J Optim Theory Appl (2010) 145: 373–386
DOI 10.1007/s10957-009-9644-y

Existence and Boundedness of the Kuhn-Tucker Multipliers in Nonsmooth Multiobjective Optimization

X.F. Li · J.Z. Zhang

Published online: 29 January 2010

Abstract Using the idea of upper convexificators, we propose constraint qualifications and study existence and boundedness of the Kuhn-Tucker multipliers for a nonsmooth multiobjective optimization problem with inequality constraints and an arbitrary set constraint. We show that, at locally weak efficient solutions where the objective and constraint functions are locally Lipschitz, the constraint qualifications are necessary and sufficient conditions for the Kuhn-Tucker multiplier sets to be nonempty and bounded under certain semiregularity assumptions on the upper convexificators of the functions.

Keywords Upper convexificators · Constraint qualifications · Existence and boundedness of Kuhn-Tucker multipliers · Nonsmooth multiobjective optimization

1 Introduction and Preliminaries

Investigating the nonemptiness and boundedness of the Kuhn-Tucker multiplier sets for optimization problems is not only intrinsically interesting, but also useful in certain stability and duality studies for nonconvex minimization problems (see for example Ref. [1]). For a differentiable scalar optimization problem with equality and

Communicated by H.P. Benson.

The study was supported by the National Science Foundation of China and Program 985 of Jilin University.

X.F. Li (✉)
Department of Applied Mathematics, School of Mathematical Sciences, Jilin University, Changchun, China
e-mail: xf-li@jlu.edu.cn

J.Z. Zhang
BJNU-HKBU United International College, Zhuhai, China

inequality constraints, Gauvin (Ref. [2]) showed that the Mangasarian-Fromovitz constraint qualification (Ref. [3]) is a necessary and sufficient condition for the set of Kuhn-Tucker multipliers to be nonempty and bounded. Later, by means of the Clarke subdifferentials (Ref. [4]), Nguyen, Strodiot and Mifflin (Ref. [5]) generalized Gauvin's result to the case of a nonsmooth scalar optimization problem with equality and inequality constraints as well as a proper set constraint, assuming that the objective function and the functions defining the inequality constraints are locally Lipschitz. By means of the Clarke subdifferentials, Pappalardo (Ref. [6]) used a generalized Mangasarian-Fromovitz qualification to establish a result concerning the bound for the Kuhn-Tucker multipliers for a nonsmooth scalar optimization problem with equality and inequality constraints, assuming that all the functions involved in the problem are locally Lipschitz. In the more general setting of Banach spaces, by means of the Clarke subdifferentials, Jourani (Ref. [7]) introduced several constraint qualifications that ensure the nonemptiness and boundedness of the Kuhn-Tucker multiplier sets for a nonsmooth scalar optimization problem with equality and inequality constraints as well as a closed set constraint, under the assumption that all the functions involved in the problem are locally Lipschitz.

Recently, the concept of convexificators of real-valued functions and its extension, approximate Jacobians, to vector-valued maps have been used to extend and sharpen various results in nonsmooth analysis and optimization (Refs. [8–17]). As has been noted in Ref. [11] (see also Refs. [9] and [15]), the Clarke subdifferential, the Michel-Penot subdifferential and some other well-known subdifferentials of a locally Lipschitz function are examples of convexificators and these subdifferentials may contain the convex hull of a convexificator. Therefore, the descriptions of the optimality conditions, mean-value theorems, and calculus rules in terms of convexificators provide sharper results. For optimization problems involving nonsmooth functions, various results concerning necessary optimality conditions that use convexificators or approximate Jacobians have been established in Refs. [10, 12, 15–17], and [18–20].

In this paper, by using the idea of upper convexificators, we study the nonemptiness and boundedness of the Kuhn-Tucker multiplier sets for a nonsmooth multiobjective optimization problem with inequality constraints and an arbitrary set constraint. We propose constraint qualifications and show (Theorems 2.1 and 3.1) that, at locally weak efficient solutions where the objective and constraint functions are locally Lipschitz, the qualifications are necessary and sufficient conditions for the Kuhn-Tucker multiplier set to be nonempty and bounded, provided that the upper convexificators of the functions satisfy certain semiregularity conditions. We give examples to show that the semiregularity conditions on the upper convexificators are necessary for our qualifications to guarantee nonemptiness of the Kuhn-Tucker multiplier sets.

Since the Clarke subdifferentials and the Michel-Penot subdifferentials of a locally Lipschitz function are convexificators, the results in this work are valid with the convexificators being replaced respectively by the Clarke subdifferentials and the Michel-Penot subdifferentials. However, by an example given in this paper (see Example 3.2), we show that, while the constraint qualifications considered in this paper hold at a locally weak efficient solution, they may fail to hold when the convexificators in the constraint qualifications are replaced by the Clarke subdifferentials or the

Michel-Penot subdifferentials. By the same example, we also show that, at a locally weak efficient solution, while the set of all Kuhn-Tucker multiplier vectors associated with the Kuhn-Tucker type necessary optimality conditions in terms of convexificators is bounded, it may not necessarily be bounded when the convexificators in the Kuhn-Tucker type necessary optimality conditions are replaced by the Clarke subdifferentials or the Michel-Penot subdifferentials.

The constraint qualifications proposed in this paper are of the Mangasarian-Fromovitz type in the case of optimization problems without equality constraints. We derive the Kuhn-Tucker type necessary optimality conditions (nonemptiness of the Kuhn-Tucker multiplier sets) in a direct way, which is different from the method that derives the Fritz-John type necessary conditions first and then imposes some constraint qualifications to obtain Kuhn-Tucker type necessary conditions.

