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# Existence and Stability of Solutions for Generalized Ky Fan Inequality Problems with Trifunctions

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**Abstract** In this paper, an existence theorem for solutions to the generalized Ky Fan Inequality problem is obtained by means of the Kakutani-Fan-Glicksberg fixed-point theorem without imposing the condition that the dual of the ordering cone has a weak* compact base. In addition, the stability of the solution set is shown.

**Keywords** Generalized Ky Fan inequality problems · Fixed-point theorems · $C$-continuity · Properly $C$-quasiconvex functions · Stability

## 1 Introduction

Throughout this paper, let $X$, $Y$ and $Z$ be real locally convex Hausdorff topological vector space and let $C \subset Z$ be a closed convex cone. The cone $C$ induces a partially ordering on $Z$ defined by $z_1 \leq z_2$ if and only if $z_2 - z_1 \in C$.

Let $D$ be a nonempty subset of $Y$ and let $E$ be a nonempty subset of $X$. Let $S : E \to 2^E$ and $T : E \to 2^D$ be set-valued maps. Let $f : E \times D \times E \to Z$ be a trifunction.

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The generalized Ky Fan inequality problem consists in finding a pair $(\bar{x}, \bar{y}) \in E \times D$ such that $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ and

$$f(\bar{x}, \bar{y}, x) \geq 0, \quad \forall x \in S(\bar{x}). \tag{1}$$

This inequality had been introduced in a simpler format by Ky Fan (see [1, 2]).

So far there are only a few papers dealing with the generalized inequality problem (1) in the strong sense. Fu [3] studied a similar problem under the condition that $C^*$ has a weak* compact base, where $C^*$ is the topological dual cone of the ordering cone $C$. Tan [4] gave an existence theorem of solutions for quasivariational inclusion problem with trivalent mapping under the same condition.

In fact, Fu made use of the following lemma in Jeyakumar, Oettle and Natividad [5] to show that, if $\bar{x}$ is a solution of a related scalar equilibrium problem, then $\bar{x}$ is a solution to the inequality problem (1).

**Lemma 1.1** *Let $B$ be a weak* compact base of $C^*$. Then*:

(i) $z \geq 0 \Leftrightarrow \langle z^*, z\rangle \geq 0, \forall z^* \in C^*$.
(ii) $z \geq 0 \Leftrightarrow \langle z^*, z\rangle \geq 0, \forall z^* \in B$.

As we know, in a normed space the condition that $C^*$ has a weak* compact base is equivalent to $\operatorname{int} C \neq \emptyset$ (see [6]). However, in many cases the ordering cone $C$ has an empty interior. For example, in the normed spaces $\ell^p$ and $L^p(\Omega)$, where $1 < p < \infty$, the standard ordering cone has an empty interior.

In this paper, we do not impose the condition that $C^*$ has a weak* compact base. We employ a new alternative approach to establish the existence theorem of solutions for the inequality problem (1). It will be shown that our results generalize those of Fu [3].

The rest of the paper is organized as follows. In Sect. 2, we recall some definitions, properties and results of set-valued maps which will be used in the sequel. By using the Kakutani-Fan-Glicksberg fixed-point theorem, an existence result for (1) is established in Sect. 3. Some applications of this existence result are given in Sect. 4. We conclude with some stability result for the solution set of (1) in Sect. 5.

## 2 Preliminaries

Let us now begin with some definitions and properties of set-valued maps (see [7–9]).

Let $F$ be a set-valued map from a Hausdorff topological space $W$ to a topological space $Q$.

We say that $F$ is upper semicontinuous at $x_0 \in W$ iff, for any neighborhood $V$ of $F(x_0)$, there exists a neighborhood $U(x_0)$ of $x_0$ such that

$$F(x) \subset V, \quad \forall x \in U(x_0).$$

$F$ is said to be upper semicontinuous on $W$ iff it is upper semicontinuous at every point of $W$.

We say that $F$ is lower semicontinuous at $x_0$ iff, for any $y_0 \in F(x_0)$ and any neighborhood $V$ of $y_0$, there exists a neighborhood $U(x_0)$ of $x_0$ such that

$$F(x) \cap V \neq \emptyset, \quad \forall x \in U(x_0).$$

$F$ is said to be lower semicontinuous on $W$ iff it is lower semicontinuous at every point $x \in W$.

