J Optim Theory Appl (2010) 146: 255–266
DOI 10.1007/s10957-010-9671-8

# Constant-Rank Condition and Second-Order Constraint Qualification

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Published online: 24 February 2010


**Abstract** The constant-rank condition for feasible points of nonlinear programming problems was defined by Janin (Math. Program. Study 21:127–138, 1984). In that paper, the author proved that the constant-rank condition is a first-order constraint qualification. In this work, we prove that the constant-rank condition is also a second-order constraint qualification. We define other second-order constraint qualifications.

**Keywords** Nonlinear programming · Constraint qualifications

## 1 Introduction

We are concerned with the general nonlinear programming problems with equality and inequality constraints in the form

$$\min f(x), \quad \text{s.t.} \quad h(x) = 0, \quad g(x) \leq 0, \tag{1}$$

Communicated by F. Giannessi.

This research was supported by PRONEX-Optimization 76.79.1008-00, FAPESP (Grant 01-04597-4) and CNPq.

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where $f:\mathbb{R}^n \to \mathbb{R}$, $h:\mathbb{R}^n \to \mathbb{R}^m$, $g:\mathbb{R}^n \to \mathbb{R}^p$ are twice continuously differentiable on $\mathbb{R}^n$. For a feasible point $x$, we define the set of active inequality constraints:

$$A(x) := \{i \in \{1,\dots,p\} : g_i(x) = 0\}.$$

In constrained optimization, one is generally interested in finding global minimizers, but we realize that this is very difficult. This is the main reason why we study optimality conditions and constraint qualifications in constrained optimization. The idea behind this is to find good necessary optimality conditions for a minimum point of the problem (1).

The desirable and most important first-order condition that combines the objective function with the constraints is the well-known Karush/Kuhn-Tucker condition (KKT condition, [2, 3]); given a feasible point $\hat{x}$ of the problem (1), there are vectors $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^p$ such that

$$\nabla f(\hat{x}) + \sum_{i=1}^{m} \lambda_i \nabla h_i(\hat{x}) + \sum_{i=1}^{p} \mu_i \nabla g_i(\hat{x}) = 0,$$
$$\mu_i \geq 0,\ \mu_i g_i(\hat{x}) = 0,\ i = 1,\dots,p. \tag{2}$$

The vectors $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^p$ are known as Karush/Kuhn-Tucker (KKT) multipliers and we say that $\hat{x}$ is a stationary point of the problem (1). See for example [4, 5]. A point that verifies (2) is a stationary point of the Lagrangian function associated to the problem (1),

$$l(x,\lambda,\mu) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x) + \sum_{i=1}^{p} \mu_i g_i(x).$$

Unfortunately, condition (2) is not a first-order necessary optimality condition for a minimum point. For example, the solution of the problem of finding a minimizer of the function $f(x) = x$, subject to $h(x) = x^2$, does not verify the KKT condition. *First-order constraint qualifications* are conditions over the constraints under which it can be claimed that, if $x$ is a feasible minimum point, then $x$ is a stationary point of the Lagrangian function associated to each objective function which it minimizes.

The most widely used first-order constraint qualification is the linear independence of the gradients of the equality and inequality active constraints at a given feasible point (LICQ). It is well known that LICQ is a first-order constraint qualification and it implies the existence and uniqueness of KKT multipliers for a given solution. There are weaker first-order constraint qualifications in the literature. The Mangasarian-Fromovitz condition (MFCQ), defined in [6], establishes the positive linear independence of the gradients of the equality and inequality active constraints at a given feasible point and it is weaker than LICQ. Another first-order constraint qualification weaker than LICQ is the constant-rank constraint qualification (CRCQ) defined in [1]. We say that, a feasible point verifies the constant-rank constraint qualification if there exists a neighbourhood of the feasible point in which the rank of any subset of the gradients of the equality and inequality active constraints does not change. There are simple examples showing that CRCQ neither implies nor is implied by MFCQ; see [1]. More recently, the constant positive linear dependence (CPLD) condition was defined in [7] and it was established as a first-order constraint qualification in [8]. CPLD is weaker than CRCQ and MFCQ, see [8]. We can find other first-order constraint qualifications in the literature. Some of them are, in order of weakness: quasinormality defined by Hestenes in [9], pseudonormality [10, 11], Abadie [12] and Guignard [13]. Guignard condition is the weakest first-order constraint qualification for differentiable problems as shown in [14]. When a first-order constraint qualification holds, it is possible to think in terms of KKT multipliers and efficient algorithms based on duality ideas can be defined. The discovery of new and weaker first-order constraint qualifications and necessary optimality conditions is an open issue in nonlinear optimization. Recently [15, 16], a necessary optimality condition and a constraint qualification were defined by means of the image space analysis [17].