We conclude this section by providing some notations that will be used in the sequel. Throughout the paper, let $\mathbb{R}^n$ be the usual n-dimensional Euclidean space and let $\mathbb{R}^n_+$ be its nonnegative orthant, namely,

$$\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0,\ \forall i = 1, \dots, n\}.$$

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two vectors in $\mathbb{R}^n$. Then,

$$\begin{aligned} x &\leqq y, \quad \text{iff } x_i \leq y_i,\ i = 1, \dots, n, \\ x &\leq y, \quad \text{iff } x_i \leqq y_i \text{ and } x \neq y, \\ x &< y, \quad \text{iff } x_i < y_i,\ i = 1, \dots, n. \end{aligned}$$

Let S be a nonempty subset of $\mathbb{R}^n$. The closure of S, the convex hull of S, and the convex cone (containing the origin of $\mathbb{R}^n$) generated by S are denoted respectively by $\operatorname{cl} S$, $\operatorname{co} S$, and $\operatorname{cone} S$. The negative polar cone S^-, the strictly negative polar cone S^s, and the positive polar cone S^+ of S are defined respectively by

$$\begin{aligned} S^- &= \{v \in \mathbb{R}^n \mid \langle x, v\rangle \leqq 0,\ \forall x \in S\}, \\ S^s &= \{v \in \mathbb{R}^n \mid \langle x, v\rangle < 0,\ \forall x \in S\}, \\ S^+ &= -S^-. \end{aligned}$$

The adjacent cone $A(S, x)$ and the contingent cone $T(S, x)$ to S at $x \in \operatorname{cl} S$ are defined respectively by

$$\begin{aligned} A(S, x) &= \{v \in \mathbb{R}^n \mid \forall t_n \downarrow 0,\ \exists v_n \to v \text{ such that } x + t_n v_n \in S\}, \\ T(S, x) &= \{v \in \mathbb{R}^n \mid \exists t_n \downarrow 0 \text{ and } v_n \to v \text{ such that } x + t_n v_n \in S\}. \end{aligned}$$

It is well known that $A(S, x)$ and $T(S, x)$ are always closed, but not necessarily convex, and that, if S is a convex set, then $A(S, x)$ and $T(S, x)$ coincide and are convex. Moreover, in general, $A(S, x) \subseteq T(S, x)$. For more information on adjacent and contingent cones, the readers may consult Refs. [4] and [21–24].

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}} =: \mathbb{R} \cup \{\pm\infty\}$ be an extended real-valued function. Let $x \in \mathbb{R}^n$ where f is finite. The lower and upper Dini derivatives of f at x in the direction

$v \in \mathbb{R}^n$ are defined, respectively, by

$$f^-(x; v) = \liminf_{t \downarrow 0} t^{-1}[f(x + tv) - f(x)],$$

$$f^+(x; v) = \limsup_{t \downarrow 0} t^{-1}[f(x + tv) - f(x)].$$

The notion of the convexificators was introduced in Ref. [8] and studied in Refs. [9, 11, 15], and [18]. On the lines of Ref. [11] (see also Ref. [15]), we now give the definitions of the upper convexificators.

Definition 1.1 Let the function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be finite at $x \in \mathbb{R}^n$.

(a) The function f is said to admit an upper convexificator $\partial^* f(x)$ at x if $\partial^* f(x) \subset \mathbb{R}^n$ is closed and, for every $v \in \mathbb{R}^n$,

$$f^-(x; v) \leqq \sup_{\xi \in \partial^* f(x)} \langle v, \xi \rangle.$$

(b) The function f is said to admit an upper semiregular convexificator $\partial^* f(x)$ at x if $\partial^* f(x) \subset \mathbb{R}^n$ is closed and, for every $v \in \mathbb{R}^n$,

$$f^+(x; v) \leqq \sup_{\xi \in \partial^* f(x)} \langle v, \xi \rangle.$$

Obviously, an upper semiregular convexificator of f at a point is an upper convexificator of f at the point. However, the converse does not necessarily hold. As has been noted in Refs. [9] and [11], if f is locally Lipschitz at x, then the Clarke subdifferential $\partial^c f(x)$ (see Ref. [4]) and the Michel-Penot subdifferential $\partial^{mp} f(x)$ of f at x (see Ref. [25]) are upper semiregular convexificators of f at x. The two subdifferentials are given respectively by

$$\partial^{mp} f(x) = \{\xi \in \mathbb{R}^n \mid f^+_{mp}(x; v) \geq \langle \xi, v \rangle, \ \forall\, v \in \mathbb{R}^n\},$$

$$\partial^c f(x) = \{\xi \in \mathbb{R}^n \mid f^+_c(x; v) \geq \langle \xi, v \rangle, \ \forall\, v \in \mathbb{R}^n\},$$

where $f^+_{mp}(x; v)$ and $f^+_c(x; v)$ are respectively the Clarke and Michel-Penot upper generalized directional derivatives of f at x in the direction v defined by

$$f^+_{mp}(x; v) = \sup_{z \in \mathbb{R}^n} \liminf_{t \downarrow 0} t^{-1}[f(x + tz + tv) - f(x + tz)],$$

$$f^+_c(x; v) = \limsup_{x' \to x,\ t \downarrow 0} t^{-1}[f(x' + tv) - f(x')].$$

Moreover, for locally Lipschitz functions, one may generate upper semiregular convexificators which are smaller than the Michel-Penot subdifferential and the Clarke subdifferential as an example given in Ref. [11] shows (see also Example 3.2 in this paper). The readers may consult Refs. [8, 9, 11, 15, 18] and references therein for more information on the definitions and properties of convexificators as well as the relationship of convexificators and some other well-known subdifferentials including the Michel-Penot and Clarke subdifferentials.