Moreover, $F$ is lower semicontinuous at $x_0 \in W$ if and only if, for any $y_0 \in F(x_0)$ and any net $\{x_\alpha\}$ with $x_\alpha \to x_0$, there exists a subnet of indexes $\{\alpha_\lambda\}_{\lambda \in \Lambda}$ and $y_\lambda \in F(x_{\alpha_\lambda})$ for each $\lambda \in \Lambda$ such that $y_\lambda \to y_0$ (see [10]).

$F$ is said to be continuous on $W$ iff it is both upper semicontinuous and lower semicontinuous on $W$.

We say that $F$ is closed or has a closed graph iff

$$\mathrm{Graph}(F) = \{(w, y) : w \in W,\ y \in F(w)\}$$

is a closed set in $W \times Q$.

**Definition 2.1** ([9]) Let $W$ be a topological vector space and let $G$ be a nonempty subset of $W$. A mapping $g : G \to Z$ is said to be $C$-continuous at $w_0 \in G$ iff, for any neighborhood $V$ of 0 in $Z$, there is a neighborhood $U(w_0)$ of $w_0$ in $W$ such that

$$g(w) \in g(w_0) + V + C, \quad \forall w \in U(w_0) \cap G.$$

We say that $g$ is $C$-continuous on $G$ iff it is $C$-continuous at every point of $G$.

Let $D \subset Y$ be a convex set. Recall that a vector-valued function $g : D \to Z$ is said to be $C$-quasiconvex on $D$ iff, for any $z \in Z$, the set $\{u \in D : g(u) \leq z\}$ is convex (see [9]).

Ferro [11] introduced the following concept on quasiconvexity:

**Definition 2.2** Let $D \subset Y$ be a convex subset and let $g : D \to Z$ be a vector-valued function. The function $g$ is said to be properly $C$-quasiconvex on $D$ iff, for every $u_1, u_2 \in D$ and $t \in [0, 1]$, we have that either

$$g(tu_1 + (1-t)u_2) \leq g(u_1) \quad \text{or} \quad g(tu_1 + (1-t)u_2) \leq g(u_2).$$

It is easy to see that $g$ is $C$-quasiconvex on $D$ whenever it is properly $C$-quasiconvex on $D$.

The concept of properly $C$-quasiconvex vector-valued function is important in the study of the minimax theorem for vector-valued functions, the generalized vector quasiequilibrium problems, and vector quasivariational inclusion problems (see [3, 4, 11]).

**Lemma 2.1** *Let $D$ be a nonempty convex subset of $Y$. Assume that $g : D \to Z$ is $C$-continuous and properly $C$-quasiconvex on $D$. Then, for any $u_1, u_2 \in D$, there exists $t_0 \in [0, 1]$ such that*

$$g(t_0u_1 + (1-t_0)u_2) \leq g(u_1), \qquad g(t_0u_1 + (1-t_0)u_2) \leq g(u_2).$$

*Proof* Since $g$ is properly $C$-quasiconvex, for any $u_1, u_2 \in D$ we have

$$[0,1] = \Gamma_1 \cup \Gamma_2, \tag{2}$$

where $\Gamma_i := \{t \in [0,1] : g(u(t)) \leq g(u_i)\}, i = 1, 2$, and $u(t) := tu_1 + (1-t)u_2, t \in [0,1]$. It is clear that $\Gamma_1$ and $\Gamma_2$ are nonempty. We now show that the set $\Gamma_1$ is closed.

Let $t_n \in \Gamma_1$ and $t_n \to t \in [0,1]$. We have

$$g(t_n u_1 + (1-t_n)u_2) \leq g(u_1), \quad \forall n. \tag{3}$$

We claim that

$$g(tu_1 + (1-t)u_2) \leq g(u_1).$$

Suppose to the contrary that

$$g(u_1) - g(tu_1 + (1-t)u_2) \notin C.$$

Since $C$ is closed, there exists some neighborhood $U$ of 0 in $Z$ such that

$$[g(u_1) - (g(tu_1 + (1-t)u_2) + U)] \cap C = \emptyset.$$

As $C$ is a convex cone, we have

$$[g(u_1) - (g(tu_1 + (1-t)u_2) + U + C)] \cap C = \emptyset. \tag{4}$$

The fact that $g$ is $C$-continuous on $D$ and $t_n u_1 + (1-t_n)u_2 \to tu_1 + (1-t)u_2$ implies that there exists $n_0$ such that

$$g(t_n u_1 + (1-t_n)u_2) \in g(tu_1 + (1-t)u_2) + U + C, \quad \forall n \geq n_0.$$

