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An Improved Delay-Dependent Criterion for Asymptotic Stability of Uncertain Dynamic Systems with Time-Varying Delays

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Abstract In this paper, the problem of stability analysis for uncertain dynamic systems with time-varying delays is considered. The parametric uncertainties are assumed to be bounded in magnitude. Based on the Lyapunov stability theory, a new delay-dependent stability criterion for the system is established in terms of linear matrix inequalities, which can be solved easily by various efficient convex optimization algorithms. Two numerical examples are illustrated to show the effectiveness of proposed method.

Keywords Dynamic systems · Time-varying delays · LMI · Lyapunov method

1 Introduction

During the past several decades, the problem of stability and stabilization on dynamical systems is the most important issue in control society [1, 2]. One of most interesting research topic in control society is the stability analysis of dynamical systems with time-delays [3–6]. Time-delays exist in many industrial systems such as neural networks, chemical processes, networked control systems, laser models, large-scale

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systems, and so on. Since the occurrence of time delay may deteriorate system performance or cause instability, many efforts have been paid to the stability analysis of dynamic systems with time-delays. For extensive research works, see the papers [7–19] and references therein. In recent years, delay-dependent stability criteria, which are less conservative than delay-independent ones when the size of time-delays are small, have been extensively studied by many researchers during the last decade [11–22]. An important issue in this field is to enlarge the feasible region of stability criteria. Therefore, how to choose Lyapunov-Krasovskii functional and derive the time derivative of this with appropriate free-weighting matrices play key roles to increase the delay bounds for guaranteeing stability.

In this regard, Park [18] proposed a new bounding technique of cross terms and showed that the proposed stability criteria of time-delay systems have larger maximum allowable delay bound. Yue et al. [19] introduced neutral model transformation to get new stability criteria. Parameter neutral model transformation was used to reduce the conservatism of stability criteria [12, 20]. Park and Ko [21] increase delay bounds by including all possible information of states when constructing some appropriate integral inequalities proposed in [18].

In this paper, we propose a new delay-dependent stability criterion of uncertain dynamic systems with time-varying delays. The considered system is assumed to have norm bounded parameter uncertainties. In order to derive less conservative results, a new Lyapunov functional which divides delay interval is proposed and different free-weighting matrices in divided delay intervals are included in taking upper bounds of integral terms of time-derivative Lyapunov functionals. Then, a novel condition for delay-dependent stability criterion is established in terms of LMIs which can be solved efficiently by various convex optimization algorithms [23]. For real computation of LMIs, we use Matlab LMI Toolbox, which implements the interior-point algorithm to solve convex optimization problem. Two numerical examples are given to show the effectiveness of the proposed method.

In the sequel, the following notation will be used. $\mathcal{R}^n$ is the n-dimensional Euclidean space. $\mathcal{R}^{m\times n}$ denotes the set of $m \times n$ real matrix. $\star$ denotes the symmetric part. $X > 0$ ($X \geq 0$) means that X is a real symmetric positive definitive matrix (positive semi-definite). I denotes the identity matrix with appropriate dimensions. $\|\cdot\|$ refers to the induced matrix 2-norm. diag$\{\cdots\}$ denotes the block diagonal matrix. $\mathcal{C}_{n,h} \equiv \mathcal{C}([-h,0],\mathcal{R}^n)$ denotes the Banach space of continuous functions mapping the interval $[-h,0]$ into $\mathcal{R}^n$, with the topology of uniform convergence.

2 Problem Statements

Consider the following uncertain dynamic system with time-varying delays:

$$\begin{aligned} \dot{x}(t) &= (A + \Delta A(t))x(t) + (A_d + \Delta A_d(t))x(t-h(t)), \\ x(s) &= \phi(s), \quad s \in [-h_U, 0]. \end{aligned} \tag{1}$$