Second-order necessary optimality conditions are important because they take into account the curvature of the Lagrangian function over the set of feasible directions. The desirable second-order condition that combines the objective function with the constraints is the KKT condition plus the strong second-order necessary condition (SSONC): given a feasible point $\hat{x}$ of the problem (1), there are vectors $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^p$ such that condition (2) holds at $\hat{x}$ and SSONC:

$$d^T\left[\nabla^2 f(\hat{x}) + \sum_{i=1}^{m}\lambda_i \nabla^2 h_i(\hat{x}) + \sum_{i=1}^{p}\mu_i \nabla^2 g_i(\hat{x})\right]d \geq 0, \tag{3}$$

for all direction $d \in \mathbb{R}^n$ in the following tangent subspace:

$$\begin{aligned}\tilde{T}(\hat{x}) := \{d \in \mathbb{R}^n : \ & \nabla h_i(\hat{x})^T d = 0,\ i = 1, \dots, m,\\ & \nabla g_j(\hat{x})^T d = 0,\ j \in A^+(\hat{x}),\\ & \nabla g_j(\hat{x})^T d \leq 0,\ j \in A^0(\hat{x})\},\end{aligned}$$

where

$$A^+(\hat{x}) = \{j \in A(\hat{x}) : \mu_j > 0\},\ A^0(\hat{x}) = \{j \in A(\hat{x}) : \mu_j = 0\}.$$

Condition (3) says that the Hessian of the Lagrangian function at $(\hat{x}, \lambda, \mu)$, restricted to the tangent subspace $\tilde{T}(\hat{x})$, is positive semidefinite.

Unfortunately, the combination KKT + SSONC is not always a second-order necessary optimality condition. It is true that, under an appropriate constraint qualification, it is possible to show that a minimum point of the problem (1) verifies KKT + SSONC.

Observe that, if a feasible point $\hat{x}$ is a KKT point, then

$$\begin{aligned}\tilde{T}(\hat{x}) = \{d \in \mathbb{R}^n : \nabla f(\hat{x})^T d \leq 0;\ \nabla h_i(\hat{x})^T d = 0,\ i = 1, \dots, m;\\ \nabla g_j(\hat{x})^T d \leq 0,\ j \in A(\hat{x})\}.\end{aligned}$$

Thus, the tangent subspace in SSONC depends on the objective function of the problem.

Some of the second-order practical algorithms (see for example [18]) take into account the analysis of the Hessian of the Lagrangian function in the following tangent subspace:

$$T(x) := \{d \in \mathbb{R}^n : \nabla h_i(x)^T d = 0, i = 1, \dots, m, \nabla g_j(x)^T d = 0, j \in A(x)\}.$$

Clearly, for a given feasible point $x$, $T(x) \subseteq \tilde{T}(x)$ and $T(x)$ does not depend on the objective function. By considering $T(x)$, we can define another second-order condition that combines the objective function and the constraints—KKT condition plus the weak second-order necessary condition (WSONC)—given a feasible point $\hat{x}$ of the problem (1), there are vectors $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^p$ such that condition (2) holds at $\hat{x}$ and WSONC:

$$d^T\left[\nabla^2 f(\hat{x}) + \sum_{i=1}^{m} \lambda_i \nabla^2 h_i(\hat{x}) + \sum_{i=1}^{p} \mu_i \nabla^2 g_i(\hat{x})\right] d \geq 0,$$

for all $d \in T(\hat{x})$.

