}, x_i]; \\ \tilde{h}^3 + 3\tilde{h}^2(x_{i+1} - x) + 3\tilde{h}(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & \text{if } x \in [x_i, x_{i+1}]; \\ (x_{i+2} - x)^3, & \text{if } x \in [x_{i+1}, x_{i+2}]; \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Suppose $V = \text{span}\{B_{-1}(x), B_0(x), \dots, B_N(x), B_{N+1}(x)\}$. It is well known that cubic B-splines $\{B_i(x)\}_{i=-1}^{N+1}$ are linearly independent and $\dim V = N + 3$.

We seek a function $u_N \in V$, an approximate solution to the problem (3), which may be represented as

$$u_N(x) = \sum_{i=-1}^{N+1} \alpha_i B_i(x),$$

where α_i are the unknown parameters. These parameters are determined by forcing u_N to satisfy the differential equation in (3) at the mesh points of the partition Ω^N as well as the boundary conditions at $x = x_0$ and $x = x_N$. Thus, we have

$$L_\varepsilon u_N(x_i) = g(x_i, u_N(x_i)), \quad i = 0, \dots, N, \; u_N(x_0) = 0, \; u_N(x_N) = 0, \tag{8}$$

where

$$L_\varepsilon u_N(x_i) := -\varepsilon^2 u_N''(x_i) + b(x_i)u_N(x_i).$$

Explicitly, the collocation equations in (8) are

$$\alpha_{i-1}(-\varepsilon^2 B_{i-1}''(x_i) + b(x_i)B_{i-1}(x_i)) + \alpha_i(-\varepsilon^2 B_i''(x_i) + b(x_i)B_i(x_i)) \\ + \alpha_{i+1}(-\varepsilon^2 B_{i+1}''(x_i) + b(x_i)B_{i+1}(x_i)) = g(x_i, u_N(x_i)), \quad i = 0, \ldots, N.$$

Each basis function B_i is twice continuously differentiable. Putting the values of B-spline functions B_i and their derivatives at the mesh points Ω^N, we obtain

$$\alpha_{i-1}\left(\frac{-6\varepsilon^2}{\tilde{h}^2} + b(x_i)\right) + \alpha_i\left(\frac{12\varepsilon^2}{\tilde{h}^2} + 4b(x_i)\right) + \alpha_{i+1}\left(\frac{-6\varepsilon^2}{\tilde{h}^2} + b(x_i)\right) \\ = g(x_i, u_N(x_i)), \quad i = 0, \ldots, N \tag{9}$$

and the boundary conditions in (8) become

$$\alpha_{-1} + 4\alpha_0 + \alpha_1 = 0, \qquad \alpha_{N-1} + 4\alpha_N + \alpha_{N+1} = 0. \tag{10}$$

On eliminating α_{-1} and α_{N+1}, the resulting system (9)–(10) can be represented as

$$A\alpha = g(\alpha), \tag{11}$$

where A is an $(N+1) \times (N+1)$ tridiagonal matrix, α is an $N+1$-dimensional column vector with components α_i and $g(\alpha)$ is the right-hand side vector of dimension $N+1$.

The elements of the tridiagonal matrix $A = (a_{ij})$ are

$$\begin{cases} a_{0,0} = \frac{36\varepsilon^2}{\tilde{h}^2}, \\ a_{i,i-1} = \frac{-6\varepsilon^2}{\tilde{h}^2} + b(x_i), & \text{for } i = 1, \ldots, N-1, \\ a_{i,i} = \frac{12\varepsilon^2}{\tilde{h}^2} + 4b(x_i), & \text{for } i = 1, \ldots, N-1, \\ a_{i,i+1} = \frac{-6\varepsilon^2}{\tilde{h}^2} + b(x_i), & \text{for } i = 1, \ldots, N-1, \\ a_{N,N} = \frac{36\varepsilon^2}{\tilde{h}^2}, \\ a_{i,j} = 0, & \forall |i-j| > 1. \end{cases}$$