Here, $x(t) \in \mathcal{R}^n$ is the state vector, $A \in \mathcal{R}^n$ and $A_d \in \mathcal{R}^n$ are known constant matrices, $\phi(s) \in \mathcal{C}_{n,h_U}$ are vector-valued initial functions, $h(t)$ means time-varying delays which satisfy $0 \le h(t) \le h_U$ and $\dot{h}(t) \le h_D$, $\Delta A(t)$, and $\Delta A_d(t)$ are the uncertainties of the system matrices of the form

$$\begin{bmatrix} \Delta A(t) & \Delta A_d(t) \end{bmatrix} = DF(t)\begin{bmatrix} E & E_d \end{bmatrix}, \tag{2}$$

in which the time-varying nonlinear function $F(t)$ satisfies

$$F^T(t)F(t) \le I, \quad \forall t. \tag{3}$$

The system (1) can be rewritten as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-h(t)) + Dp(t), \\ p(t) &= F(t)q(t), \\ q(t) &= Ex(t) + E_d x(t-h(t)). \end{aligned} \tag{4}$$

The purpose of this paper is to present a delay-dependent stability criterion for the system (4).

Before deriving our main results, we need the following facts and lemma.

Fact 2.1 (Schur Complement) *Given constant matrices* $\Sigma_1, \Sigma_2, \Sigma_3$, *with* $\Sigma_1 = \Sigma_1^T$ *and* $0 < \Sigma_2 = \Sigma_2^T$, *then* $\Sigma_1 + \Sigma_3^T \Sigma_2^{-1} \Sigma_3 < 0$ *if and only if*

$$\begin{bmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{bmatrix} < 0.$$

Fact 2.2 *For any real vectors* a, b *and any matrix* $Q > 0$ *with appropriate dimensions, we have*

$$\pm 2a^T b \le a^T Q a + b^T Q^{-1} b.$$

To derive a less conservative stability criterion, we use the following lemma, to be utilized in deriving an upper bound of integral terms.

Lemma 2.1 *For any scalar* $h(t) \ge 0$ *and any constant matrix* $Q \in \mathcal{R}^{n\times n}$, $Q = Q^T > 0$, *the following inequality holds*:

$$-\int_{t-h(t)}^{t} \dot{x}^T(s) Q \dot{x}(s) ds \le h(t)\zeta^T(t)\mathcal{X} Q^{-1} \mathcal{X}^T \zeta(t) + 2\zeta^T(t)\left[x(t) - x(t-h(t))\right], \tag{5}$$

where

$$\zeta^T(t) = \begin{bmatrix} x^T(t) & x^T(t-h(t)) & x^T(t-\frac{h_U}{2}) & x^T(t-h_U) & \dot{x}^T(t) & p(t) \end{bmatrix} \tag{6}$$

and $\mathcal{X}$ *is free weighting matrix with appropriate dimensions.*

Proof From Fact 2.2, the following inequality holds:

$$-2\int_{t-h(t)}^{t}(\mathcal{X}^T\zeta(t))^T\dot{x}(s)ds \le \int_{t-h(t)}^{t}[\zeta^T(t)\mathcal{X}Q^{-1}\mathcal{X}^T\zeta(t)+\dot{x}^T(s)Q\dot{x}(s)]ds. \quad (7)$$

From (7), we obtain

$$\begin{aligned}
&-\int_{t-h(t)}^{t}\dot{x}^T(s)Q\dot{x}(s)ds\\
&\quad\le \int_{t-h(t)}^{t}\zeta^T(t)\mathcal{X}Q^{-1}\mathcal{X}^T\zeta(t)ds+2\int_{t-h(t)}^{t}(\mathcal{X}^T\zeta(t))^T\dot{x}(s)ds\\
&\quad= h(t)\zeta^T(t)\mathcal{X}Q^{-1}\mathcal{X}^T\zeta(t)ds+2\zeta^T(t)\mathcal{X}[x(t)-x(t-h(t))]. \qquad (8)
\end{aligned}$$

This completes the proof. □

3 Main Results

In this section, a new delay-dependent stability criterion for the uncertain dynamic system with time-varying delays (1) will be presented by using the Lyapunov analysis. Before stating the main results, the notations of several matrices are defined for simplicity:

$$\begin{aligned}
&\Sigma=\left[\Sigma_{(i,j)}\right],\quad i,j=1,\dots,6,\\
&\Sigma_{(1,1)}=N_{11}+R_2+P_1+P_1^T,\qquad \Sigma_{(1,2)}=P_1A_d,\qquad \Sigma_{(1,3)}=N_{12},\\
&\Sigma_{(1,4)}=R_1,\qquad \Sigma_{(1,5)}=-P_1+A^TP_2^T,\qquad \Sigma_{(1,6)}=P_1D,\\
&\Sigma_{(2,2)}=-(1-h_D)R_2,\qquad \Sigma_{(2,3)}=0,\qquad \Sigma_{(2,4)}=0,\qquad \Sigma_{(2,5)}=A_d^TP_2^T,\\
&\Sigma_{(2,6)}=0,\qquad \Sigma_{(3,3)}=N_{22}-N_{11},\qquad \Sigma_{(3,4)}=-N_{12},\qquad \Sigma_{(3,5)}=0,\\
&\Sigma_{(3,6)}=0,\qquad \Sigma_{(4,4)}=-N_{22},\qquad \Sigma_{(4,5)}=0,\qquad \Sigma_{(4,6)}=0,\\
&\Sigma_{(5,5)}=\frac{h_U}{2}(Q_1+Q_2)-P_2-P_2^T,\qquad \Sigma_{(5,6)}=P_2D,\qquad \Sigma_{(6,6)}=-\varepsilon I,\\
&\mathcal{X}=\begin{bmatrix}X_1^T & X_2^T & 0 & 0 & 0 & 0\end{bmatrix}^T,\qquad \mathcal{Y}=\begin{bmatrix}0 & Y_1^T & Y_2^T & 0 & 0 & 0\end{bmatrix}^T,\\
&\mathcal{Z}=\begin{bmatrix}0 & 0 & Z_1^T & Z_2^T & 0 & 0\end{bmatrix}^T,\qquad \bar{\mathcal{X}}=\begin{bmatrix}\bar{X}_1^T & 0 & \bar{X}_2^T & 0 & 0 & 0\end{bmatrix}^T,\\
&\bar{\mathcal{Y}}=\begin{bmatrix}0 & \bar{Y}_1^T & \bar{Y}_2^T & 0 & 0 & 0\end{bmatrix}^T,\qquad \bar{\mathcal{Z}}=\begin{bmatrix}0 & \bar{Z}_1^T & 0 & \bar{Z}_2^T & 0 & 0\end{bmatrix}^T,\\
&\Upsilon=\begin{bmatrix}\mathcal{X} & -\mathcal{X}+\mathcal{Y} & -\mathcal{Y}+\mathcal{Z} & -\mathcal{Z} & 0 & 0\end{bmatrix},\\
&\bar{\Upsilon}=\begin{bmatrix}\bar{\mathcal{X}} & -\bar{\mathcal{Y}}+\bar{\mathcal{Z}} & -\bar{\mathcal{X}}+\bar{\mathcal{Y}} & -\bar{\mathcal{Z}} & 0 & 0\end{bmatrix},\\
&\Psi=\begin{bmatrix}E & E_d & 0 & 0 & 0 & 0\end{bmatrix}. \qquad (9)
\end{aligned}$$

We have the following theorem.

Theorem 3.1 *For given scalars $h_U > 0$, and h_D, the system* (4) *is asymptotically stable for $0 \le h(t) \le h_U$ and $\dot{h}(t) \le h_D$ if there exist positive-definite matrices $R_1 > 0$, $R_2 > 0$, $Q_1 > 0$, $Q_2 > 0$, $\left[\begin{smallmatrix} N_{11} & N_{12} \\ \star & N_{22} \end{smallmatrix}\right] > 0$, a positive scalar ε and matrices $P_1, P_2, X_i, Y_i, Z_i, \bar{X}_i, \bar{Y}_i, \bar{Z}_i$ $(i = 1, 2)$ satisfying the following LMIs*:

$$\begin{bmatrix} \Sigma + \Upsilon + \Upsilon^T & \varepsilon\Psi^T & \frac{h_U}{2}\mathcal{Y} & \frac{h_U}{2}\mathcal{Z} \\ \star & -\varepsilon I & 0 & 0 \\ \star & \star & -\frac{h_U}{2}Q_1 & 0 \\ \star & \star & \star & -\frac{h_U}{2}Q_2 \end{bmatrix} < 0, \tag{10}$$

$$\begin{bmatrix} \Sigma + \Upsilon + \Upsilon^T & \varepsilon\Psi^T & \frac{h_U}{2}\mathcal{X} & \frac{h_U}{2}\mathcal{Z} \\ \star & -\varepsilon I & 0 & 0 \\ \star & \star & -\frac{h_U}{2}Q_1 & 0 \\ \star & \star & \star & -\frac{h_U}{2}Q_2 \end{bmatrix} < 0, \tag{11}$$