Together with (4), this gives

$$g(u_1) - g(t_n u_1 + (1-t_n)u_2) \notin C, \quad \forall n \geq n_0,$$

a contradiction to (3). Thus, the set $\Gamma_1$ is closed. Similarly, the set $\Gamma_2$ is also closed. $\Gamma_1 \cup \Gamma_2 = [0,1]$ being connected, we conclude that $\Gamma_1 \cap \Gamma_2 \neq \emptyset$. Hence, there exists $t_0 \in [0,1]$ such that

$$g(t_0 u_1 + (1-t_0)u_2) \leq g(u_1), \qquad g(t_0 u_1 + (1-t_0)u_2) \leq g(u_2). \qquad \square$$

## 3 Existence of Solutions

In this section, we prove an existence theorem of solutions for the generalized Ky Fan inequality problem (1) by means of the Kakutani-Fan-Glicksberg fixed-point theorem.

**Definition 3.1** ([12]) A topological vector space $Y$ is called quasicomplete iff every bounded closed subset of $Y$ is complete.

**Theorem 3.1** *Let $X, Y, Z$ be real locally convex Hausdorff topological vector spaces with $Y$ quasicomplete. Let $E \subset X$ be a compact convex set, $D \subset Y$ a closed convex set, and $C \subset Z$ a closed convex cone. Let $S : E \to 2^E$ be a continuous set-valued map such that, for each $x \in E$, $S(x)$ is a nonempty closed convex set. Let $T : E \to 2^D$ be a upper semicontinuous map such that, for each $x \in E$, $T(x)$ is a nonempty compact convex set of $D$. Let $f : E \times D \times E \to Z$ be a trifunction that is $C$-continuous and $(-C)$-continuous on $E \times D \times E$. Suppose that:*

(i) *For any $x \in E$ and $y \in T(x)$, $f(x, y, x) \geq 0$.*
(ii) *For any $(x, y) \in E \times D$, $f(x, y, u)$ is properly $C$-quasiconvex in $u$.*

*Then, there exists $\bar{x} \in E$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and*

$$f(\bar{x}, \bar{y}, x) \geq 0, \quad \forall x \in S(\bar{x}).$$

*Proof* We divide the proof into five steps. First let us define the set-valued map $A : E \times D \to 2^E$ by

$$A(x, y) := \{v \in S(x) : f(x, y, v) \leq f(x, y, u),\ \forall u \in S(x)\}, \quad (x, y) \in E \times D.$$

(I) For any $(x, y) \in E \times D$, the set $A(x, y)$ is nonempty. Indeed, for every $u \in S(x)$, let us set

$$H(u) := \{v \in S(x) : f(x, y, v) \leq f(x, y, u)\}.$$

Then $u \in H(u)$ and so $H(u) \neq \emptyset$.

Now, we show by induction that the family $\{H(u) : u \in S(x)\}$ has the finite intersection property. Let $u_1, u_2 \in S(x)$. By the assumptions on $f$ and Lemma 2.1, there exists some $t \in [0, 1]$ such that

$$f(x, y, tu_1 + (1-t)u_2) \leq f(x, y, u_1),$$
$$f(x, y, tu_1 + (1-t)u_2) \leq f(x, y, u_2).$$

Since $S(x)$ is convex, we have $v := tu_1 + (1-t)u_2 \in S(x)$. Hence, $v \in H(u_1) \cap H(u_2)$; therefore, $H(u_1) \cap H(u_2) \neq \emptyset$.

Let $u_1, \dots, u_n \in S(x)$ and assume $\bigcap_{i=1}^n H(u_i) \neq \emptyset$. Then, there exists $v \in S(x)$ such that

$$f(x, y, v) \leq f(x, y, u_i), \quad i = 1, \dots, n. \tag{5}$$

Let $u_{n+1} \in S(x)$. By the assumptions on $f$ and Lemma 2.1, there exists some $t \in [0, 1]$ such that

$$f(x, y, tv + (1-t)u_{n+1}) \leq f(x, y, u_{n+1}), \tag{6}$$
$$f(x, y, tv + (1-t)u_{n+1}) \leq f(x, y, v). \tag{7}$$

Thus, by (5), (6), (7), we get

$$f(x, y, tv + (1-t)u_{n+1}) \leq f(x, y, u_i), \quad i = 1, \dots, n+1.$$

Since $S(x)$ is convex, we have $tv+(1-t)u_{n+1} \in S(x)$. Therefore $tv+(1-t)u_{n+1} \in \bigcap_{i=1}^{n+1} H(u_i)$. This completes the induction step and the finite intersection property of the family $\{H(u) : u \in S(x)\}$ is established.