The tangent $T(x)$ is associated in practice with some of the well known nonlinear programming algorithms; see [19] and references therein. In [19], the authors have shown a class of nonlinear optimization algorithms, using barrier functions, for which the strong second-order necessary condition may not hold at the limit points, even if the sequence of the subproblems minimizers satisfies the second-order sufficient condition. This implies the practical importance of the weak second-order condition.

Observe that, if the strict complementarity condition holds at a stationary point $\hat{x}$ (this means that $\mu_i - g_i(\hat{x}) > 0, i = 1, \dots, p$), then $\tilde{T}(\hat{x}) = T(\hat{x})$ and SSONC is equivalent to WSONC. The strict complementarity condition is used for the convergence analysis of many nonlinear programming algorithms.

Thus, we can say that, from a theoretical point of view, it is important to analyze conditions that imply SSONC; also, from a practical point of view, it is important to analyze conditions that imply WSONC.

It is well established in the literature that, if a minimum point of the problem (1) verifies the linear independence of the equality and inequality active constraints gradients, then there is a unique KKT multiplier vector for which SSONC holds (see [20]). The important question is: is it possible to relax the linear independence constraint qualification and still have the possibility that a minimum point verifies KKT + SSONC (or KKT + WSONC)? All the first-order constraint qualifications that are weaker than LICQ imply the existence of a set of KKT multipliers at the solution. Thus, we do not have uniqueness of the multipliers at the solution. *Strong second-order constraint qualifications* (respectively *weak second-order constraint qualifications*) are conditions over the constraints under which it can be claimed that, if $x$ is a minimum point (for any objective function $f$), then $x$ verifies the KKT condition and there is, at least, a KKT multiplier vector that verifies SSONC (respectively WSONC).

As the example in [21] (rediscovered by Anitescu in [22]) shows, the Mangasarian-Fromovitz constraint qualification is neither a strong nor a weak second-order constraint qualification. Thus, from those first-order constraint qualifications mentioned before, the only condition that can be a strong or a weak second-order constraint qualification is the constant-rank condition. Here, we prove that CRCQ is, in fact, a strong second-order constraint qualification. This is the main goal of the present work.

In the article where the CRCQ was defined [1], the condition was used in the context of nonconvex mathematical programming problems under general perturbation. In the last years, CRCQ has been used to obtain theoretical results with practical relevance in the context of bilevel problems; see for example [23]. In [24], CRCQ has been used to achieve global and superlinear convergence of an infeasible interior-point algorithm for monotone variational inequality problems. In [25], an augmented Lagrangian method with convergence under a weaker constraint qualification (CPLD) was defined. This convergence result implies convergence under CRCQ. In [26], the CRCQ was used to investigate the properties of the parametric set defined by the equality and inequality constraints. The author of [26] shows that, in the absence of parameters, the CRCQ implies that the Mangasarian-Fromovitz constraint qualification holds in some alternative expression of the feasible set.

As mentioned before, it is important, from the practical point of view, to define weak second-order constraint qualifications. We consider the weak constant-rank condition introduced in [27] to define a new weak second-order constraint qualification. We believe that this new second-order constraint qualification can be used in the future in the study of problems having complementarity constraints.

This paper is organized as follows. In Sect. 2, we state the main definitions. In Sect. 3, we prove that the constant-rank constraint qualification is a strong second-order constraint qualification. Also in this section, we prove that a generalization of the second-order constraint qualification introduced in [27] implies WSONC for any KKT multiplier vector. The conclusions are given in Sect. 4.

## 2 Definitions

Let $h:\mathbb{R}^n \to \mathbb{R}^m$ and $g:\mathbb{R}^n \to \mathbb{R}^p$ be twice continuously differentiable functions. Define the *feasible set* $\Omega$ as

$$\Omega := \{x \in \mathbb{R}^n \ : \ h(x) = 0,\ g(x) \le 0\}.$$

**Definition 2.1** (Ref. [1]) Given a family of differentiable functions $\{f_i(x) : i = 1, \dots, r\}$, $f_i : \mathbb{R}^n \to \mathbb{R}$, we say that the *constant-rank condition* holds at $x^*$ if and only if, for any subset $K \subset \{i \in \{1, \dots, r\} : f_i(x^*) = 0\}$, the family of gradients

$$\{\nabla f_i(x)\}_{i \in K}$$

remains of constant rank near the point $x^*$.