2 Existence and Boundedness

Consider the following nonsmooth multiobjective optimization problem with inequality and an arbitrary set constraints:

$$\begin{array}{lll}\text{(MPIS)} & \text{Min} & f(x)=(f_1(x),\dots,f_p(x)),\\ & \text{s.t.} & g(x)=(g_1(x),\dots,g_q(x))\leqq 0,\\ & & x\in S,\end{array}$$

where f_i and $g_j:\mathbb{R}^n\to\overline{\mathbb{R}}$ are extended real-valued functions for all $i\in I:=\{1,2,\dots,p\}$ and $j\in J:=\{1,2,\dots,q\}$ and where S is an arbitrary subset of $\mathbb{R}^n$. Denote by X the feasible region of problem (MPIS), namely,

$$X=\{x\in\mathbb{R}^n \mid x\in S \text{ and } g(x)\leqq 0\}.$$

A feasible point $x\in X$ is said to be a locally weak efficient solution to problem (MPIS) if there is a real number $\delta>0$ such that there is no $y\in X\cap B(x,\delta)$ satisfying $f(y)<f(x)$, where $B(x,\delta)$ is the open unit ball centered at x with radius δ. For the sake of simplicity in our presentation, in the sequel we assume without loss of generality that all the inequality constraints are binding at the feasible point $x\in X$, namely, $g_j(x)=0$ for all $j\in J$.

Let x be a feasible point for problem (MPIS) at which f_i and g_j admit, respectively, upper convexificators $\partial^* f_i(x)$ and $\partial^* g_j(x)$ for each $i\in I$ and $j\in J$. The constraint qualifications considered in this paper are

(CQ1) $\exists v\in T(S,x)$ such that $v\in(\bigcup_{j\in J}\partial^* g_j(x))^s$,
(CQ2) $\exists v\in A(S,x)$ such that $v\in(\bigcup_{j\in J}\partial^* g_j(x))^s$.

Let K be a closed convex subcone of $T(S,x)$ or $A(S,x)$. Denote by $\Lambda(K,x)$ the set of all Kuhn-Tucker multiplier vectors $(\alpha_1,\dots,\alpha_p,\beta_1,\dots,\beta_q)$ in $\mathbb{R}^p_+\times\mathbb{R}^q_+$ associated with the Kuhn-Tucker type necessary optimality conditions with respect to K, namely, $(\alpha_1,\dots,\alpha_p,\beta_1,\dots,\beta_q)\in\Lambda(K,x)$ if and only if

$$(\alpha_1,\dots,\alpha_p,\beta_1,\dots,\beta_q)\in\mathbb{R}^p_+\times\mathbb{R}^q_+,\qquad \sum_{i\in I}\alpha_i=1,$$

$$0\in\sum_{i\in I}\alpha_i\,\mathrm{co}\,\partial^* f_i(x)+\sum_{j\in J}\beta_j\,\mathrm{co}\,\partial^* g_j(x)+K^-.$$

The aim of this paper is to show that, under some conditions, the constraint qualifications (CQ1) and (CQ2) are necessary and sufficient conditions for the set of Kuhn-Tucker multipliers to be nonempty and bounded at locally weak efficient solutions. In this section, we show that, at a locally weak efficient solution x, the qualification (CQ1) is a necessary and sufficient condition for the Kuhn-Tucker multiplier set $\Lambda(K,x)$ with respect to the closed convex subcone K of the contingent cone $T(S,x)$ to be nonempty and bounded at a locally weak efficient solution x under certain conditions.

Theorem 2.1 *Let x be a locally weak efficient solution to problem (MPIS). Suppose that, at x, f_i and g_j admit bounded upper convexificators $\partial^* f_i(x)$ and $\partial^* g_j(x)$ for each $i \in I$ and $j \in J$.*

(a) *Assume that f_i and g_j are locally Lipschitz at x and that $\partial^* f_i(x)$ and $\partial^* g_j(x)$ are upper semiregular for all $i \in I$ and $j \in J$. If (CQ1) holds at x, then for every closed convex cone K_v which is included in $T(S,x)$ and contains v, the set $\Lambda(K_v, x)$ is nonempty.*
(b) *If (CQ1) holds at x, and if, for every closed convex cone K_v which is included in $T(S,x)$ and contains v, the set $\Lambda(K_v,x)$ is nonempty, then $\Lambda(K_v,x)$ is bounded.*
(c) *If for some closed convex subcone K of $T(S,x)$, the Kuhn-Tucker multiplier set $\Lambda(K,x)$ with respect to K is nonempty and bounded, then (CQ1) holds at x.*

Proof (a) We prove that, at the locally weak efficient solution x, (CQ1) ensures the nonemptiness of the set $\Lambda(K_v,x)$ for every closed convex cone K_v which is included in $T(S,x)$ and contains v. Assume that (CQ1) holds at x. Suppose on the contrary that $\Lambda(K_v,x) = \emptyset$ for some K_v such that $v \in K_v \subseteq T(S,x)$. Then, by the definition of the set $\Lambda(K_v,x)$, it follows that, for every vector $(\alpha_1,\dots,\alpha_p,\beta_1,\dots,\beta_q)$ in $\mathbb{R}^p_+ \times \mathbb{R}^q_+$ satisfying $\sum_{i\in I}\alpha_i = 1$,

$$0 \notin \sum_{i\in I}\alpha_i \operatorname{co}\partial^* f_i(x) + \sum_{j\in J}\beta_j \operatorname{co}\partial^* g_j(x) + K_v^-. \tag{1}$$

For notational simplicity, we denote

$$F = \bigcup_{i\in I}\operatorname{co}\partial^* f_i(x) \quad \text{and} \quad G = \bigcup_{j\in J}\operatorname{co}\partial^* g_j(x).$$

We assert that

$$0 \notin \operatorname{co}(F \cup G) + K_v^-. \tag{2}$$

Indeed, if the assertion is not true, then $\exists(\bar\alpha_1,\dots,\bar\alpha_p,\bar\beta_1,\dots,\bar\beta_q)$ in $\mathbb{R}^p_+ \times \mathbb{R}^q_+$, with $\sum_{i\in I}\bar\alpha_i + \sum_{j\in J}\bar\beta_j = 1$, $\xi_i \in \operatorname{co}\partial^* f_i(x)$, $\eta_j \in \operatorname{co}\partial^* g_j(x)$, and $\gamma \in K_v^-$ such that

$$\sum_{i\in I}\bar\alpha_i\xi_i + \sum_{j\in J}\bar\beta_j\eta_j + \gamma = 0. \tag{3}$$

In view of (CQ1) and hypothesis (a), we have that $\langle \eta_j, v\rangle < 0$ for all $j \in J$ and that $\langle \gamma, v\rangle \le 0$. Hence, $(\bar\alpha_1,\dots,\bar\alpha_p) \neq 0$. By dividing both sides of (3) by $\sum_{i\in I}\bar\alpha_i$, we obtain a contradiction to (1). Thus, the assertion (2) is true.

