J Optim Theory Appl (2010) 145: 355–372
DOI 10.1007/s10957-009-9625-1

Vector Variational Inequalities Involving Set-valued Mappings via Scalarization with Applications to Error Bounds for Gap Functions

J. Li · G. Mastroeni

Published online: 30 October 2009

Abstract In this paper, by using the scalarization approach of Konnov, several kinds of strong and weak scalar variational inequalities (SVI and WVI) are introduced for studying strong and weak vector variational inequalities (SVVI and WVVI) with set-valued mappings, and their gap functions are suggested. The equivalence among SVVI, WVVI, SVI, WVI is then established under suitable conditions and the relations among their gap functions are analyzed. These results are finally applied to the error bounds for gap functions. Some existence theorems of global error bounds for gap functions are obtained under strong monotonicity and several characterizations of global (respectively local) error bounds for the gap functions are derived.

Keywords Vector variational inequalities · Set-valued mappings · Scalarization · Gap functions · Error bounds

1 Introduction

The notion of vector variational inequality (VVI) was introduced first by Giannessi [1] in finite-dimensional spaces. Since then, VVI have been studied extensively by

Communicated by F. Giannessi.

This work was supported by the National Natural Science Foundation of China (60804065), the Applied Research Project of Sichuan Province, the National Natural Science Foundation of Sichuan Province (07ZA123) and the Fundation of China West Normal University (08B075).

J. Li (✉)
School of Mathematics and Information, China West Normal University, Nanchong, Sichuan 637002, P.R. China
e-mail: junli1026@163.com

G. Mastroeni
Department of Mathematics, University of Pisa, Largo B. Pontecorvo 5, 56127 Pisa, Italy
e-mail: mastroen@mail.dm.unipi.it

many authors in finite or infinite-dimensional spaces under generalized monotonicity and convexity assumptions (see for example [2–11] and the references therein). Among the approaches in the analysis of a VVI, the traditional scalarization approach consists in replacing the VVI with an equivalent scalar variational inequality (VI).

The concept of gap function was first introduced for the study of convex optimization problems and subsequently applied to variational inequalities. Since gap functions transform a variational inequality into an equivalent optimization problem, powerful optimization solution methods and algorithms can be applied for finding solutions of variational inequalities. Recently, gap functions have been extended to vector variational inequalities [3, 7, 9, 11, 12] and vector equilibrium problems [13–17]. On the other hand, the study of error bounds is closely related to the sensitivity analysis of optimization problems and the convergence analysis of some algorithms [18–21]. It is therefore of interest to investigate error bounds for gap functions associated with VVI.

Let Y be a real Banach space. A nonempty subset P of Y is said to be a cone if $\lambda P \subseteq P$ for all $\lambda > 0$. P is said to be a convex cone if P is a cone and $P + P = P$. P is called a pointed cone if P is a cone and $P \cap \{-P\} = \{0\}$. An ordered Banach space (Y, P) is a real Banach space Y with an ordering defined by a closed, convex and pointed cone $P \subseteq Y$ with apex at the origin, in the form

$$x \geq y \quad \Leftrightarrow \quad x - y \in P, \quad \forall x, y \in Y,$$

and

$$x \ngeq y \quad \Leftrightarrow \quad x - y \notin P, \quad \forall x, y \in Y.$$

If the interior of P, say $\operatorname{int} P$, is nonempty, then a weak ordering in X is also defined by

$$y < x \quad \Leftrightarrow \quad x - y \in \operatorname{int} P, \quad \forall x, y \in Y,$$

and

$$y \nless x \quad \Leftrightarrow \quad x - y \notin \operatorname{int} P, \quad \forall x, y \in Y.$$

We remark that, for any $x, y \in Y$,

$$x \geq y \quad \Leftrightarrow \quad y \leq x; \qquad x \ngeq y \quad \Leftrightarrow \quad y \nleq x;$$
$$y < x \quad \Leftrightarrow \quad x > y; \qquad y \nless x \quad \Leftrightarrow \quad x \ngtr y.$$

In this paper, without other specifications, let X be a real Banach space and let X^* be its dual space. Let K be a nonempty, closed and convex subset of X and let (Y, P) be an ordered Banach space induced by a closed convex pointed cone P. Denote by $L(X, Y)$ the space of all the continuous linear mappings from X to Y and by $\langle l, x\rangle$ the value of $l \in L(X, Y)$ at $x \in X$. Let $T : K \to 2^{L(X,Y)}$ be a set-valued mapping. In this paper, we consider two kinds of vector variational inequalities with set-valued mappings as follows.