For $i = 1, \ldots, N-1$, we have

$$|a_{i,i}| - |a_{i,i-1}| - |a_{i,i+1}| \geq 2b(x_i) \geq 2b_* > 0.$$

Hence the coefficient matrix A is strictly diagonally dominant. We know that, if $B = [b_{i,j}]_{N\times N}$ is strictly diagonally dominant, then [22]

$$\|B^{-1}\| \leq \left(\min_{1\leq i\leq N}\left(|b_{i,i}| - \sum_{i\neq j}|b_{i,j}|\right)\right)^{-1}.$$

Let N_0 be the smallest positive integer such that

$$\frac{2b_*}{9f_*} \le N_0^2/\ln^2 N_0.$$

Then, for any $N \ge N_0$, we have

$$\|A^{-1}\| \le \frac{1}{2b_*} \equiv K(say). \tag{12}$$

This bound on A^{-1} will be useful in the convergence analysis of the present B-spline collocation method.

4 Error Analysis

In this section, we establish the parameter-uniform convergence of the B-spline collocation method described in Sect. 3. For this we need sharp bounds on the derivatives of the exact solution u. Moreover, we require a special decomposition of the exact solution into its regular component and layer components. This is given by the following lemma.

Lemma 4.1 *The solution u to the problem (1) can be decomposed into two parts: $u = v + w$, where v is the regular component of u satisfying*

$$|v^{(j)}(x)| \le C$$

and w is the layer component satisfying

$$|w^{(j)}(x)| \le C\varepsilon^{-i}(\exp(-xf_*/\varepsilon) + \exp(-(1-x)f_*/\varepsilon)),$$

for $x \in \Omega$ and $j = 0, \ldots, 4$.

Proof See [13]. □

Theorem 4.2 *Let b, g be sufficiently smooth functions satisfying (4)–(5). Let u be the exact solution of the problem (3) and let u_N be the cubic B-spline collocation approximate solution on a piecewise-uniform Shishkin mesh. Then, for any $N \ge N_0$*

$$\|u - u_N\|_\Omega \le CN^{-2}\ln^2 N.$$

Proof Let u_N be the cubic spline collocation approximate solution of (3) given by

$$u_N(x) = \sum_{i=-1}^{N+1} \alpha_i B_i(x), \tag{13}$$

and let $\overline{u}_N$ be the unique cubic spline interpolate from the space V to the exact solution u of (3) given by

$$\overline{u}_N(x) = \sum_{i=-1}^{N+1} \overline{\alpha}_i B_i(x). \tag{14}$$

By the triangle inequality, we have

$$\|u - u_N\|_\Omega \le \|u - \overline{u}_N\|_\Omega + \|\overline{u}_N - u_N\|_\Omega. \tag{15}$$

Both terms on the right-hand side of (15) are estimated separately. First we estimate the interpolation error $\|u - \overline{u}_N\|_\Omega$. Let $\overline{y}$ be the unique cubic spline interpolant of $y \in C^4(\Omega)$. Then by the standard cubic spline interpolation error estimates [23, 24], for $x \in \Omega_i := [x_{i-1}, x_i] \subset \Omega$,

$$|(\overline{y} - y)^{(k)}(x)| \le \begin{cases} C h_i^{4-k} \|y^{(4)}\|_{\Omega_i}, \\ C\|y^{(k)}\|_{\Omega_i}, \quad k = 0, 2. \end{cases} \tag{16}$$

If Ω_i lies in the layer regions then $h_i \le C\varepsilon N^{-1} \ln N$. Furthermore, by Lemma 4.1 $\|u^{(4)}\|_\Omega \le C\varepsilon^{-4}$. Therefore, using the first interpolation error estimate of (16) with $k = 0$, we obtain

$$|(u - \overline{u}_N)(x)| \le CN^{-4} \ln^4 N \quad \text{for } x \in \Omega_i \subset [0, \sigma] \cup [1 - \sigma, 1].$$

For $x \in \Omega_i \subset [\sigma, 1 - \sigma]$, using the solution decomposition of Lemma 4.1 and the triangle inequality, we obtain

$$|(u - \overline{u}_N)(x)| \le |(v - \overline{v}_N)(x)| + |(w - \overline{w}_N)(x)|. \tag{17}$$