We now show that assertion (2) implies

$$(F \cup G)^s \cap K_v \neq \emptyset. \tag{4}$$

Since $\partial^* f_i(x)$ and $\partial^* g_j(x)$, $i \in I$ and $j \in J$, are closed by definition and bounded by hypothesis, the set $F \cup G$ is compact; hence, the convex set on the right-hand side of (2) is closed. By separating this convex set from the origin, we find some vector $w \in \mathbb{R}^n$ such that $\langle a + \gamma, w\rangle < 0$ for all $a \in F \cup G$ and $\gamma \in K_v^-$. Since K_v is a closed convex cone, we deduce that $w \in (K_v^-)^- = K_v$ and $\langle a, w\rangle < 0$ for all $a \in F \cup G$. This shows that (4) is true. This also gives

$$f_i^+(x, w) < 0, \quad \forall i \in I, \tag{5}$$

$$g_j^+(x, w) < 0, \quad \forall j \in J, \tag{6}$$

by noting that $\partial^* f_i(x)$ and $\partial^* g_j(x)$, $i \in I$ and $j \in J$, are upper semiregular by hypothesis.

Since $w \in K_v \subseteq T(S, x)$, by the definition of contingent cones it follows that there exists $(w_n, t_n) \to (w, 0^+)$ such that $x + t_n w_n \in S$. For each $i \in I$, since f_i is locally Lipschitz near x, we have

$$\lim_{t_n \downarrow 0} t_n^{-1} |f_i(x + t_n w_n) - f_i(x + t_n w)| \leqq \lim_{n\to+\infty} L_i |w_n - w| \to 0, \tag{7}$$

where L_i is the Lipschitz constant for f_i near x. Therefore, for each $i \in I$, writing

$$\begin{aligned} t_n^{-1}[f_i(x + t_n w_n) - f_i(x)] = t_n^{-1}&[f_i(x + t_n w_n) - f_i(x + t_n w)] \\ &+ t_n^{-1}[f_i(x + t_n w) - f_i(x)] \end{aligned}$$

and noting (5) and (7), we have

$$\limsup_{t_n \downarrow 0} t_n^{-1}[f_i(x + t_n w_n) - f_i(x)] \leqq 0 + f_i^+(x, w) < 0.$$

It then follows that, for all sufficiently large n,

$$f_i(x + t_n w_n) < f_i(x), \quad \forall i \in I. \tag{8}$$

Similarly, in view of inequalities (6), we have that, for all sufficiently large n,

$$g_j(x + t_n w_n) < g_j(x), \quad \forall j \in J, \tag{9}$$

which together with the fact that $x + t_n w_n \in S$ imply that $x + t_n w_n$ are feasible points for problem (MPIS) for all sufficiently large n. But now inequalities (8) contradict the assumption that x is a locally weak efficient solution to the problem. This completes the proof of (a).

(b) We show that (CQ1) ensures the boundedness of $\Lambda(K_v, x)$ for every closed convex cone K_v which is included in $T(S, x)$ and contains v. Suppose on the contrary that $\Lambda(\hat{K}_v, x)$ is not bounded for a cone $\hat{K}_v$ of this kind. Then, there exist $\xi_i^{(n)} \in \operatorname{co} \partial^* f_i(x)$, $i \in I$, $\eta_j^{(n)} \in \operatorname{co} \partial^* g_j(x)$, $j \in J$, $\gamma^{(n)} \in \hat{K}_v^-$, and

$$(\alpha_1^{(n)}, \dots, \alpha_p^{(n)}, \beta_1^{(n)}, \dots, \beta_q^{(n)}) \in \Lambda(\hat{K}_v, x)$$

satisfying $\beta_{j_0}^{(n)} \to +\infty$ (as $n \to +\infty$) for some $j_0 \in J$, such that

$$\sum_{i\in I} \alpha_i^{(n)} \xi_i^{(n)} + \sum_{j\in J} \beta_j^{(n)} \eta_j^{(n)} + \gamma^{(n)} = 0, \quad \forall n = 1, 2, \dots.$$

It follows that

$$\sum_{i\in I}\alpha_i^{(n)}\langle\xi_i^{(n)},v\rangle+\sum_{j\in J}\beta_j^{(n)}\langle\eta_j^{(n)},v\rangle+\langle\gamma^{(n)},v\rangle=0,\quad\forall n=1,2,\ldots,\tag{10}$$

from which and by noting that $v\in(\bigcup_{j\in J}\partial^*g_j(x))^s\cap\hat{K}_v$ in view of (CQ1), we have

$$\sum_{i\in I}\alpha_i^{(n)}\langle\xi_i^{(n)},v\rangle+\beta_{j_0}^{(n)}\langle\eta_{j_0}^{(n)},v\rangle\geqq 0,\quad\forall n=1,2,\ldots.\tag{11}$$

Since $\partial^*f_i(x)$ $(i\in I)$ and $\partial^*g_{j_0}(x)$ are closed by the definition of convexificators and bounded by hypothesis, so are their convex hulls $\operatorname{co}\partial^*f_i(x)$ and $\operatorname{co}\partial^*g_{j_0}(x)$. Without loss of generality, we may assume that, as $n\to+\infty$, $\xi_i^{(n)}\to\xi_i$ $(i\in I)$ and $\eta_{j_0}^{(n)}\to\eta_{j_0}$. Note that ξ_i is in $\operatorname{co}\partial^*f_i(x)$ and η_{j_0} in $\operatorname{co}\partial^*g_{j_0}(x)$. Now, by dividing the left-hand side of (11) by $\beta_{j_0}^{(n)}$ and passing to the limit as $n\to+\infty$, we derive that

$$\left(\sum_{i\in I}\alpha_i^{(n)}\langle\xi_i^{(n)},v\rangle+\beta_{j_0}^{(n)}\langle\eta_{j_0}^{(n)},v\rangle\right)/\beta_{j_0}^{(n)}\to\sum_{i\in I}0\cdot\langle\xi_i,v\rangle+\langle\eta_{j_0},v\rangle<0,$$

contradicting inequality (11). This proves (b).