$$\begin{bmatrix} \Sigma + \bar{\Upsilon} + \bar{\Upsilon}^T & \varepsilon\Psi^T & \frac{h_U}{2}\bar{\mathcal{X}} & \frac{h_U}{2}\bar{\mathcal{Z}} \\ \star & -\varepsilon I & 0 & 0 \\ \star & \star & -\frac{h_U}{2}Q_1 & 0 \\ \star & \star & \star & -\frac{h_U}{2}Q_2 \end{bmatrix} < 0, \tag{12}$$

$$\begin{bmatrix} \Sigma + \bar{\Upsilon} + \bar{\Upsilon}^T & \varepsilon\Psi^T & \frac{h_U}{2}\bar{\mathcal{X}} & \frac{h_U}{2}\bar{\mathcal{Y}} \\ \star & -\varepsilon I & 0 & 0 \\ \star & \star & -\frac{h_U}{2}Q_1 & 0 \\ \star & \star & \star & -\frac{h_U}{2}Q_2 \end{bmatrix} < 0. \tag{13}$$

Proof For positive-definite matrices R_i $(i = 1, 2)$, Q_i $(i = 1, 2)$, $\left[\begin{smallmatrix} N_{11} & N_{12} \\ \star & N_{22} \end{smallmatrix}\right]$, let us consider the Lyapunov-Krasovskii functional candidate

$$V = \sum_{i=1}^{3} V_i, \tag{14}$$

where

$$\begin{aligned} V_1 &= x^T(t)R_1x(t) \\ &\quad + \int_{t-\frac{h_U}{2}}^{t} \begin{bmatrix} x(s) \\ x(s-\frac{h_U}{2}) \end{bmatrix}^T \begin{bmatrix} N_{11} & N_{12} \\ \star & N_{22} \end{bmatrix} \begin{bmatrix} x(s) \\ x(s-\frac{h_U}{2}) \end{bmatrix} ds, \\ V_2 &= \int_{t-h(t)}^{t} x^T(s)R_2x(s)ds, \\ V_3 &= \int_{t-\frac{h_U}{2}}^{t}\int_{s}^{t} \dot{x}^T(u)Q_1\dot{x}(u)duds + \int_{t-h_U}^{t-\frac{h_U}{2}}\int_{s}^{t} \dot{x}^T(u)Q_2\dot{x}(u)duds. \end{aligned} \tag{15}$$

First, the time-derivative of V_1 can be calculated as

$$\dot{V}_1 = 2x^T(t)R_1\dot{x}(t) + \begin{bmatrix} x(t) \\ x(t-\frac{h_U}{2}) \end{bmatrix}^T \begin{bmatrix} N_{11} & N_{12} \\ \star & N_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-\frac{h_U}{2}) \end{bmatrix} - \begin{bmatrix} x(t-\frac{h_U}{2}) \\ x(t-h_U) \end{bmatrix}^T \begin{bmatrix} N_{11} & N_{12} \\ \star & N_{22} \end{bmatrix} \begin{bmatrix} x(t-\frac{h_U}{2}) \\ x(t-h_U) \end{bmatrix}. \tag{16}$$

Second, an upper bound of the time-derivative of V_2 can be obtained as

$$\dot{V}_2 \le x^T(t)R_2x(t) - (1-h_D)x^T(t-h(t))R_2x(t-h(t)). \tag{17}$$

Third, calculating $\dot{V}_3$ leads to

$$\dot{V}_3 = \frac{h_U}{2}\dot{x}^T(t)Q_1\dot{x}(t) - \int_{t-\frac{h_U}{2}}^{t} \dot{x}^T(s)Q_1\dot{x}(s)ds + \frac{h_U}{2}\dot{x}^T(t)Q_2\dot{x}(t) - \int_{t-h_U}^{t-\frac{h_U}{2}} \dot{x}^T(s)Q_2\dot{x}(s)ds. \tag{18}$$

Using Lemma 2.1, the upper bounds of the integral terms of $\dot{V}_3$ can be obtained as follows:

(i) When $0 \le h(t) \le \frac{h_U}{2}$, we have

$$\begin{aligned} -\int_{t-\frac{h_U}{2}}^{t} \dot{x}^T(s)Q_1\dot{x}(s)ds &= -\int_{t-h(t)}^{t} \dot{x}^T(s)Q_1\dot{x}(s)ds - \int_{t-\frac{h_U}{2}}^{t-h(t)} \dot{x}^T(s)Q_1\dot{x}(s)ds \\ &\le h(t)\zeta^T(t)\mathcal{X}Q_1^{-1}\mathcal{X}^T\zeta(t) + 2\zeta^T(t)\mathcal{X}[x(t)-x(t-h(t))] \\ &\quad + \left(\frac{h_U}{2}-h(t)\right)\zeta^T(t)\mathcal{Y}Q_1^{-1}\mathcal{Y}^T\zeta(t) \\ &\quad + 2\zeta^T(t)\mathcal{Y}\left[x(t-h(t)) - x\left(t-\frac{h_U}{2}\right)\right] \end{aligned} \tag{19}$$

and

$$\begin{aligned} -\int_{t-h_U}^{t-\frac{h_U}{2}} \dot{x}^T(s)Q_2\dot{x}(s)ds &\le \frac{h_U}{2}\zeta^T(t)\mathcal{Z}Q_2^{-1}\mathcal{Z}^T\zeta(t) \\ &\quad + 2\zeta^T(t)\mathcal{Z}\left[x\left(t-\frac{h_U}{2}\right) - x(t-h_U)\right]. \end{aligned} \tag{20}$$

Note that $\zeta(t)$ is defined in (6) and that $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ are in (9).

To derive less conservative results, we add the following equation with free variables P_1 and P_2:

$$0 = 2[x^T(t)P_1 + \dot{x}^T(t)P_2][-\dot{x}(t) + Ax(t) + A_dx(t-h(t)) + Dp(t)]. \tag{21}$$

Since the following inequality holds from (3) and (4),

$$p^T(t)p(t) \le q^T(t)q(t), \tag{22}$$

there exists a positive scalar ε satisfying the following inequality:

$$\varepsilon[\zeta^T(t)\Psi^T\Psi\zeta(t) - p^T(t)p(t)] \geq 0, \tag{23}$$

where Ψ are defined in (9).

From (15)–(23) and by applying the S-procedure [23], $\dot{V} = \sum_{i=1}^{3} \dot{V}_i$ has a new upper bound as

$$\dot{V} \leq \zeta^T(t)\Omega_1\zeta(t), \tag{24}$$

where

$$\begin{aligned}\Omega_1 &= \Sigma + \Upsilon + \Upsilon^T + \varepsilon\Psi^T\Psi + h(t)\mathcal{X}Q_1^{-1}\mathcal{X}^T \\ &\quad + \left(\frac{h_U}{2} - h(t)\right)\mathcal{Y}Q_1^{-1}\mathcal{Y}^T + \frac{h_U}{2}\mathcal{Z}Q_2^{-1}\mathcal{Z}^T,\end{aligned} \tag{25}$$

with Σ, Υ, Ψ defined in (9).

Since

$$h(t)\mathcal{X}Q_1^{-1}\mathcal{X}^T + \left(\frac{h_U}{2} - h(t)\right)\mathcal{Y}Q_1^{-1}\mathcal{Y}^T \tag{26}$$

is a convex combination of the matrices $\mathcal{X}Q_1^{-1}\mathcal{X}^T$, and $\mathcal{Y}Q_1^{-1}\mathcal{Y}^T$ on $h(t)$, $\Omega_1 < 0$ for $0 \leq h(t) \leq \frac{h_U}{2}$ can be handled via two corresponding boundary LMIs,

$$\Sigma + \Upsilon + \Upsilon^T + \varepsilon\Psi^T\Psi + \frac{h_U}{2}\mathcal{Y}Q_1^{-1}\mathcal{Y}^T + \frac{h_U}{2}\mathcal{Z}Q_2^{-1}\mathcal{Z} < 0, \tag{27}$$

$$\Sigma + \Upsilon + \Upsilon^T + \varepsilon\Psi^T\Psi + \frac{h_U}{2}\mathcal{X}Q_1^{-1}\mathcal{X}^T + \frac{h_U}{2}\mathcal{Z}Q_2^{-1}\mathcal{Z} < 0. \tag{28}$$

Using Fact 2.1, the inequalities (27) and (28) are equivalent to the LMIs (10) and (11), respectively.