For the first term on the right-hand side of (17), using the first interpolation error estimate of (16) with $k = 0$, $h_i \le CN^{-1}$ and $\|v^{(4)}\|_\Omega \le C$, we get

$$|(v - \overline{v}_N)(x)| \le CN^{-4} \quad \text{for } x \in \Omega_i \subset [\sigma, 1 - \sigma].$$

For the second term on the right-hand side of (17), using the second interpolation error estimate of (16) with $k = 0$, and Lemma 4.1, we obtain

$$\begin{aligned} |(w - \overline{w}_N)(x)| &\le C\|w\|_{\Omega_i} \\ &\le C \max_{x \in [\sigma, 1-\sigma]} |(\exp(-x f_*/\varepsilon) + \exp(-(1 - x) f_*/\varepsilon))| \\ &\le C(\exp(-\sigma f_*/\varepsilon) + \exp(-(1 - (1 - \sigma)) f_*/\varepsilon)) \\ &= C(2\exp(-\sigma f_*/\varepsilon)) \le CN^{-2}. \end{aligned}$$

On collecting all the estimates we have

$$\|u - \overline{u}_N\|_\Omega \le CN^{-2}. \tag{18}$$

Now we estimate $\|\overline{u}_N - u_N\|_\Omega$. For this consider the quantities $L_\varepsilon \overline{u}_N(x_i)$, $i = 0, \dots, N$. Using interpolation error estimates (16) and the arguments that we have used to estimate $\|u - \overline{u}_N\|_\Omega$, we obtain $\|L_\varepsilon \overline{u}_N - L_\varepsilon u\|_\Omega \le C N^{-2} \ln^2 N$. At the mesh points, in particular, we write

$$L_\varepsilon \overline{u}_N(x_i) = g(x_i, \overline{u}_N(x_i)) + r(x_i), \quad i = 0, \dots, N,$$

where r is the error function with the order of magnitude $O(N^{-2} \ln^2 N)$. With the boundary conditions $\overline{u}_N(x_0) = 0$, $\overline{u}_N(x_N) = 0$, this leads to the nonlinear system

$$A\overline{\alpha} = g(\overline{\alpha}) + R, \tag{19}$$

where A is the same matrix as in (11), $\overline{\alpha}$, $g(\overline{\alpha})$ and R are vectors of dimension $(N+1)$ with components $\overline{\alpha}_i$, $g(x_i, \overline{u}_N(x_i))$ and $r(x_i)$, respectively.

Recall (11) and (19), to get

$$Ae = R + g(\overline{\alpha}) - g(\alpha), \tag{20}$$

where e is an $(N+1)$-dimensional error vector with components $e_i = \overline{\alpha_i} - \alpha_i$.

Using Lipschitz condition (4), we have

$$\begin{aligned} g(x_i, \overline{u}_N(x_i)) - g(x_i, u_N(x_i)) &= M_i(\overline{u}_N(x_i) - u_N(x_i)) \\ &= \begin{cases} M_i[B_{i-1}(x_i)e_{i-1} + B_i(x_i)e_i + B_{i+1}(x_i)e_{i+1}], & \text{for } 1 \le i \le N-1 \\ 0, & \text{for } i = 0, N \end{cases} \end{aligned} \tag{21}$$

for some constants M_i where $|M_i| \le M$, for $i = 0, \dots, N$. From (20)–(21), we write

$$Ae = R + \widetilde{M} T e, \tag{22}$$

where $\widetilde{M} = \operatorname{diag}(M_0, M_1, \dots, M_N)$, and

$$T = \begin{pmatrix} 0 & 0 & 0 & & & & & \\ B_0(x_1) & B_1(x_1) & & B_2(x_1) & & & & \\ & B_1(x_2) & & B_2(x_2) & & B_3(x_2) & & \\ & & \cdot & & \cdot & & \cdot & \\ & & & \cdot & & \cdot & & \cdot \\ & & & & \cdot & & \cdot & & \cdot \\ & & & & & B_{N-1}(x_N) & & B_N(x_N) & & B_{N+1}(x_N) \\ & & & & & 0 & & 0 & & 0 \end{pmatrix}.$$

$\widetilde{M}$ and T satisfy

$$\|\widetilde{M}\| \le M, \qquad \|T\| = 6. \tag{23}$$

Since A is strictly diagonally dominant, A^{-1} exists. Thus from (22) we have

$$e = A^{-1}R + A^{-1}\widetilde{M} T e.$$

Using (23) and (12), we obtain $\|e\| \le K\|R\| + 6KM\|e\|$.