Strong Vector Variational Inequality (SVVI): Find $x^* \in K$ such that

$$\exists t^* \in T(x^*) : \langle t^*, y - x^*\rangle \nless 0, \quad \text{for all } y \in K.$$

This has been investigated by Giannessi [5], Hadjisavvas and Schaible [6], Konnov [7], Konnov and Yao [8] with a moving cone, Yang and Yao [11].

Weak Vector Variational Inequality (WVVI): Find $x^* \in K$ such that

$$\forall y \in K, \exists t^* \in T(x^*) : \langle t^*, y - x^* \rangle \not< 0.$$

This has been studied by Konnov [7], Yang and Yao [11].

We denote by S_{SVVI} and S_{WVVI} the solution sets of SVVI and WVVI, respectively. It is easy to see that $S_{\text{SVVI}} \subseteq S_{\text{WVVI}}$.

In [11], Yang and Yao presented gap functions for SVVI and WVVI. By using a scalarization approach, Konnov [7] converted SVVI and WVVI into equivalent strong and weak scalar variational inequalities (SVI and WVI respectively) and suggested a gap function for SVVI, WVVI, SVI, WVI. Since the solutions sets of SVVI, WVVI, SVI, WVI coincide under suitable conditions, it is interesting to investigate the relations between their gap functions.

The purpose of this paper is to continue the study of SVVI and WVVI by using the scalarization approach of Konnov [7]. To this end, we consider several kinds of SVI and WVI and their gap functions. Under certain conditions, we show the equivalence of SVVI, WVVI, SVI, WVI and analyze the relations among their gap functions. Finally, we apply the results to error bounds for gap functions. We obtain some existence theorems of global error bounds for gap functions under strong monotonicity and derive some characterizations of global (respectively, local) error bounds for the gap functions.

2 Gap Functions for SVVI and WVVI

We first recall some basic definitions and lemmas which play an important role in our main results.

Definition 2.1 Let K be the domain of SVVI (respectively WVVI). A function $p : K \to \mathbb{R} \cup \{+\infty\}$ is said to be a gap function for SVVI (respectively WVVI) if it satisfies the following properties:

(i) $p(x) \geq 0, \quad \forall x \in K$.
(ii) $p(x^*) = 0$, if and only if x^* solves SVVI (respectively WVVI).

Lemma 2.1 *Let A_j, $j = 1, 2, \dots, n$, be nonempty subsets of a Banach space. Then*:

(i) *If A_j, $j = 1, 2, \dots, n$, is compact, then* $\prod_{j=1}^n A_j$, $\bigcup_{j=1}^n A_j$, $\text{conv}\{A_j\}_{j=1,2,\dots,n}$ *and* $\text{conv}\{\bigcup_{j=1}^n A_j\}$ *are compact*;
(ii) *The following assertion is true*:

$$\bigcup_{j=1}^n A_j \subseteq \text{conv}\{A_j\}_{j=1,2,\dots,n} = \text{conv}\left\{\bigcup_{j=1}^n A_j\right\},$$

where $\text{conv}\{A_j\}_{j=1,2,\dots,n}$ *denotes the convex hull of* $\{A_j\}_{j=1,\dots,n}$.

Proof It is obvious that statement (i) holds. It suffices to show that statement (ii) is true. Since for each $j = 1, 2, \dots, n$, $A_j \subseteq \mathrm{conv}\{A_j\}_{j=1,2,\dots,n}$, we have

$$\bigcup_{j=1}^{n} A_j \subseteq \mathrm{conv}\{A_j\}_{j=1,2,\dots,n}.$$

We now prove that $\mathrm{conv}\{A_j\}_{j=1,2,\dots,n} = \mathrm{conv}\{\bigcup_{j=1}^{n} A_j\}$. Since $\bigcup_{j=1}^{n} A_j \subseteq \mathrm{conv}\{A_j\}_{j=1,2,\dots,n}$ and $\mathrm{conv}\{A_j\}_{j=1,2,\dots,n}$ is convex, we have $\mathrm{conv}\{\bigcup_{j=1}^{n} A_j\} \subseteq \mathrm{conv}\{A_j\}_{j=1,2,\dots,n}$. Conversely, let $x \in \mathrm{conv}\{A_j\}_{j=1,2,\dots,n}$. Then, there exist $x_j \in A_j$, $j = 1, 2, \dots, n$, and $\lambda_j \ge 0$, $j = 1, 2, \dots, n$, with $\sum_{j=1}^{n} \lambda_j = 1$, such that

$$x = \sum_{j=1}^{n} \lambda_j x_j.$$

Since

$$x_j \in \bigcup_{j=1}^{n} A_j, \quad \forall j = 1, 2, \dots, n,$$

we have $x \in \mathrm{conv}\{\bigcup_{j=1}^{n} A_j\}$, implying that $\mathrm{conv}\{A_j\}_{j=1,2,\dots,n} \subseteq \mathrm{conv}\{\bigcup_{j=1}^{n} A_j\}$. The proof is complete. □