(ii) When $\frac{h_U}{2} < h(t) \leq h_U$, we have upper bounds of the integral terms in $\dot{V}_3$ as

$$\begin{aligned}-\int_{t-\frac{h_U}{2}}^{t} \dot{x}^T(s)Q_1\dot{x}(s)ds &\leq \frac{h_U}{2}\zeta^T(t)\bar{\mathcal{X}}Q_1^{-1}\bar{\mathcal{X}}^T\zeta(t) \\ &\quad + 2\zeta^T(t)\bar{\mathcal{X}}\left[x(t) - x\left(t - \frac{h_U}{2}\right)\right]\end{aligned} \tag{29}$$

and

$$\begin{aligned}&-\int_{t-h_U}^{t-\frac{h_U}{2}} \dot{x}^T(s)Q_2\dot{x}(s)ds \\ &= -\int_{t-h(t)}^{t-\frac{h_U}{2}} \dot{x}^T(s)Q_2\dot{x}(s)ds - \int_{t-h_U}^{t-h(t)} \dot{x}^T(s)Q_2\dot{x}(s)ds\end{aligned}$$

$$
\begin{aligned}
&\le \left(h(t)-\frac{h_U}{2}\right)\zeta^T(t)\bar{\mathcal{Y}}Q_2^{-1}\bar{\mathcal{Y}}^T\zeta(t)\\
&\quad+2\zeta^T(t)\bar{\mathcal{Y}}\left[x\left(t-\frac{h_U}{2}\right)-x(t-h(t))\right]\\
&\quad+(h_U-h(t))\zeta^T(t)\bar{\mathcal{Z}}Q_2^{-1}\bar{\mathcal{Z}}^T\zeta(t)\\
&\quad+2\zeta^T(t)\bar{\mathcal{Z}}[x(t-h(t))-x(t-h_U)]. \qquad (30)
\end{aligned}
$$

Note that $\bar{\mathcal{X}}$, $\bar{\mathcal{Y}}$, $\bar{\mathcal{Z}}$ are defined in (9).

From (15)–(18), (21)–(23), (29)–(30) and by applying the S-procedure [23], $\dot{V}=\sum_{i=1}^{3}\dot{V}_i$ has a new upper bound as

$$
\dot{V}\le\zeta^T(t)\Omega_2\zeta(t), \qquad (31)
$$

where

$$
\begin{aligned}
\Omega_2&=\Sigma+\bar{\Upsilon}+\bar{\Upsilon}^T+\varepsilon\Psi^T\Psi+\frac{h_U}{2}\bar{\mathcal{X}}Q_1^{-1}\bar{\mathcal{X}}^T+\left(h(t)-\frac{h_U}{2}\right)\bar{\mathcal{Y}}Q_2^{-1}\bar{\mathcal{Y}}^T\\
&\quad+(h_U-h(t))\bar{\mathcal{Z}}Q_2^{-1}\bar{\mathcal{Z}}^T \qquad (32)
\end{aligned}
$$

and $\bar{\Upsilon}$ is defined in (9).

Using a similar method to that first employed in (26)–(28), $\Omega_2<0$ for $\frac{h_U}{2}<h(t)\le h_U$ can be handled via two corresponding boundary LMIs,

$$
\Sigma+\bar{\Upsilon}+\bar{\Upsilon}^T+\varepsilon\Psi^T\Psi+\frac{h_U}{2}\bar{\mathcal{X}}Q_1^{-1}\bar{\mathcal{X}}^T+\frac{h_U}{2}\bar{\mathcal{Z}}Q_2^{-1}\bar{\mathcal{Z}}^T<0, \qquad (33)
$$

$$
\Sigma+\bar{\Upsilon}+\bar{\Upsilon}^T+\varepsilon\Psi^T\Psi+\frac{h_U}{2}\bar{\mathcal{X}}Q_1^{-1}\bar{\mathcal{X}}^T+\frac{h_U}{2}\bar{\mathcal{Y}}Q_2^{-1}\bar{\mathcal{Y}}^T<0. \qquad (34)
$$