As $(1-6KM)>0$ and $\|R\|=\max_{0\le i\le N}|r(x_i)|\le CN^{-2}\ln^2 N$, we have

$$\|e\|\le CN^{-2}\ln^2 N. \tag{24}$$

We have

$$(\overline{\alpha}_{-1}-\alpha_{-1})B_{-1}(x_0)+(\overline{\alpha}_0-\alpha_0)B_0(x_0)+(\overline{\alpha}_1-\alpha_1)B_1(x_0)=0,$$

$$(\overline{\alpha}_{N-1}-\alpha_{N-1})B_{N-1}(x_N)+(\overline{\alpha}_N-\alpha_N)B_N(x_N)+(\overline{\alpha}_{N+1}-\alpha_{N+1})B_{N+1}(x_N)=0.$$

Using (24), we get

$$|\overline{\alpha}_{-1}-\alpha_{-1}|\le CN^{-2}\ln^2 N,\qquad |\overline{\alpha}_{N+1}-\alpha_{N+1}|\le CN^{-2}\ln^2 N.$$

Hence

$$\max_{-1\le i\le N+1}|\overline{\alpha}_i-\alpha_i|\le CN^{-2}\ln^2 N. \tag{25}$$

From (13)–(14), we have

$$|\overline{u}_N(x)-u_N(x)|\le \max_{-1\le i\le N+1}|\overline{\alpha}_i-\alpha_i|\sum_{i=-1}^{N+1}|B_i(x)|. \tag{26}$$

The cubic B-splines $\{B_{-1}(x),B_0(x),\dots,B_N(x),B_{N+1}(x)\}$ satisfy the following inequality [20, 21]

$$\sum_{i=-1}^{N+1}|B_i(x)|\le 10,\quad 0\le x\le 1. \tag{27}$$

Thus from (25)–(27), we get

$$\|\overline{u}_N-u_N\|_\Omega\le CN^{-2}\ln^2 N. \tag{28}$$

Thus from (15), (18) and (28), we have

$$\|u-u_N\|_\Omega\le CN^{-2}\ln^2 N.$$

This completes the proof. □

5 Numerical Results

In this section, the present B-spline collocation method on a piecewise-uniform Shishkin mesh is implemented on three test problems. The maximum pointwise errors E_ε^N and the numerical rates of convergence p_ε^N are computed for different values of ε and N. Numerical results of the present method are compared with the results obtained with uniform mesh.

Example 5.1 Consider the singularly perturbed problem [25]

$$-\varepsilon^2 u'' + (1 + x^2 + \cos(x))u - x^{9/2} - \sin(x) = 0, \quad \text{in } \Omega,\ u(0) = 1,\ u(1) = 1.$$

We rewrite the problem to an equivalent form

$$-\varepsilon^2 u'' + (1 + x^2 + \cos(x))u = x^{9/2} + \sin(x), \quad \text{in } \Omega,\ u(0) = 1,\ u(1) = 1.$$

Example 5.2 Consider the singularly perturbed problem [14]

$$-\varepsilon^2 u'' + (1 + u)(1 + (1 + u)^2) = 0, \quad \text{in } \Omega,\ u(0) = 0,\ u(1) = 0.$$

We rewrite the problem to an equivalent form

$$-\varepsilon^2 u'' + 1.4u = -1 + 0.4u - (1 + u)^3, \quad \text{in } \Omega,\ u(0) = 0,\ u(1) = 0.$$