**Definition 2.2** (Ref. [1]) Given the family of differentiable functions $\{h_i(x) : i = 1, \dots, m; g_i(x) : i = 1, \dots, p\}$ associated with problem (1), we say that a feasible point $x^* \in \Omega$ satisfies the *constant-rank constraint qualification* (CRCQ) if and only if the constant-rank condition holds at $x^*$.

As we mentioned before, in [1] it was proved that CRCQ is a first-order constraint qualification.

Several well-known constraint qualifications with practical relevance imply the CRCQ. For example, the CRCQ is clearly implied by the LICQ. If all the constraints are defined by affine functions, the CRCQ is obviously fulfilled. Moreover, if $x \in \Omega$ satisfies the CRCQ and some equality constraint $h_i(x) = 0$ is replaced by two inequality constraints ($h_i(x) \le 0$ and $-h_i(x) \le 0$) the CRCQ still holds with the new description of the feasible set. We observe that the Mangasarian-Fromovitz constraint qualification does not accomplish those properties.

## 3 Main Results

The proof that constant-rank is a strong second-order constraint qualification needs the following two technical results. Both propositions were established in [1] in the context of nonconvex mathematical programming problems under general perturbations. For the sake of completeness, we will rewrite the results differently but in an equivalent form.

**Proposition 3.1** (Ref. [1]) *Let $\{f_i(x); i \in K\}$ be a family of differentiable functions on $\mathbb{R}^n$ such that the Jacobian matrix $(\nabla f_i(x))_{i\in K}$ is of constant rank in a neighborhood of $x^*$. Define the linear subspace*

$$E = \{y \in \mathbb{R}^n : \nabla f_i(x^*)^T y = 0,\ i \in K\}.$$

*Then, there exists some local diffeomorphism $\phi : V_1 \to V_2$, where $V_1, V_2$ are neighborhoods of $x^*$, such that:*

(i) $\phi(x^*) = x^*$,
(ii) *the Jacobian matrix of $\phi$ at $x^*$ is the identity matrix,*
(iii) *the functions $f_i \circ \phi^{-1} (i \in K)$ are of constant value for all $y \in E$.*

*Remark 3.1* As it was mentioned in [1], this result is a special case of the constant-rank theorem in Malliavin [28]. Observe that the hypothesis required is not the constant-rank condition but the constant rank of the Jacobian matrix in a neighborhood of the point. This is the hypothesis that appears in the constant-rank theorem in Malliavin.

Proposition 3.2 is a consequence of Proposition 3.1 and it was used in [1] to prove that CRCQ is a first-order constraint qualification. It will be rewritten in a new form that will be more useful for us to prove the strong second-order necessary condition. The second part of the proposition was not stated in [1] and it can be deduced straightforwardly from the proof made in it. We will explain it here for completeness.

**Proposition 3.2** *Assume that the constant-rank constraint qualification holds at $x^* \in \Omega$. Then, for each $y \in \mathbb{R}^n$ such that*

$$\nabla h_i(x^*)^T y = 0, i = 1, \dots, m;\ \nabla g_i(x^*)^T y \le 0, \quad i \in A(x^*), \tag{4}$$

*there exists some arc* $t \to \xi(t), t \in (0, \bar{t}), \bar{t} > 0$, *such that* $\xi(t) \subset \Omega$ *and*

(a) $\lim_{t\to 0^+} \xi(t) = x^*$, $\lim_{t\to 0^+} \frac{\xi(t)-x^*}{t} = y$,
(b) *for all* $j \in A(x^*)$ *such that* $\nabla g_j(x^*)^T y = 0$ *then* $g_j(\xi(t)) = 0$.