(c) We now prove that the nonemptiness and boundedness of the set $\Lambda(K,x)$ ensure that (CQ1) holds at x. Suppose on the contrary that this is not true. Then, in view of (CQ1), $(\bigcup_{j\in J}\partial^*g_j(x))^s\cap T(S,x)=\emptyset$. It follows that

$$\left(\bigcup_{j\in J}\operatorname{co}\partial^*g_j(x)\right)^s\cap K=\emptyset.\tag{12}$$

By a similar argument to that used in showing that assertion (2) implies (4), we know that (12) implies

$$0\in\operatorname{co}\bigcup_{j\in J}\operatorname{co}\partial^*g_j(x)+K^-.\tag{13}$$

Thus, there exist $(\hat{\beta}_1,\ldots,\hat{\beta}_q)\in\mathbb{R}_+^q$ with $\sum_{j\in J}\hat{\beta}_j=1$ such that

$$0\in\sum_{j\in J}\hat{\beta}_j\operatorname{co}\partial^*g_j(x)+K^-.\tag{14}$$

Now, since the Kuhn-Tucker multiplier set $\Lambda(K,x)$ with respect to K is nonempty by hypothesis (c), there exist $(\alpha_1,\ldots,\alpha_p)\in\mathbb{R}_+^p$ and $(\beta_1,\ldots,\beta_q)\in\mathbb{R}_+^q$ with $\sum_{i\in I}\alpha_i=1$ such that

$$0\in\sum_{i\in I}\alpha_i\operatorname{co}\partial^*f_i(x)+\sum_{j\in J}\beta_j\operatorname{co}\partial^*g_j(x)+K^-.\tag{15}$$

From relations (14) and (15), we know that, for any $\lambda>0$,

$$0\in\sum_{i\in I}\alpha_i\operatorname{co}\partial^*f_i(x)+\sum_{j\in J}\beta_j\operatorname{co}\partial^*g_j(x)+K^-+\sum_{j\in J}\lambda\hat{\beta}_j\operatorname{co}\partial^*g_j(x)+\lambda K^-,$$

which implies that, for any $\lambda > 0$,

$$0 \in \sum_{i\in I} \alpha_i \operatorname{co} \partial^* f_i(x) + \sum_{j\in J} (\beta_j + \lambda\hat{\beta}_j) \operatorname{co} \partial^* g_j(x) + K^-, \tag{16}$$

since $K^- + \lambda K^- = K^-$ and since

$$\beta_j \operatorname{co} \partial^* g_j(x) + \lambda\hat{\beta}_j \operatorname{co} \partial^* g_j(x) = (\beta_j + \lambda\hat{\beta}_j) \operatorname{co} \partial^* g_j(x).$$

From the relation (16), we obtain

$$(\alpha_1, \dots, \alpha_p, \beta_1 + \lambda\hat{\beta}_1, \dots, \beta_q + \lambda\hat{\beta}_q) \in \Lambda(K, x), \quad \forall\lambda > 0,$$

which, by noting that $\hat{\beta}_j > 0$ for at least one $j \in J$, contradicts the hypothesis that $\Lambda(K, x)$ is bounded. □

The example below illustrates that hypothesis of (a) in Theorem 2.1, that $\partial^* f_i(x)$ and $\partial^* g_j(x)$ are upper semiregular for all $i \in I$ and $j \in J$, is necessary for the Kuhn-Tucker multiplier set $\Lambda(K_v, x)$ to be nonempty.

Example 2.1 Consider the nonsmooth scalar optimization problem

$$\text{(SPIS1)} \quad \text{Min}\{f(x) \mid g(x) \leqq 0, x \in S\},$$

where the functions $f,\ g : \mathbb{R} \to \mathbb{R}$ and the subset S of $\mathbb{R}$ are defined as

$$f(x) = \begin{cases} |x| \sin \ln |x|, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

$$g(x) = -x,$$

$$S = \{e^{-[(2n+1)\pi + 2^{-1}\pi]} \mid n = 1, 2, \dots\}.$$

Then, $\bar{x} = 0$ is a locally weak efficient solution and $f,\ g$ are globally Lipschitz. Moreover,

$$\begin{aligned} &f^+(\bar{x}; v) = 1 \quad \text{and} \quad f^-(\bar{x}; v) = -1, \quad \forall v \in \mathbb{R}\backslash\{0\}, \\ &g^+(\bar{x}; v) = g^-(\bar{x}; v) = -1, \quad \forall v \in \mathbb{R}\backslash\mathbb{R}_+, \\ &g^+(\bar{x}; v) = g^-(\bar{x}; v) = 1, \quad \forall v \in \mathbb{R}_+\backslash\{0\}, \end{aligned}$$

$\partial^* f(\bar{x}) = [-1, -2^{-1}]$ is an upper convexificator of f at $\bar{x}$, and $\partial^* g(\bar{x}) = \{-1\}$ is an upper semiregular convexificator of g at $\bar{x}$. Obviously, $T(S, \bar{x}) = \mathbb{R}_+$. So,

$$v = 1 \in T(S, \bar{x}) \cap (\partial^* g(\bar{x}))^s,$$

and hence (CQ1) holds at $\bar{x}$. Now, let K_v be an arbitrary nonempty closed convex cone such that $v \in K_v$ and $K_v \subseteq T(S, \bar{x})$. Then, $K_v = T(S, \bar{x}) = \mathbb{R}_+$; hence, $(K_v)^- = \{x \in \mathbb{R} \mid x \leqq 0\}$. Therefore, for any $\beta \geqq 0$,

$$0 \notin \operatorname{co} \partial^* f(\bar{x}) + \beta \operatorname{co} \partial^* g(\bar{x}) + (K_v)^-,$$

which implies that $\Lambda(K_v, \bar{x}) = \emptyset$. We observe that $\partial^* f(\bar{x}) = [-1, -2^{-1}]$ is not an upper semiregular convexificator of f at $\bar{x}$.