Next, we show that each $H(u)$ is closed. Let $\{v_\alpha : \alpha \in I\} \subset H(u)$ be a net with $v_\alpha \to v$. Then, $\{v_\alpha\} \subset S(x)$ and

$$f(x, y, v_\alpha) \leq f(x, y, u), \quad \forall \alpha \in I. \tag{8}$$

Since $S(x)$ is closed, $v \in S(x)$. We claim that $f(x, y, v) \leq f(x, y, u)$. Suppose to the contrary that

$$f(x, y, u) - f(x, y, v) \notin C.$$

Since $C$ is a closed convex cone, there exists some neighborhood $U$ of 0 in $Z$ such that

$$(f(x, y, u) - (f(x, y, v) + U)) \cap C = \emptyset,$$

whence

$$(f(x, y, u) - (f(x, y, v) + U + C)) \cap C = \emptyset. \tag{9}$$

The fact that $f$ is $C$-continuous at $(x, y, v)$ and $v_\alpha \to v$ implies that there exists $\alpha_0 \in I$ such that

$$f(x, y, v_\alpha) \in f(x, y, v) + U + C, \quad \forall \alpha \geq \alpha_0.$$

Thus by (9), we get

$$f(x, y, u) - f(x, y, v_\alpha) \notin C, \quad \forall \alpha \geq \alpha_0.$$

This contradicts (8). Hence, $f(x, y, v) \leq f(x, y, u)$ and so $v \in H(u)$. Therefore $H(u)$ is closed for every $u \in S(x)$.

The facts that $S(x)$ is a closed subset of $E$ and $E$ is compact imply that $S(x)$ is a compact subset of $E$. Thus, we have

$$\bigcap_{u \in S(x)} H(u) \neq \emptyset.$$

Let $v \in \bigcap_{u \in S(x)} H(u)$. Then $v \in S(x)$ and

$$f(x, y, v) \leq f(x, y, u), \quad \forall u \in S(x),$$

so that $v \in A(x, y)$. Hence $A(x, y) \neq \emptyset$.

(II) For any $(x, y) \in E \times D$, $A(x, y)$ is a closed subset of $E$. In fact, let $\{v_\alpha : \alpha \in I\} \subset A(x, y)$ with $v_\alpha \to v \in E$. Then, $\{v_\alpha : \alpha \in I\} \subset S(x)$ and, for any $u \in S(x)$, we have

$$f(x, y, v_\alpha) \leq f(x, y, u), \quad \forall \alpha \in I.$$

Since $S(x)$ is a closed set, $v \in S(x)$. It follows from the $C$-continuity of $f$ that

$$f(x, y, v) \leq f(x, y, u), \quad \forall u \in S(x).$$

Hence, $v \in A(x, y)$.

(III) For any fixed $(x, y) \in E \times D$, the set $A(x, y)$ is a convex subset of $E$. Indeed, let $v_1, v_2 \in A(x, y)$. Then, $v_1, v_2 \in S(x)$ and, for every $u \in S(x)$, we have

$$f(x, y, v_i) \le f(x, y, u), \quad i = 1, 2.$$

Since $f(x, y, \cdot)$ is $C$-quasiconvex on $E$, the set $\{v \in E : f(x, y, v) \le f(x, y, u)\}$ is therefore convex; thus,

$$f(x, y, tv_1 + (1-t)v_2) \le f(x, y, u), \quad \forall t \in [0, 1].$$

$S(x)$ being convex, we have $tv_1 + (1-t)v_2 \in S(x)$ for all $t \in [0, 1]$. Hence, $tv_1 + (1-t)v_2 \in A(x, y)$ and $A(x, y)$ is convex.