Example 5.3 Consider the singularly perturbed problem [26]

$$-\varepsilon^2 u'' + \frac{u - 4}{5 - u} = 0, \quad \text{in } \Omega,\ u(0) = 0,\ u(1) = 0.$$

We rewrite the problem to an equivalent form

$$-\varepsilon^2 u'' + 2u = 2u - \frac{u - 4}{5 - u}, \quad \text{in } \Omega,\ u(0) = 0,\ u(1) = 0.$$

6 Discussion

The cubic B-spline collocation method on a piecewise-uniform Shishkin mesh for the numerical solution of singularly perturbed semilinear reaction-diffusion problem is presented, that is much easier and more efficient for computing than other schemes available in the literature. The original problem (1) is reformulated to an equivalent problem (3). A B-spline collocation method for the modified problem (3) on a piecewise-uniform Shishkin mesh is proposed. It is interesting to note that, in all the numerical experiments, there is no difficulty in solving (1) directly by the present B-spline collocation method, and they rendered exactly the same result as the modified form (3). The reformulation of the problem (1) to (3) is solely for the theoretical purpose. The essential idea in this method is to use the cubic B-spline basis on a piecewise-uniform Shishkin mesh to approximate the solution of the modified problem (3) via collocation approach. The cubic B-spline basis function defined in Sect. 3 has a finite support on the four consecutive intervals $[x_{i+jh}, x_{i+(j+1)h}]_{j=-2}^{1}$, and results in a tridiagonal system which can be solved using the standard algorithm.

The proposed method is implemented on three test problems to demonstrate its efficiency. To solve the corresponding nonlinear systems, the Newton's method is used with the initial guess $u_N^{(0)} = (0, u_0(x_1), \ldots, u_0(x_{N-1}), 0)^T$, where u_0 is the solution of the reduced problem. The stopping criterion is $\|u_N^{(k)} - u_N^{(k-1)}\| < 10^{-12}$. Here $u_N^{(k)}$,

Table 1 Errors E_ε^N for Example 5.1 with uniform mesh

$\varepsilon = 10^{-K}$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$K = 0$	2.13e−06	5.34e−07	1.33e−07	3.34e−08	8.34e−09
1	6.00e−05	1.50e−05	3.75e−06	9.37e−07	2.34e−07
2	5.61e−04	1.40e−04	3.51e−05	8.76e−06	2.19e−06
3	5.63e−03	1.42e−03	3.51e−04	8.77e−05	2.19e−05
4	6.08e−02	1.61e−02	3.61e−03	8.80e−04	2.19e−04
5	2.27e−01	1.18e−01	4.03e−02	9.54e−003	2.22e−03
10	3.39e−01	3.39e−01	3.39e−01	3.39e−01	3.39e−01
15	3.39e−01	3.39e−01	3.39e−01	3.39e−01	3.39e−01

for $k = 1, 2, \dots,$ represent the successive approximates to u_N computed iteratively. For each N and ε in the tables, it takes only about 5 iterations to satisfy this criterion.

As the exact solution of the test problem is not known, the double mesh method is used to compute the numerical rate of convergence. For this we compute not only u_N, but also another approximate solution $\tilde{u}_N$ to the problem (3) on the mesh $\widetilde{\Omega}^N$ with a slightly altered mesh parameter $\tilde{\sigma}$, where

$$\tilde{\sigma} = \min\left\{\frac{1}{4}, \frac{2\varepsilon}{f_*}\ln(N/2)\right\}.$$

Here the altered mesh parameter is used such that the ith mesh point of the mesh Ω^N coincides with the $(2i)$th mesh point of the mesh $\widetilde{\Omega}^{2N}$.

Assuming the convergence of order $(N^{-1}\ln N)^p$ for some p, the numerical rate of convergence p_ε^N, for each fixed ε, is calculated by

$$p_\varepsilon^N = \frac{\ln(E_\varepsilon^N) - \ln(E_\varepsilon^{2N})}{\ln(2\ln N) - \ln(\ln(2N))},$$

where

$$E_\varepsilon^N = \max_{0\le i\le N} |(u_N)_i - (\tilde{u}_{2N})_{2i}|.$$

The parameter-uniform numerical rate of convergence p^N is calculated by

$$p^N = \frac{\ln(E^N) - \ln(E^{2N})}{\ln(2\ln N) - \ln(\ln(2N))},$$

where $E^N = \max_{\forall\varepsilon} E_\varepsilon^N$.