*Proof* Let us suppose that $A(x^*) = \{1, \dots, r\}$ and define $f_i(x) = h_i(x)$, $\forall i = 1, \dots, m$, $f_{m+i}(x) = g_i(x)$, $\forall i = 1, \dots, r$. Take $K = \{i \in \{1, \dots, m + r\} : \nabla f_i(x^*)^T y = 0\}$ and consider the linear subspace $E = \{d \in \mathbb{R}^n : \nabla f_i(x^*)^T d = 0, i \in K\}$. Since the constant-rank constraint qualification holds at $x^*$, according to the hypothesis, we can use Proposition 3.1. Thus, there exists some local diffeomorphism $\phi$ verifying items (i), (ii) and (iii) of Proposition 3.1.

Define the arc $\xi(t)$ by

$$\xi(t) = \phi^{-1}(x^* + ty),$$

for $t > 0$ sufficiently small. Then, by continuity and conditions (i) and (ii) of Proposition 3.1, we have that

$$\lim_{t\to 0^+} \xi(t) = x^*, \qquad \lim_{t\to 0^+} \frac{\xi(t) - x^*}{t} = y.$$

Thus, the arc $\xi(t)$ verifies (a).

For any index $j \in K$, using item (iii) of Proposition 3.1, we have that, for $t > 0$ sufficiently small,

$$f_j(\xi(t)) = f_j(\phi^{-1}(x^* + ty)) = f_j(\phi^{-1}(x^*)) = f_j(x^*) = 0.$$

Thus, if $j \in A(x^*)$ is such that $\nabla g_j(x^*)^T y = 0$, then we have that $g_j(\xi(t)) = 0$ for $t > 0$ small enough. This proves that (b) holds.

Let us prove that the arc is feasible. For $i \in A(x^*)\backslash K$, we have that $\nabla f_i(x^*)^T d < 0$. Thus,

$$f_i(\xi(t)) = t\nabla f_i(x^*)^T y + t\varepsilon(t), \qquad \text{with } \varepsilon(t) \text{ such that } \varepsilon(t) \to 0.$$

Then, for $\bar{t}_1 > 0$ sufficiently small, we have that

$$f_i(\xi(t)) \le 0, \quad 0 < t < \bar{t}.$$

A similar argument, based on the continuity of each $f_i(x)$ would show that, for $i \in \{1, \dots, p\}\backslash A(x^*)$, there exists $\bar{t}_2 > 0$ sufficiently small such that

$$f_i(\xi(t)) \le 0, \quad 0 < t < \bar{t}_2.$$

Thus, for $\bar{t} = \min\{\bar{t}_1, \bar{t}_2\}$, we have that $\xi(t) \subset \Omega$, $0 < t < \bar{t}$. □

In the next theorem, we prove that, if a minimum point verifies the constant-rank constraint qualification, then the strong second-order necessary condition holds for any KKT multiplier.

**Theorem 3.1** *Let $x^*$ be a minimum point of problem (1) that verifies the constant-rank constraint qualification. Then, for any KKT multiplier vector $(\lambda^*, \mu^*) \in \mathbb{R}^{m+p}$, $x^*$ verifies the strong second-order necessary optimality condition.*

*Proof* The existence of a set of KKT multipliers at $x^*$ was clearly established in [1]. Let us prove that the strong second-order necessary condition is verified for any of those multipliers. Let $(\lambda, \mu) \in \mathbb{R}^{m+p}$ be any fixed KKT multiplier.

Take a direction $y \in \tilde{T}(x^*)$. Then, $y$ verifies

$$\begin{aligned} \nabla h_i(x^*)^T y = 0, &\quad i = 1, \dots, m, \\ \nabla g_j(x^*)^T y = 0, &\quad j \in A^+(x^*), \\ \nabla g_j(x^*)^T y \le 0, &\quad j \in A^0(x^*). \end{aligned}$$

Thus,

$$\left(\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla h_i(x^*) + \sum_{i \in A^+(x^*)} \mu_i \nabla g_i(x^*)\right)^T y = 0$$

and we have that

$$\nabla f(x^*)^T y = 0.$$

By Proposition 3.2, there exist a feasible arc $\xi(t), t \in [0, \bar{t})$ such that

$$\xi(0) = x^*, \qquad \xi'(0) = \lim_{t \to 0^+} \frac{\xi(t) - x^*}{t} = y, \qquad g_i(\xi(t)) = 0, \quad \forall i \in A^+(x^*).$$