In the rest of the paper, let $Y = \mathbb{R}^n$,

$$\mathbb{R}^n_+ = \{y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n : y_j \ge 0,\ j = 1, 2, \dots, n\},$$

$$\mathrm{int}\,\mathbb{R}^n_+ = \{y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n : y_j > 0,\ j = 1, 2, \dots, n\},$$

$$T(x) = \prod_{j=1}^{n} T_j(x), \quad \text{where } T_j : K \to 2^{X^*}.$$

We define set-valued mappings $T_0, G_0 : K \to 2^{X^*}$ by

$$T_0(x) = \mathrm{conv}\{T_j(x)\}_{j=1,2,\dots,n}, \qquad G_0(x) = \bigcup_{j=1}^{n} T_j(x),$$

respectively.

In [11], Yang and Yao suggested gap functions for SVVI and WVVI. Following similar ideas, we next introduce gap functions for SVVI and WVVI with $T(x) = \prod_{j=1}^{n} T_j(x)$ for any $x \in K$.

Let $x, y \in K$ and let $t = (t_1, t_2, \dots, t_n) \in T(x)$. Then, $t_j \in T_j(x)$ for each $j = 1, 2, \dots, n$. Denote

$$\langle t, y - x\rangle = (\langle t_1, y - x\rangle, \dots, \langle t_n, y - x\rangle),$$

i.e., $\langle t_j, y - x\rangle$ is the jth component of $\langle t, y - x\rangle$, $j = 1, 2, \dots, n$. Now, we define $\varphi_T : K \to \mathbb{R} \cup \{+\infty\}$,

$$\varphi_T(x) = -\sup_{t\in T(x)} \inf_{y\in K} \max_{1\le j\le n} \langle t_j, y-x\rangle$$
$$= \inf_{t\in T(x)} \sup_{y\in K} \min_{1\le j\le n} \langle t_j, x-y\rangle, \quad \forall x\in K. \tag{1}$$

For $x\in K$, denote $B_x^T = \{t|t: K\to T(x)\}$, i.e., B_x^T is the set of all operators t from K to $T(x)$. Let $x\in K$ and $t\in B_x^T$. Then,

$$t(y) = (t_1(y), t_2(y), \dots, t_n(y)) \in T(x), \quad \forall y\in K,$$

where $t_j(y)\in T_j(x)$ for each $j=1,2,\dots,n$. We also define $\phi_T: K\to \mathbb{R}\cup\{+\infty\}$ as follows:

$$\phi_T(x) = -\sup_{t\in B_x^T} \inf_{y\in K} \max_{1\le j\le n} \langle t_j(y), y-x\rangle$$
$$= \inf_{t\in B_x^T} \sup_{y\in K} \min_{1\le j\le n} \langle t_j(y), x-y\rangle, \quad \forall x\in K. \tag{2}$$

Theorem 2.1 *Suppose that, for each $x\in K$, $T_j(x)$ is nonempty, convex and weakly* compact for $j=1,2,\dots,n$. Then, the following statements are true*:

(i) *The function φ_T defined by* (1) *is a gap function for SVVI.*
(ii) *The function ϕ_T defined by* (2) *is a gap function for WVVI.*

Proof Since for each $x\in K$, $T_j(x)$ is nonempty, convex and weakly* compact for $j=1,2,\dots,n$, from Lemma 2.1(i), we have that $T(x)$ is weakly* compact. Then, the function φ_T given by (1) is well defined and

$$\varphi_T(x) = \min_{t\in T(x)} \sup_{y\in K} \min_{1\le j\le n} \langle t_j, x-y\rangle, \quad \forall x\in K.$$

It is immediate that

$$\sup_{y\in K} \min_{1\le j\le n} \langle t_j, x-y\rangle \ge 0, \quad \forall x\in K,\ \forall t\in T(x),$$

so that $\varphi_T(x)\ge 0, \forall x\in K$. Let $x^*\in K$. We observe that $\varphi_T(x^*)=0$ iff there exists $t^*\in T(x^*)$ such that

$$\sup_{y\in K} \min_{1\le j\le n} \langle t_j^*, x^*-y\rangle = 0,$$

or

$$\min_{1\le j\le n} \langle t_j^*, x^*-y\rangle \le 0, \quad \forall y\in K,$$

which is equivalent to

$$\langle t^*, y-x^*\rangle \notin -\operatorname{int}\mathbb{R}_+^n, \quad \forall y\in K,$$

that is, $x^*\in S_{\mathrm{SVVI}}$.