For the different values of ε and N, Tables 1, 2 and 3 represent the errors E_ε^N of the present B-spline collocation method on a uniform mesh for Examples 5.1, 5.2 and 5.3, respectively. Numerical results given in Tables 1, 2 and 3, clearly show that the proposed method on a uniform mesh is not uniformly convergent. For the small values of ε, the error E_ε^N does not decrease as the mesh points increase. This shows that the method is not behaving properly with respect to parameter ε. This can be

Table 2 Errors E_ε^N for Example 5.2 with uniform mesh

$\varepsilon = 10^{-K}$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$K = 0$	7.25e−06	1.81e−06	4.53e−07	1.13e−07	2.83e−08
1	8.97e−05	2.24e−05	5.95e−06	1.40e−06	3.50e−07
2	8.49e−04	2.10e−04	5.24e−05	1.31e−05	3.27e−06
3	1.02e−02	2.20e−03	5.28e−04	1.31e−04	3.27e−05
4	1.15e−01	2.93e−02	5.80e−03	1.30e−03	3.29e−04
5	3.62e−01	2.12e−01	7.58e−02	1.73e−02	3.50e−03
10	5.88e−01	5.88e−01	5.88e−01	5.87e−01	5.86e−01
15	5.88e−01	5.88e−01	5.88e−01	5.88e−01	5.88e−01

Table 3 Errors E_ε^N for Example 5.3 with uniform mesh

$\varepsilon = 10^{-K}$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$K = 0$	6.17e−08	1.54e−08	3.86e−09	9.65e−10	2.41e−10
1	6.78e−06	1.69e−06	4.24e−07	1.06e−07	2.65e−08
2	1.70e−04	4.24e−05	1.06e−05	2.65e−06	6.63e−07
3	8.41e−04	2.14e−04	5.35e−05	1.34e−05	3.35e−06
4	7.90e−03	2.10e−03	5.32e−04	1.34e−04	3.34e−05
5	8.00e−03	1.86e−02	4.70e−03	1.30e−03	3.32e−04
10	2.38e−01	2.37e−01	2.37e−01	2.37e−01	2.36e−01
15	2.38e−01	2.38e−01	2.38e−01	2.38e−01	2.38e−01

avoided by considering a special class of piecewise-uniform mesh known as Shishkin mesh. For the different values of ε and N, Tables 4, 5 and 6 represent the errors E_ε^N and the numerical rates of convergence p_ε^N of the present B-spline collocation method on a piecewise-uniform Shishkin mesh for Examples 5.1, 5.2 and 5.3, respectively. From Tables 4, 5 and 6, it can be observed that, for fixed value of ε, the errors E_ε^N decrease as the mesh points increase, and the rate of convergence is almost two. The last row in each of the tables (Tables 4, 5 and 6) represents the parameter-uniform numerical rate of convergence p^N of the present method.

7 Conclusions

A Dirichlet boundary value problem for a class of singularly perturbed semilinear reaction-diffusion equations is considered. To solve such problems numerically, a B-spline collocation method on a piecewise-uniform Shishkin meshes is developed. The error analysis is given and the method is shown to be almost second order parameter-uniform in nature. Numerical results are given in support of the theoretical results. Furthermore, this method ensures that the solution is, at least, continuous in the domain Ω, whereas the finite difference methods give the solution only at the chosen mesh points.