3 Existence and Boundedness under (CQ2)

In this section, we show that, under some conditions, at a locally weak efficient solution x to problem (MPIS), the constraint qualification (CQ2) is a necessary and sufficient condition for the Kuhn-Tucker multiplier set $\Lambda(K, x)$ with respect to the closed convex subcone K of the adjacent cone $A(S, x)$ to be nonempty and bounded.

Theorem 3.1 *Let x be a locally weak efficient solution to problem (MPIS). Suppose that, at x, f_i and g_j admit bounded upper convexificators $\partial^* f_i(x)$ and $\partial^* g_j(x)$ for each $i \in I$ and $j \in J$.*

(a) *Assume that f_i and g_j are locally Lipschitz at x for all $i \in I$ and $j \in J$ and that there is an $i_0 \in I$ (or $j_0 \in J$) such that $\partial^* f_i(x)$ and $\partial^* g_j(x)$ are semiregular for all $i \in I \backslash \{i_0\}$ and $j \in J$ (or $i \in I$ and $j \in J \backslash \{j_0\}$). If (CQ2) holds at x, then for every closed convex cone K_v which is included in $A(S, x)$ and contains v, the set $\Lambda(K_v, x)$ is nonempty.*

(b) *If (CQ2) holds at x, and if, for every closed convex cone K_v which is included in $A(S, x)$ and contains v, the set $\Lambda(K_v, x)$ is nonempty, then $\Lambda(K_v, x)$ is bounded.*

(c) *For some closed convex subcone K of $A(S, x)$, if the Kuhn-Tucker multiplier set $\Lambda(K, x)$ with respect to K is nonempty and bounded, then (CQ2) holds at x.*

Proof (a) We show that, if there is an $i_0 \in I$ such that $\partial^* f_i(x)$ and $\partial^* g_j(x)$ are semiregular for all $i \in I \backslash \{i_0\}$ and $j \in J$, then (CQ2) ensures the nonemptiness of $\Lambda(K_v, x)$ for every closed convex cone K_v which is included in $A(S, x)$ and contains v. The proof for the case when there is a $j_0 \in J$ such that $\partial^* f_i(x)$ and $\partial^* g_j(x)$ are semiregular for all $i \in I$ and $j \in J \backslash \{j_0\}$ is similar. Suppose on the contrary that there is a closed convex cone K_v which is included in $A(S, x)$ and contains v such that $\Lambda(K_v, x) = \emptyset$. Then, in view of the proof of (a) of Theorem 2.1, there exists a vector w in $\mathbb{R}^n$ such that

$$w \in \left(\bigcup_{i \in I} \partial^* f_i(x) \right)^s, \tag{17}$$

$$w \in \left(\bigcup_{j \in J} \partial^* g_j(x) \right)^s, \tag{18}$$

$$w \in K_v. \tag{19}$$

In view of the definitions of upper (semiregular) convexificators, it follows respectively from relations (17) and (18) that

$$f_{i_0}^-(x, w) < 0, \tag{20}$$

$$f_i^+(x, w) < 0, \quad \forall i \in I \backslash \{i_0\}, \tag{21}$$

$$g_j^+(x,w) < 0, \quad \forall j \in J. \tag{22}$$

By the definition of the lower Dini derivatives, it follows from inequality (20) that there exist $t_n \to 0^+$ such that

$$\lim_{t_n \downarrow 0} t_n^{-1}[f_{i_0}(x+t_n w) - f_{i_0}(x)] = f_{i_0}^-(x,w) < 0. \tag{23}$$

By the definition of adjacent cones, for $t_n \to 0^+$, there exists $w_n \to w$ such that

$$x + t_n w_n \in S. \tag{24}$$

Since f_{i_0} is locally Lipschitz near x,

$$\lim_{t_n \downarrow 0} t_n^{-1}|f_{i_0}(x+t_n w_n) - f_{i_0}(x+t_n w)| \leqq \lim_{n\to+\infty} L_{i_0}|w_n - w| \to 0, \tag{25}$$

where L_{i_0} is a Lipschitz constant for f_{i_0} near x. Therefore, writing

$$\begin{aligned} t_n^{-1}[f_{i_0}(x+t_n w_n) - f_{i_0}(x)] &= t_n^{-1}[f_{i_0}(x+t_n w_n) - f_{i_0}(x+t_n w)] \\ &\quad + t_n^{-1}[f_{i_0}(x+t_n w) - f_{i_0}(x)] \end{aligned}$$

and noting (23) and (25), we get

$$\lim_{t_n \downarrow 0} t_n^{-1}[f_{i_0}(x+t_n w_n) - f_{i_0}(x)] = 0 + f_{i_0}^-(x,w) < 0.$$

It then follows that, for all sufficiently large n,

$$f_{i_0}(x+t_n w_n) < f_{i_0}(x). \tag{26}$$

Moreover, in view of the proof of inequalities (8) and (9), it follows respectively from inequalities (21) and (22) that, for all sufficiently large n,

$$f_i(x+t_n w_n) < f_i(x), \quad \forall\, i \in I\backslash\{i_0\}. \tag{27}$$

$$g_j(x+t_n w_n) < g_j(x), \quad \forall j \in J. \tag{28}$$

Relation (24) and inequalities (28) imply that $x+t_n w_n$ are feasible points for problem (MPIS), which together with inequalities (26) and (27) contradicts that x is a locally weak efficient solution to the problem.