(IV) $A(x, y)$ is upper semicontinuous on $E \times D$. Since $E$ is compact, we need only to show that $A$ is closed (see [7, 8]). Let a net $\{(x_\alpha, y_\alpha) : \alpha \in I\} \subset E \times D$ be given such that

$$(x_\alpha, y_\alpha) \to (x, y) \in E \times D$$

and let $v_\alpha \in A(x_\alpha, y_\alpha)$ with $v_\alpha \to v$. We will show that $v \in A(x, y)$. Since $S$ is upper semicontinuous and $S(x)$ is closed for each $x \in E$, it follows that $S$ is closed (see [7, 8]). Thus, $(x_\alpha, v_\alpha) \in \text{Graph}(S)$ and $(x_\alpha, v_\alpha) \to (x, v)$ imply $(x, v) \in \text{Graph}(S)$, and whence $v \in S(x)$.

It remains to show that

$$f(x, y, v) \le f(x, y, u), \quad \forall u \in S(x). \tag{10}$$

Suppose to the contrary that, for some $u_0 \in S(x)$,

$$f(x, y, u_0) - f(x, y, v) \notin C.$$

Since $C$ is closed, there exists some neighborhood $U$ of 0 in $Z$ such that

$$[f(x, y, u_0) - f(x, y, v) + U] \cap C = \emptyset.$$

Pick some balanced neighborhood $U_1$ of 0 in $Z$ with $U_1 + U_1 \subset U$. Then,

$$[(f(x, y, u_0) + U_1) - (f(x, y, v) + U_1)] \cap C = \emptyset.$$

Therefore,

$$[(f(x, y, u_0) + U_1 - C) - (f(x, y, v) + U_1 + C)] \cap C = \emptyset. \tag{11}$$

Since $S$ is lower semicontinuous at $x$, $x_\alpha \to x$ and $u_0 \in S(x)$. Thus, by passing to a subnet if necessary, we can assume that there exists a net $\{u_\alpha\}$ with $u_\alpha \in S(x_\alpha)$ for each $\alpha$ and $u_\alpha \to u_0$. Thus, we have $(x_\alpha, y_\alpha, u_\alpha) \to (x, y, u_0)$ and $(x_\alpha, y_\alpha, v_\alpha) \to (x, y, v)$. $f$ being $C$-continuous and $(-C)$-continuous on $E \times D \times E$, there exists $\alpha_0 \in I$ such that

$$f(x_\alpha, y_\alpha, u_\alpha) \in f(x, y, u_0) + U_1 - C, \quad \forall \alpha \ge \alpha_0,$$

$$f(x_\alpha, y_\alpha, v_\alpha) \in f(x, y, v) + U_1 + C, \quad \forall \alpha \ge \alpha_0.$$

Combining this with (11), we have

$$f(x_\alpha, y_\alpha, u_\alpha) - f(x_\alpha, y_\alpha, v_\alpha) \notin C, \quad \forall \alpha \geq \alpha_0. \tag{12}$$

But this contradicts

$$f(x_\alpha, y_\alpha, v_\alpha) \leq f(x_\alpha, y_\alpha, u_\alpha),$$

as $v_\alpha \in A(x_\alpha, y_\alpha)$ and $u_\alpha \in S(x_\alpha)$. We conclude that (10) must be true, so that $v \in A(x, y)$ and $A$ has closed graph. Hence, $A$ is upper semicontinuous.

(V) Finally, we finish the proof of the theorem. Let $T(E) := \bigcup_{x\in E} T(x)$ and $L := \overline{\text{co}}(T(E)) \subset D$. Since $Y$ is quasicomplete and $T(E)$ is compact (see [7]), it follows that $L$ is a compact convex subset of $Y$ (see [12]). Then, the set-valued map $F : E \times L \to 2^{E\times L}$, defined by

$$F(x, y) := (A(x, y), T(x)), \quad (x, y) \in E \times L,$$

is upper semicontinuous and, for each $(x, y) \in E \times L$, the set $F(x, y)$ is a nonempty closed convex subset of $E \times L$. By the Kakutani-Fan-Glicksberg fixed-point theorem (see [13]), there exists a point $(\bar{x}, \bar{y}) \in E \times L$ such that $(\bar{x}, \bar{y}) \in F(\bar{x}, \bar{y})$, that is,

$$\bar{x} \in A(\bar{x}, \bar{y}), \quad \bar{y} \in T(\bar{x}).$$

This means that $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ and

$$f(\bar{x}, \bar{y}, u) \geq f(\bar{x}, \bar{y}, \bar{x}), \quad \forall u \in S(\bar{x}). \tag{13}$$

By condition (i), $f(\bar{x}, \bar{y}, \bar{x}) \geq 0$. This and (13) imply

$$f(\bar{x}, \bar{y}, u) \geq 0, \quad \forall u \in S(\bar{x}).$$

The theorem is proved. □

*Remark 3.1* If $D$ is a compact convex subset of $Y$, then we can drop the condition that $Y$ be quasicomplete.