By using Fact 2.1, the inequalities (33) and (34) are equivalent to the LMIs (12) and (13), respectively. Therefore, if the LMIs (10)–(13) are satisfied, then the system (4) is guaranteed to be asymptotically stable for $0\le h(t)\le h_U$ and $\dot{h}(t)\le h_D$. This completes our proof. □

Remark 3.1 In the field of delay-dependent stability analysis, the main concern is to find the maximum delay bounds for guaranteeing stability. Recently, in [22], a discretization scheme of the delay was proposed to improve the feasible region of stability criteria. However, if a discretization number is increased, then the computational burden is large and the solving of the concerned LMIs much time-consuming. As a tradeoff between time-consuming and improved results, a new Lyapunov functional which divides the delay interval $[0,h_U]$ into two ones $[0,\frac{h_U}{2}]$ and $(\frac{h_U}{2},h_U]$ is proposed. And in deriving an upper bound of $\dot{V}_3$ in each intervals, different free-weighting matrices $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ and $\bar{\mathcal{X}}$, $\bar{\mathcal{Y}}$, $\bar{\mathcal{Z}}$ are introduced at each intervals for the first time. This can give an improved feasible region for delay-dependent stability criterion. In Sect. 4, we will show this tendency by two numerical examples.

Remark 3.2 If V_2 in (15) is not considered, the stability criterion do not need the information of time derivative of $h(t)$.

Remark 3.3 The LMI problem given in Theorem 3.1 is a convex problem with respect to solution variables. The problem can be solved by various convex optimization algorithms. In this paper, a software, Matlab LMI Toolbox, for solving convex optimization problem is used in which the interior-point algorithm was implemented.

4 Numerical Examples

Here, we give two numerical examples to verify the goodness of our work.

Example 4.1 Consider the following nominal systems with time-varying delays:

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - h(t)). \tag{35}$$

To the authors' knowledge, up to date the best result of delay bound for guaranteeing stability with unknown h_D was 1.8681 presented in [21]. However, by applying Theorem 3.1 and Remark 3.2 with unknown h_D, one can obtain h_U as 2.118. For different values of h_D, comparison of delay bounds with recent results in [15, 16, 21] is shown in Table 1. From Table 1, it is clear that our results for this example provide larger delay bounds than the ones in [15, 16, 21].

Example 4.2 Consider the following system (1) with parameters

$$\begin{aligned} A &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \\ \Delta A(t) &= \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{bmatrix}, \qquad \Delta A_d(t) = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}, \end{aligned} \tag{36}$$

where δ_i and γ_i $(i = 1, 2)$ denote the parameter uncertainties satisfying

$$|\delta_1| \le 1.6, \qquad |\delta_2| \le 0.05, \qquad |\gamma_1| \le 0.1, \qquad |\gamma_2| \le 0.3. \tag{37}$$

By applying Theorem 3.1 to the above system, the obtained results of delay bounds are listed in Table 2. This shows that Theorem 3.1 provides improved delay bounds than the ones in other literature [15, 16, 21].

Table 1 Upper bounds of time-varying delays for different h_D (Example 4.1)

h_D	0	0.1	0.5	0.9	unknown
Lien [16]	4.472	3.604	2.008	1.180	0.999
Kwon et al. [15]	4.472	3.604	2.008	1.180	0.999
Park and Ko [21]	4.472	3.669	2.337	1.873	1.863
Theorem 1	5.207	4.171	2.402	2.118	2.118

Table 2 Upper bounds of time-varying delays for different h_D (Example 4.2)

h_D	0	0.2	0.4	0.6	0.8	unknown
Lien [16]	1.149	1.063	0.973	0.873	0.760	–
Kwon et al. [15]	1.209	1.133	1.057	0.978	0.900	0.769
Park and Ko [21]	1.149	1.099	1.077	1.070	1.068	1.068
Theorem 1	1.273	1.186	1.105	1.087	1.087	1.087

5 Conclusions

In this paper, a novel delay-dependent stability criterion for uncertain dynamic systems with time-varying delays has been proposed. To obtain a less conservative result, a new Lyapunov-Krasovskii functional is proposed and different free weighting matrices in two divided delay intervals has been introduced by utilizing a bounding technique of integral terms (Lemma 2.1) with the LMI framework for obtaining the stability criterion of the system. The effectiveness of the proposed stability criterion is verified by two numerical examples.

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