Table 4 Errors E_ε^N and rates p_ε^N for Example 5.1 with piecewise-uniform Shishkin mesh

$\varepsilon = 10^{-K}$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$K = 0$	2.13e−06	5.34e−07	1.33e−07	3.34e−08	8.34e−09
	2.57	2.48	2.41	2.36	
1	6.00e−05	1.50e−05	3.75e−06	9.37e−07	2.34e−07
	2.57	2.48	2.41	2.36	
2	5.61e−04	1.40e−04	3.51e−05	8.76e−06	2.19e−06
	2.57	2.48	2.41	2.36	
3	3.20e−03	1.06e−03	3.46e−04	8.77e−05	2.19e−05
	2.04	2.01	2.39	2.36	
4	3.20e−03	1.06e−03	3.46e−04	1.09e−04	3.37e−05
	2.04	2.01	2.00	2.00	
5	3.20e−03	1.06e−03	3.46e−04	1.09e−04	3.37e−05
	2.04	2.01	2.00	2.00	
10	3.20e−03	1.06e−03	3.46e−04	1.09e−04	3.37e−05
	2.04	2.01	2.00	2.00	
15	3.20e−03	1.06e−03	3.46e−04	1.09e−04	3.37e−05
	2.04	2.01	2.00	2.00	
p^N	2.04	2.01	2.00	2.00	

Table 5 Errors E_ε^N and rates p_ε^N for Example 5.2 with piecewise-uniform Shishkin mesh

$\varepsilon = 10^{-K}$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$K = 0$	7.25e−06	1.81e−06	4.53e−07	1.13e−07	2.83e−08
	2.57	2.48	2.41	2.36	
1	8.97e−05	2.24e−05	5.60e−06	1.40e−06	3.50e−07
	2.57	2.48	2.41	2.36	
2	8.49e−04	2.10e−04	5.24e−05	1.31e−05	3.27e−06
	2.59	2.48	2.41	2.36	
3	1.02e−02	2.20e−03	5.28e−04	1.31e−04	3.27e−05
	2.84	2.55	2.42	2.36	
4	1.16e−02	3.39e−03	1.06e−03	3.28e−04	1.01e−04
	2.28	2.08	2.04	2.01	
5	1.16e−02	3.39e−03	1.06e−03	3.28e−04	1.01e−04
	2.28	2.08	2.04	2.01	
10	1.16e−02	3.39e−03	1.06e−03	3.28e−04	1.01e−04
	2.28	2.08	2.04	2.01	
15	1.16e−02	3.39e−03	1.06e−03	3.28e−04	1.01e−04
	2.28	2.08	2.04	2.01	
p^N	2.28	2.08	2.04	2.01	

Table 6 Errors E_ε^N and rates p_ε^N for Example 5.3 with piecewise-uniform Shishkin mesh

$\varepsilon = 10^{-K}$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
$K = 0$	6.17e−08	1.54e−08	3.86e−09	9.65e−10	2.41e−10
	2.58	2.47	2.41	2.36	
1	6.78e−06	1.69e−06	4.24e−07	1.06e−07	2.65e−08
	2.58	2.47	2.41	2.36	
2	1.70e−04	4.24e−05	1.06e−05	2.65e−06	6.63e−07
	2.58	2.48	2.41	2.36	
3	8.41e−04	2.14e−04	5.35e−05	1.34e−05	3.35e−06
	2.54	2.48	2.41	2.36	
4	2.10e−03	7.23e−04	2.37e−04	7.49e−05	2.32e−05
	1.98	1.99	2.00	1.99	
5	2.10e−03	7.23e−04	2.37e−04	7.49e−05	2.32e−05
	1.98	1.99	2.00	1.99	
10	2.10e−03	7.23e−04	2.37e−04	7.49e−05	2.32e−05
	1.98	1.99	2.00	1.99	
15	2.10e−03	7.23e−04	2.37e−04	7.49e−05	2.32e−05
	1.98	1.99	2.00	1.99	
p^N	1.98	1.99	2.00	1.99	