*Remark 3.2* Comparing with Theorem 1 of [3], here we do not require that $C^*$ have a weak* base and $f$ be continuous. It is easy to see that a continuous function must be $C$-continuous and $(-C)$-continuous, but the converse is not necessarily true as demonstrated by the example below. Thus, our result generalizes the main result of [3].

*Example 3.1* Let $Z = R^2$ and $C = \{(x, y) : x \in R, y \geq 0\}$. Then, $C$ is a closed convex cone in $Z$. Define $f : [-1, 1] \times [-1, 1] \to Z$ by

$$f(x, y) := \begin{cases} (1/y, y), & y \neq 0, \\ (0, 0), & y = 0. \end{cases}$$

It is clear that $f$ is $C$-continuous and $(-C)$-continuous at $(0, 0)$, but not continuous there.

## 4 Applications

In this section, we give some applications of Theorem 3.1.

**Theorem 4.1** *Let $X$, $Y$, $Z$, $E$, $D$, $C$, $S$, $T$ be given as in Theorem 3.1. Let $\varphi : E \times D \to Z$ be $C$-continuous and $(-C)$-continuous on $E \times D$. Let $h : E \times E \to Z$ be $C$-continuous and $(-C)$-continuous on $E \times E$. Suppose that:*

(i) *For any $x \in E$ and $y \in T(x)$, $\varphi(x, y) + h(x, x) \geq 0$.*
(ii) *For any $x \in E$, $h(x, u)$ is properly $C$-quasiconvex in $u$.*

*Then, there exist $\bar{x} \in E$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and*

$$h(\bar{x}, u) \geq -\varphi(\bar{x}, \bar{y}), \quad \forall u \in S(\bar{x}).$$

*Proof* Define $f : E \times D \times E \to Z$ by

$$f(x, y, u) := \varphi(x, y) + h(x, u), \quad (x, y, u) \in E \times D \times E.$$

Then, $f$ is $C$-continuous and $(-C)$-continuous on $E \times D \times E$. By assumption (i), for any $x \in E$ and $y \in T(x)$, we have $f(x, y, x) \geq 0$. By assumption (ii), $h(x, u)$ is properly $C$-quasiconvex in $u$ for any $x \in E$. It is easy to see that for any $(x, y) \in E \times D$, $f(x, y, u)$ is properly $C$-quasiconvex in $u$. We can now apply Theorem 3.1 to conclude that there exist $\bar{x} \in E$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and

$$f(\bar{x}, \bar{y}, u) \geq 0, \quad \forall u \in S(\bar{x}),$$

or equivalently,

$$h(\bar{x}, u) \geq -\varphi(\bar{x}, \bar{y}), \quad \forall u \in S(\bar{x}).$$

□

**Theorem 4.2** *Let $X$, $Y$, $Z$, $E$, $D$, $C$, $S$, $T$ be given as in Theorem 3.1. Let $\psi : E \times D \times E \to Z$ be $C$-continuous and $(-C)$-continuous on $E \times D \times E$. Suppose that:*

(i) *There exists $c \in Z$ such that for any $x \in E$ and $y \in T(x)$, $\psi(x, y, x) \geq c$.*
(ii) *For any $(x, y) \in E \times D$, $\psi(x, y, u)$ is properly $C$-quasiconvex in $u$.*

*Then, there exist $\bar{x} \in E$ and $\bar{y} \in T(\bar{x})$ such that $\bar{x} \in S(\bar{x})$ and*

$$\psi(\bar{x}, \bar{y}, u) \geq c, \quad \forall u \in S(\bar{x}).$$

*Proof* This follows directly from Theorem 3.1 by setting $f : E \times D \times E \to Z$ by

$$f(x, y, u) := \psi(x, y, u) - c, \quad (x, y, u) \in E \times D \times E.$$

□

The following corollary is a vector version of the Walras excess demand theorem.