Define $\varphi(t) = f(\xi(t))$. Since $x^*$ is a minimum point of $f$ in $\Omega$ and since

$$\varphi'(0) = \nabla f(x^*)^T y = 0,$$

then

$$\varphi''(0) = \frac{d^2}{dt^2} f(\xi(t))|_{t=0} = y^T \nabla^2 f(x^*) y + \nabla f(x^*)^T \xi''(0) \ge 0. \tag{5}$$

Furthermore, by the feasibility of the arc and using $g_i(\xi(t)) = 0,\ \forall i \in A^+(x^*),\ 0 \le t < \bar{t}$, we have that

$$R(t) = \sum_{i=1}^{m} \lambda_i h_i(\xi(t)) + \sum_{i \in A(x^*)} \mu_i g_i(\xi(t)) = 0, \quad 0 \le t < \bar{t}.$$

So, differentiating that relation twice and taking $t = 0$, we obtain

$$\begin{aligned} R''(0) = y^T &\left(\sum_{i=1}^{m} \lambda_i \nabla^2 h_i(x^*) + \sum_{i \in A(x^*)} \mu_i \nabla^2 g_i(x^*)\right) y \\ &+ \left(\sum_{i=1}^{m} \lambda_i \nabla h_i(x^*) + \sum_{i \in A(x^*)} \mu_i \nabla g_i(x^*)\right)^T \xi''(0) = 0. \end{aligned} \tag{6}$$

Adding (6) to (5), we obtain

$$y^T\left(\nabla^2 f(x^*)+\sum_{i=1}^{m}\lambda_i\nabla^2 h_i(x^*)+\sum_{i\in A(x^*)}\mu_i\nabla^2 g_i(x^*)\right)y\geq 0.$$

Thus, the strong second-order necessary condition holds at $x^*$ as we wanted to prove. □

*Remark 3.2* Observe that, given a direction $y\in\tilde{T}(x^*)$, it can happens that not all the gradients of the inequality active constraints verify $\nabla g_i(x^*)^T y=0$. Because of this, the hypothesis of constant-rank constraint qualification is necessary.

The existence of the feasible arc $\xi(t)$ shows that the constant-rank constraint qualification can be seen as a natural generalization of the linear independence constraint qualification. Theorem 3.1 shows that CRCQ captures correctly the geometry of the tangent subspace $\tilde{T}(x^*)$, just as LICQ does. Observe that, under the CRCQ, the SSONC condition holds for any KKT multiplier.

Due to the counterexample in [21], most of the new second-order constraint qualifications have the form: MFCQ + some condition, which gives the validity of the (strong or weak) second-order necessary condition. In this scope, we can mention the constraint qualifications defined in [29] and [27]. Both conditions imply the existence of some KKT multiplier vector for which a second-order necessary condition holds.

In [29], the authors proved that there is a KKT multiplier for which the SSONC condition holds at $x^*$, if $x^*$ is a minimum point of (1) verifying the following second-order constraint qualification:

(i) MFCQ holds at $x^*$.
(ii) The rank of the gradients of the equality and inequality active constraints at $x^*$ is $m+q-1$, where $q$ is the number of active inequality constraints at $x^*$.
(iii) There exists at most only one index $i_0\in A(x^*)$ such that, if $(\lambda,\mu)$ is a KKT multiplier, then $\mu_{i_0}=0$.

The third condition is a complementarity condition and the authors conjecture that it is not necessary.

In [27], the authors proved that there is a KKT multiplier for which the WSONC condition holds at $x^*$, if $x^*$ is a minimum point of (1) verifying the following second-order constraint qualification:

(i) MFCQ holds at $x^*$.
(ii) The weak constant-rank condition (WCR) holds at $x^*$: The rank of the Jacobian matrix made of the gradients $\{\nabla h_i(x)\}_{i=1,\dots,m}\cup\{\nabla g_i(x)\}_{i\in A(x^*)}$ does not change in a neighborhood of $x^*$.

The proof that the last condition is a weak second-order constraint qualification was achieved using penalty ideas and this constraint qualification was used in the convergence analysis of the second-order augmented Lagrangian method defined in [27].