The proof of (b) and that of (c) are, respectively, similar to their counterparts in Theorem 2.1. □

The following example illustrates that the semiregularity hypothesis of (a) of Theorem 3.1 on $\partial^* f_i(x)$ and $\partial^* g_j(x)$, $i \in I$ and $j \in J$, is necessary for the Kuhn-Tucker multiplier set $\Lambda(K_v, x)$ to be nonempty.

Example 3.1 Consider nonsmooth scalar optimization problem

$$(\text{SPIS2}) \quad \text{Min}\{f(x) \mid g(x) \leqq 0, x \in S\},$$

where the functions $f,\ g:\mathbb{R}\to\mathbb{R}$ are defined respectively by

$$f(x)=\begin{cases}|x|\sin\ln|x|, & x\neq 0;\\ 0, & x=0,\end{cases}$$

$g(x)=-f(x)$, and where the subset S of $\mathbb{R}$ is defined by $S=\mathbb{R}_+$. Then, $\bar{x}=0$ is a locally weak efficient solution, and f and g are globally Lipschitz. Moreover,

$$f^+(\bar{x};v)=g^+(\bar{x};v)=1,\quad \forall v\in\mathbb{R}\backslash\{0\},$$
$$f^-(\bar{x};v)=g^-(\bar{x};v)=-1,\quad \forall v\in\mathbb{R}\backslash\{0\},$$

$\partial^*f(\bar{x})=\partial^*g(\bar{x})=[-1,-2^{-1}]$ is an upper convexificator of both f and g at $\bar{x}$. Obviously, $A(S,\bar{x})=\mathbb{R}_+$. So, $v=1$ is in $A(S,\bar{x})\cap(\partial^*g(\bar{x}))^s$; hence, (CQ2) holds at $\bar{x}$. Now, let K_v be an arbitrary nonempty closed convex cone such that $v\in K_v$ and $K_v\subseteq A(S,\bar{x})$. Then, $K_v=A(S,\bar{x})=\mathbb{R}_+$; hence, $(K_v)^-=\{x\in\mathbb{R}\mid x\leqq 0\}$. Therefore, for any $\beta\geqq 0$,

$$0\notin \operatorname{co}\partial^*f(\bar{x})+\beta\operatorname{co}\partial^*g(\bar{x})+(K_v)^-,$$

which implies that $\Lambda(K_v,\bar{x})=\emptyset$. We note that neither the upper convexificator $\partial^*f(\bar{x})$ nor the upper convexificator $\partial^*g(\bar{x})$ is semiregular.

To end our presentation, we make some remarks on the results that we obtained. Since the Clarke subdifferential and Michel-Penot subdifferential of a locally Lipschitz function are upper semiregular convexificators, the results of Theorems 2.1 and 3.1 in this work are valid with the convexificators being replaced respectively by the Clarke subdifferentials and the Michel-Penot subdifferentials. On the other hand, it has been known that a locally Lipschitz function may have a convexificator at a point which is strictly contained in the two subdifferentials. Therefore, for an optimization problem with Lipschitz data, although the constraint qualifications (CQ1) and (CQ2) in terms of convexificators hold, they may fail to hold when the convexificators are replaced respectively by the Clarke and Michel-Penot subdifferentials. Moreover, although the set of Kuhn-Tucker multipliers associated with the Kuhn-Tucker type necessary optimality conditions in terms of some convexificators is nonempty and bounded and hence, with the convexificators replaced respectively by the Clarke and Michel-Penot subdifferentials, the set of multipliers is also nonempty, it may fail to be bounded. The following example makes these points.

Example 3.2 Consider the nonsmooth scalar optimization problem

$$\text{(SPI)}\quad \operatorname{Min}\{f(x)\mid g(x)\leqq 0, x\in S=\mathbb{R}^2\},$$

where $f,\ g:\mathbb{R}^2\to\mathbb{R}$ are defined by $f(x)=x^2$ and $g(x)=|x_1|-|x_2|+x_1/2$, where $x=(x_1,x_2)$. The point $\bar{x}=(0,0)\in\mathbb{R}^2$ is a locally weak efficient solution to (SPI). At $\bar{x}$, f is locally Lipschitz, the Clarke subdifferential and the Michel-Penot subdifferential of f are

$$\partial^c f(\bar{x})=\partial^{mp}f(\bar{x})=\{(0,0)\},$$

and f admits an upper semiregular convexificator $\partial^* f(\bar{x}) = \{(0,0)\}$.

According to Example 9.1 in Ref. [9], the function g is directionally differentiable and locally Lipschitz on $\mathbb{R}^2$, the directional derivative of g at $\bar{x}$ in direction $v = (v_1, v_2)$ is

$$g'(\bar{x}, v) = |v_1| - |v_2| + v_1/2,$$

the Clarke subdifferential and the Michel-Penot subdifferential of g at $\bar{x}$ are

$$\partial^c g(\bar{x}) = \partial^{mp} g(\bar{x}) = \mathrm{co}\{(-1/2, 1), (-1/2, -1), (3/2, 1), (-3/2, -1)\},$$

and

$$\partial^* g(\bar{x}) = \mathrm{co}\{(-1/2, -1), (3/2, 1)\}$$

is an upper semiregular convexificator of g at $\bar{x}$.

Since $S = \mathbb{R}^2$, we have

$$T(S, \bar{x}) = A(S, \bar{x}) = \mathbb{R}^2.$$

Note that $\bar{v} = (-1, 1) \in (\partial^* g(\bar{x}))^s$. Thus, we have

$$\bar{v} \in T(S, \bar{x}) \cap (\partial^* g(\bar{x}))^s = A(S, \bar{x}) \cap (\partial^* g(\bar{x}))^s,$$

from which we know that (CQ1) and (CQ2) hold at $\bar{x}$. Therefore, for any fixed upper semiregular convexificator of f at $\bar{x}$ and any fixed closed convex cone $K_{\bar{v}}$ containing $\bar{v}$, by Theorem 2.1 the set $\Lambda(K_{\bar{v}}, x)$ of Kuhn-Tucker multipliers is nonempty and bounded.