**Corollary 4.1** *Let $X$, $Y$, $Z$, $E$, $D$, $C$, $T$ be given as in Theorem 3.1. Let $\varphi : E \times D \to Z$ be $C$-continuous and $(-C)$-continuous on $E \times D$. Suppose that:*

(i) *There exists* $c \in Z$ *such that for any* $x \in E$ *and* $y \in T(x)$, $\varphi(x, y) \geq c$.
(ii) *For any* $y \in D$, $\varphi(u, y)$ *is properly* $C$*-quasiconvex in* $u$.

*Then, there exist* $\bar{x} \in E$ *and* $\bar{y} \in T(\bar{x})$ *such that*

$$\varphi(u, \bar{y}) \geq c, \quad \forall u \in E.$$

*Proof* This follows directly from Theorem 4.2 by setting $S : E \to 2^E$ as $S(x) := E$ for all $x \in E$ and setting $\psi : E \times D \times E \to Z$ as

$$\psi(x, y, u) := \varphi(u, y), \quad (x, y, u) \in E \times D \times E. \qquad \square$$

## 5 Stability

In this section, we discuss the stability of the solutions for the generalized Ky Fan inequality problem (1).

Throughout this section, let $X, Y$ be Banach spaces and let $Z$ be a real locally convex Hausdorff topological vector space. Let $E \subset X$ be a compact convex subset, let $D \subset Y$ be a closed convex set, and let $C \subset Z$ be a closed convex cone. Let

$$M := \{(T, S) \mid T : E \to 2^D \text{ is upper semicontinuous with nonempty compact convex values, and } S : E \to 2^E \text{ is continuous with nonempty closed convex values}\}.$$

Let $B_1, B_2$ be compact sets in a normed space. Recall that the Hausdorff metric is defined by

$$h(B_1, B_2) := \max\{h^0(B_1, B_2), h^0(B_2, B_1)\},$$

where $h^0(B_1, B_2) := \sup_{b \in B_1} d(b, B_2)$ and $d(b, B_2) := \inf_{b' \in B_2} \|b - b'\|$.

For $(T, S), (T', S') \in M$, define

$$\rho((T, S), (T', S')) := \sup_{x \in E} h_1(T(x), T'(x)) + \sup_{x \in E} h_2(S(x), S'(x)),$$

with $h_1, h_2$ being the appropriate Hausdorff metrics. Obviously, $(M, \rho)$ is a metric space.

Assume that $f$ satisfies the assumptions of Theorem 3.1. Then, for each $(T, S) \in M$, the corresponding inequality problem (1) has a solution $\bar{x}$ by Theorem 3.1, that is, there exist $\bar{x} \in E$, $\bar{y} \in T(\bar{x})$ satisfying $\bar{x} \in S(\bar{x})$ and

$$f(\bar{x}, \bar{y}, x) \geq 0, \quad \forall x \in S(\bar{x}). \tag{14}$$

For $(T, S) \in M$, let

$$\psi(T, S) := \{\bar{x} \in E : \bar{x} \in S(\bar{x}) \text{ and there exists } \bar{y} \in T(\bar{x}) \text{ such that } f(\bar{x}, \bar{y}, x) \geq 0, \ \forall x \in S(\bar{x})\}.$$

Then, $\psi(T, S) \neq \emptyset$ and so $\psi$ defines a set-valued mapping from $M$ into $E$.

**Lemma 5.1** ([14, 15]) *Let $W$ be a metric space and let $A$, $A_n (n = 1, 2, \ldots)$ be compact sets in $W$. Suppose that for any open set $O \supset A$, there exists $n_0$ such that $A_n \subset O$ for all $n \geq n_0$. Then, any sequence $\{x_n\}$ satisfying $x_n \in A_n$ has a convergent subsequence with limit in $A$.*

**Theorem 5.1** $\psi : M \to 2^E$ *is upper semicontinuous with compact values.*

*Proof* Since $E$ is compact, we need only to show that $\psi$ has a closed graph. Let a sequence $\{(T_n, S_n, x_n)\} \subset \text{Graph}(\psi)$ be given such that $(T_n, S_n, x_n) \to (T, S, x^*) \in M \times E$. We show that $(T, S, x^*) \in \text{Graph}(\psi)$.