Using Proposition 3.2, we are able to prove the following theorem that generalizes the result obtained in [27].

**Theorem 3.2** *Let $x^*$ be a minimum point of problem (1) that verifies the KKT condition and the weak constant-rank condition. Then, for any KKT multiplier vector $(\lambda^*, \mu^*) \in \mathbb{R}^{m+p}$, the weak second-order necessary condition is verified.*

*Proof* Let take $y \in T(x^*)$. Then, $y$ verifies $\nabla h_i(x^*)^T y = 0, i = 1, \dots, m$, $\nabla g_j(x^*)^T y = 0$, $j \in A(x^*)$. Thus, $y$ verifies (4) and we can prove the existence of a feasible arc as in Proposition 3.2. The proof follows the idea of Theorem 3.1. □

*Remark 3.3* In Theorem 3.2, we prove that the weak second-order necessary condition holds at a minimum point, under any first-order constraint qualification and the weak constant-rank condition. The result of this theorem is interesting when we consider the following special class of mathematical programming problems with complementarity constraints:

$$\min f(x, y, z), \quad \text{s.t.} \quad x \geq 0, y \geq 0, xy = 0.$$

In this kind of problems, the origin (0, 0, 0) is a problematic point for most of the nonlinear programming algorithms, since it only verifies the Guignard constraint qualification [13]. We observe that this point verifies the new weak second-order constraint qualification defined by Guignard + WRC.

In [27], there is an example showing that the weak constant-rank condition is not a first-order constraint qualification. In that example, the problem has equality and inequality constraints. The KKT condition does not hold in the example due to the presence of an inequality constraint. The WCR condition was previously defined in [30], for problems with just equality constraints. In that work, the author proved that, in this case, WCR implies the Abadie constraint qualification. In the following theorem, we prove that the WCR condition is also a second-order constraint qualification whenever the problem has only equality constraints.

**Theorem 3.3** *Let $x^*$ be a minimum point of the following problem with only equality constraints*:

$$\min f(x), \quad \text{s.t.} \quad h(x) = 0.$$

*Suppose that $x^*$ verifies the weak constant-rank condition. Then, $x^*$ verifies the KKT condition and the second-order necessary condition for any KKT multiplier.*

*Proof* This theorem is a consequence of Proposition 3.1 and Proposition 3.2. □

*Remark 3.4* Observe that, when the problem has only equality constraints, $\tilde{T}(x^*) = T(x^*)$ and the weak second-order necessary condition is equivalent to the strong one. The weak constant-rank condition for equality problems is weaker than the Mangasarian-Fromovitz constraint qualification (which is, in this case, equivalent to the linear independence constraint qualification) and weaker than the constant-rank constraint qualification. Therefore, the constant-rank constraint qualification can be seen as a generalization of those conditions.

## 4 Conclusions

The constant-rank constraint qualification seems to be a useful tool for the analysis of convergence of some nonlinear programming methods; see for example [23–25]. The status of the constant-rank as a first-order constraint qualification was clearly proved in [1]. In this work, we proved that CRCQ is in fact a strong second-order constraint qualification. We proved that, under this constraint qualification, a minimum point verifies the strong second-order necessary condition for any KKT multiplier.

Given the practical importance of the weak second-order necessary condition, we generalized the second-order constraint qualification defined in [27]. We proved that, on having a minimum point that verifies any first-order constraint qualification and the weak constant-rank condition, then the WSONC holds for any KKT multiplier. WCR as well as MFCQ, were useful for obtaining the convergence of the second-order augmented Lagrangian method defined in [27]. Throughout this work, we proved that WCR is a first-order and a second-order constraint qualification when we consider equality constraint problems. We believe that this fact can be useful in the future, since WCR can be seen as a generalization of the regularity and the constant-rank constraint qualification for this kind of problems.

It is still an open issue the challenge of finding new and weaker constraint qualifications, not only of first-order but also of second-order. It is desirable that those conditions be easily verifiable and possibly be associated from a practical point of view with the convergence analysis of nonlinear optimization algorithms.

**Acknowledgements** We are grateful to the Associate Editor Franco Giannessi whose comments helped us to improve the first version of this paper.

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