References

1. Chang, K.W., Howes, F.A.: Nonlinear Singular Perturbation Phenomena. Springer, New York (1984)
2. O'Malley, R.E. Jr.: Singular Perturbation Methods for Ordinary Differential Equations. Springer, New York (1991)
3. Jacob, M.: Heat Transfer. Wiley, New York (1959)
4. Van Roosbroeck, W.V.: Theory of flows of electrons and holes in germanium and other semiconductors. Bell Syst. Tech. J. **29**, 560–607 (1950)
5. Hirsch, C.: Numerical Computation of Internal and External Flows. Wiley, New York (1988)
6. Murray, J.D.: Lectures on Nonlinear Differential Equation Models in Biology. Clarendon Press, Oxford (1977)
7. Weekman, V.W. Jr., Gorring, R.L.: Influence of volume change on gas-phase reactions in porous catalysts. J. Catal. **4**, 260–270 (1965)
8. Bakhvalov, N.S.: Towards optimization of methods for solving boundary value problems in the presence of a boundary layer. Z. Vychisl. Mat. Mat. Fiz. **9**, 841–859 (1969)
9. Vulanović, R.: Mesh construction for discretization of singularly perturbed boundary value problems. Ph.D. thesis, University of Novi Sad (1986)
10. Shishkin, G.I.: A difference scheme for a singularly perturbed parabolic equation with a discontinuous boundary condition. Z. Vychisl. Mat. Mat. Fiz. **28**, 1679–1692 (1988)
11. Roos, H.-G.: Layer-adapted grids for singular perturbation problems. Z. Angew. Math. Mech. **78**, 291–301 (1998)
12. Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Fitted Numerical Methods for Singular Perturbation Problems, Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions. World Scientific, Singapore (1996)
13. Vulanović, R.: On a numerical solution of a type of singularly perturbed boundary value problem by using a special discretization mesh. Univ. Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **13**, 187–201 (1983)
14. Surla, K., Uzelac, Z.: A uniformly accurate spline collocation method for a normalized flux. J. Comput. Appl. Math. **166**, 291–305 (2004)

15. Surla, K.: On modelling of semilinear singularly perturbed reaction-diffusion problem. Nonlinear Anal. **30**, 61–66 (1997)
16. Rao, S.C.S., Kumar, M.: Parameter-uniformly convergent exponential spline difference scheme for singularly perturbed semilinear reaction-diffusion problems. Nonlinear Anal. **71**, e1579–e1588 (2009)
17. Kadalbajoo, M.K., Reddy, Y.N.: Initial-value technique for a class of nonlinear singular perturbation problems. J. Optim. Theory Appl. **53**, 395–406 (1987)
18. Kadalbajoo, M.K., Patidar, K.C.: Spline techniques for solving singularly-perturbed nonlinear problems on nonuniform grids. J. Optim. Theory Appl. **114**, 573–591 (2002)
19. Rao, S.C.S., Kumar, M.: B-spline collocation method for nonlinear singularly-perturbed two-point boundary-value problem. J. Optim. Theory Appl. **134**, 91–105 (2007)
20. Kadalbajoo, M.K., Gupta, V.: Numerical solution of singularly perturbed convection-diffusion problem using parameter uniform B-spline collocation method. J. Math. Anal. Appl. **355**, 439–452 (2009)
21. Kadalbajoo, M.K., Gupta, V., Awasthi, A.: A uniformly convergent B-spline collocation method on a nonuniform mesh for singularly perturbed one-dimensional time-dependent linear convection-diffusion problem. J. Comput. Appl. Math. **220**, 271–289 (2008)
22. Varah, J.M.: A lower bound for the smallest singular value of a matrix. Linear Algebra Appl. **11**, 3–5 (1975)
23. Carlson, R.E., Hall, C.A.: Error bounds for bicubic spline interpolation. J. Approx. Theory **7**, 41–47 (1973)
24. Hall, C.A.: Natural cubic and bicubic spline interpolation. SIAM J. Numer. Anal. **7**, 41–47 (1973)
25. Gracia, J.L., Lisbona, F., Clavero, C.: High order ε-uniform methods for singularly perturbed reaction-diffusion problems. In: Vulkov, L., Waśniewski, J., Yalamov, P. (eds.) Lecture Notes in Computer Science, vol. 1988, pp. 350–358. Springer, Berlin (2001)
26. Bohl, E.: Finite Modelle gewööhnlicher Randwertaufgaben. Teubner, Stuttgart (1981)