Since $x_n \in \psi(T_n, S_n)$, we have that $x_n \in S_n(x_n)$ and there exists $y_n \in T_n(x_n)$ such that

$$f(x_n, y_n, x) \geq 0, \quad \forall x \in S_n(x_n). \tag{15}$$

For any open set $O \supset T(x^*)$, since $T(x^*)$ is a compact set, there exists $\varepsilon_0 > 0$ such that

$$\{y \in Y : d(y, T(x^*)) < \varepsilon_0\} \subset O, \tag{16}$$

where $d(y, T(x^*)) = \inf_{y' \in T(x^*)} \|y - y'\|$.

Since $\rho((T_n, S_n), (T, S)) \to 0$, $x_n \to x^*$, and since $T$ is upper semicontinuous at $x^*$, there exists $n_0$ such that

$$\sup_{x \in E} h_1(T_n(x), T(x)) < \varepsilon_0/2, \tag{17}$$

$$T(x_n) \subset \{y \in Y : d(y, T(x^*)) < \varepsilon_0/2\}, \quad \forall n \geq n_0. \tag{18}$$

From (16), (17), (18), we have

$$\begin{aligned} T_n(x_n) &\subset \{y \in Y : d(y, T(x_n)) < \varepsilon_0/2\} \\ &\subset \{y \in Y : d(y, T(x^*)) < \varepsilon_0\} \subset O, \quad \forall n \geq n_0. \end{aligned} \tag{19}$$

Noting (19), $T(x^*) \subset O$ and $y_n \in T_n(x_n)$, we can apply Lemma 5.1 to get a convergent subsequence of $\{y_n\}$, again denoted as $\{y_n\}$, with limit $y^* \in T(x^*)$.

We now show that $x^* \in S(x^*)$. Since $E$ is compact and $S(x^*)$, $S_n(x_n)$ are closed subsets of $E$. Therefore, $S(x^*)$ and $S_n(x_n)$ are compact. As $x_n \in S_n(x_n)$, $\rho((T_n, S_n), (T, S)) \to 0$, $x_n \to x^*$, and $S$ is continuous, it is easy to see by using the same argument as above that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x_0 \in S(x^*)$. It follows that $x^* = x_0 \in S(x^*)$.

To finish the proof of the theorem, we need only to show that $f(x^*, y^*, x) \geq 0$ for all $x \in S(x^*)$. Since $x_n \to x^*$ and $S$ is lower semicontinuous at $x*$, thus for any $x \in S(x^*)$ we have, by passing to subsequence if necessary, a sequence $\{z_n\}$ satisfying $z_n \in S(x_n)$ and $z_n \to x$. Since $\rho((T_n, S_n), (T, S)) \to 0$, we can choose a subsequence $\{S_{n_k}\}$ of $\{S_n\}$ such that

$$\sup_{x \in E} h_2(S_{n_k}(x), S(x)) < 1/k. \tag{20}$$

Thus,

$$h_2(S_{n_k}(x_{n_k}), S(x_{n_k})) < 1/k.$$

This implies that there exist $z'_{n_k} \in S_{n_k}(x_{n_k})$, $k = 1, 2, \ldots,$ such that

$$\|z'_{n_k} - z_{n_k}\| < 1/k.$$

As

$$\|z'_{n_k} - x\| \le \|z'_{n_k} - z_{n_k}\| + \|z_{n_k} - x\| < 1/k + \|z_{n_k} - x\| \to 0,$$

we have $z'_{n_k} \to x$. Since $x_{n_k} \in S_{n_k}(x_{n_k})$, $y_{n_k} \in T_{n_k}(x_{n_k})$, $z'_{n_k} \in S_{n_k}(x_{n_k})$, applying (15) yields

$$f(x_{n_k}, y_{n_k}, z'_{n_k}) \ge 0.$$

It follows from the $C$-continuity of $f$ that

$$f(x^*, y^*, x) \ge 0.$$

$x \in S(x^*)$ being arbitrary, we have

$$f(x^*, y^*, x) \ge 0, \quad \forall x \in S(x^*). \tag{21}$$

Since $x^* \in S(x^*)$ and $y^* \in T(x^*)$, (21) implies that $(T, S, x^*) \in \mathrm{Graph}(\psi)$, and so $\mathrm{Graph}(\psi)$ is closed in $M \times E$. Now $E$ being compact, we assert that $\psi$ has compact values. The theorem is proved. □

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