However, since $(0, 0)$ is contained in $\partial^c g(\bar{x})$ and $\partial^{mp} g(\bar{x})$, it follows that

$$(\partial^c g(\bar{x}))^s = (\partial^{mp} g(\bar{x}))^s = \emptyset.$$

Thus, with the convexificator $\partial^* g(\bar{x})$ in (CQ1) and (CQ2) replaced by the Clarke or Michel-Penot subdifferential, (CQ1) or (CQ2) does not hold. Moreover, since

$$0 \in \partial^c f(\bar{x}) \cap \partial^c g(\bar{x}) \cap K^- = \partial^{mp} f(\bar{x}) \cap \partial^{mp} g(\bar{x}) \cap K^-$$

for arbitrary closed convex cone K in $\mathbb{R}^2$, it follows that

$$0 \in \partial^c f(\bar{x}) + \beta \partial^c g(\bar{x}) + K^- = \partial^{mp} f(\bar{x}) + \beta \partial^{mp} g(\bar{x}) + K^-,$$

for any real number $\beta \geq 0$. So, the set of all Kuhn-Tucker multiplier vectors associated with the Kuhn-Tucker type necessary optimality conditions expressed in terms of the Clarke or Michel-Penot subdifferentials is nonempty but not bounded.

Acknowledgements The authors are grateful to the referees for valuable comments and suggestions leading to the correction of errors in the proof of a main result and to the improvements in the final version of the paper.

References

1. Gauvin, J., Tolle, J.W.: Differentiable stability in nonlinear programming. SIAM J. Control Optim. **15**, 294–311 (1977)
2. Gauvin, J.: A necessary and sufficient regularity condition to have bounded multipliers in nonconvex programming. Math. Program. **12**, 136–138 (1977)
3. Mangasarian, O.L., Fromovitz, S.: The Fritz-John necessary optimality condition in the presence of equality and inequality constraints. J. Math. Anal. Appl. **17**, 37–47 (1976)
4. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley, New York (1983)
5. Nguyen, V.H., Strodiot, J.-J., Mifflin, R.: On conditions to have bounded multipliers in locally Lipschitz programming. Math. Program. **18**, 100–106 (1980)
6. Pappalardo, M.: Error bounds for generalized Lagrange multipliers in locally Lipschitz programming. J. Optim. Theory Appl. **73**, 205–210 (1992)
7. Jourani, A.: Constraint qualifications and Lagrange multipliers in nondifferentiable programming problems. J. Optim. Theory Appl. **81**, 533–548 (1994)
8. Demyanov, V.F.: Convexification and concavification of positively homogeneous functions by the same family of linear functions. Technical Report, Univ. of Pisa, No. 3.208.802, pp. 1–11 (1994)
9. Demyanov, V.F., Jeyakumar, V.: Hunting for smaller convex subdifferential. J. Glob. Optim. **10**, 305–326 (1997)
10. Jeyakumar, V., Luc, D.T.: Approximate Jacobian matrices for nonsmooth continuous maps and C^1-optimization. SIAM J. Control Optim. **36**, 1815–1832 (1998)
11. Jeyakumar, V., Luc, D.T.: Nonsmooth calculus, minimality, and monotonicity of convexificators. J. Optim. Theory Appl. **101**, 599–621 (1999)
12. Jeyakumar, V., Wang, W.: Approximate Hessian matrices and second-order optimality conditions for nonlinear programming problem with C^1-data. J. Aust. Math. Soc. B **40**, 403–420 (1999)
13. Jeyakumar, V., Yang, X.Q.: Approximate generalized Hessians and Taylor's expansions for continuously Gateaux differentiable functions. Nonlinear Anal., Theory Methods Appl. **36**, 353–368 (1999)
14. Jeyakumar, V., Luc, D.T., Schaible, S.: Characterizations of generalized monotone nonsmooth continuous maps using approximate Jacobians. J. Convex Anal. **5**, 119–132 (1998)
15. Dutta, J., Chandra, S.: Convexificators, generalized convexity, and optimality conditions. J. Optim. Theory Appl. **113**, 41–64 (2002)
16. Li, X.F., Zhang, J.Z.: Stronger Kuhn-Tucker type conditions in nonsmooth multiobjective optimization: locally Lipschitz case. J. Optim. Theory Appl. **127**, 367–388 (2005)
17. Li, X.F., Zhang, J.Z.: Necessary optimality conditions in terms of convexificators in Lipschitz optimization. J. Optim. Theory Appl. **132** (2007)
18. Uderzo, A.: Convex approximators, convexificators, and exhausters: application to constrained extremum problem. In: Quasidifferentiability and Related Topics, Nonconvex Optimization and Application, vol. 43, pp. 297–327. Kluwer Academic, Dordrecht (2000)
19. Wang, X., Jeyakumar, V.: A sharp Lagrange multiplier rule for nonsmooth mathematical programming problems involving equality constraints. SIAM J. Optim. **10**, 1136–1148 (2000)
20. Luc, D.T.: A multiplier rule for multiobjective programming problems with continuous data. SIAM J. Optim. **13**, 168–178 (2002)
21. Bazaraa, M.S., Goode, J.J., Shetty, C.M.: Constraint qualifications revised. Manag. Sci. **18**, 567–573 (1972)
22. Martin, D.H., Watkins, G.G.: Cores of tangent cones and Clarke's tangent cone. Math. Oper. Res. **10**, 565–575 (1985)
23. Ward, D.E.: Convex subcones of the contingent cone in nonsmooth calculus and optimization. Transl. Am. Math. Soc. **302**, 661–682 (1987)
24. Ward, D.E., Lee, G.M.: Generalized properly efficient solutions of vector optimization problems. Math. Methods Oper. Res. **53**, 215–232 (2001)
25. Michel, P., Penot, J.P.: A generalized derivative for calm and stable functions. Differ. Integral Equ. **5**, 433–454 (1992)