(ii) Since for any given $x \in K$ and $t \in B_x^T$,

$$\sup_{y\in K} \min_{1\le j\le n} \langle t_j(y), x-y\rangle \ge \min_{1\le j\le n} \langle t_j(x), x-x\rangle = 0,$$

we obtain

$$\phi_T(x) = \inf_{t\in B_x^T} \sup_{y\in K} \min_{1\le j\le n} \langle t_j(y), x-y\rangle \ge 0, \quad \forall x \in K.$$

Assume that x^* solves WVVI. Then, for any $y \in K$, there is a $t^*(y) \in T(x^*)$ such that

$$\langle t^*(y), y-x^*\rangle \notin -\operatorname{int}\mathbb{R}_+^n,$$

which implies that

$$\max_{1\le j\le n} \langle t_j^*(y), y-x^*\rangle \ge 0,$$

or equivalently,

$$\min_{1\le j\le n} \langle t_j^*(y), x^*-y\rangle \le 0.$$

Thus, an operator t^* from K to $T(x^*)$ has been defined. It follows that $t^* \in B_{x^*}^T$ and

$$\min_{1\le j\le n} \langle t_j^*(y), x^*-y\rangle \le 0, \quad \forall y \in K.$$

Therefore,

$$\sup_{y\in K} \min_{1\le j\le n} \langle t_j^*(y), x^*-y\rangle \le 0$$

and so

$$\phi_T(x^*) = \inf_{t\in B_{x^*}^T} \sup_{y\in K} \min_{1\le j\le n} \langle t_j(y), x^*-y\rangle \le 0.$$

Since $\phi_T(x) \ge 0$ for all $x \in K$, it follows that $\phi_T(x^*) = 0$. Conversely, suppose that $\phi_T(x^*) = 0$. Let

$$\sup_{y\in K} \min_{1\le j\le n} \langle t_j(y), x^*-y\rangle, \quad \forall t \in B_{x^*}^T.$$

Since $\phi_T(x^*) = 0$, for $\epsilon_1 > \epsilon_2 > \cdots > \epsilon_l > \cdots > 0$ with $\epsilon_l = \frac{\epsilon_1}{2^{l-1}}$, there exist $t^l \in B_{x^*}^T$, $l = 1, 2, \ldots$, such that

$$\sup_{y\in K} \min_{1\le j\le n} \langle t_j^l(y) x^*-y\rangle \le \epsilon_l$$

or equivalently,

$$\min_{1\le j\le n} \langle t_j^l(y), x^*-y\rangle \le \epsilon_l, \quad \forall y \in K,\ \forall l \in \mathbb{N}. \tag{3}$$

Note that $\{t^l(y)\} \subset T(x^*)$ for all $y \in K$. Since $T(x^*)$ is weakly* compact, for any given $y \in K$, $\{t^l(y)\}$ has a weakly* convergent subnet with limit $t^*(y) \in T(x^*)$. Without loss of generality, we may assume that $\{t^l(y)\}$ converges weakly* to $t^*(y)$. Consequently, an operator $t^* : K \to T(x^*)$ has been defined, i.e., $t^* \in B^T_{x^*}$ and

$$\langle t^l(y), x^* - y\rangle \to \langle t^*(y), x^* - y\rangle, \quad \forall y \in K,$$

which implies

$$\min_{1\le j\le n} \langle t^l_j(y), x^* - y\rangle \to \min_{1\le j\le n} \langle t^*_j(y), x^* - y\rangle, \quad \forall y \in K.$$

Taking the limit for $l \to \infty$ in (3), we obtain

$$\min_{1\le j\le n} \langle t^*_j(y), x^* - y\rangle \le 0, \quad \forall y \in K.$$

Then, for any $y \in K$, there exists $t^*_j(y) \in T_j(x^*)$ such that

$$\langle t^*_j(y), x^* - y\rangle \le 0,$$

or equivalently,

$$\langle t^*_j(y), y - x^*\rangle \ge 0,$$

which implies that, for any $y \in K$, there is $t^*(y) \in T(x^*)$ such that

$$\langle t^*(y), y - x^*\rangle \notin -\operatorname{int}\mathbb{R}^n_+,$$

that is, $x^* \in S_{\mathrm{WVVI}}$. This completes the proof. □

Remark 2.1 The gap functions defined by (1) and (2) are slightly different from those introduced by Yang and Yao [11], since no compactness assumptions on the set K are required. Consequently, the proof of Theorem 2.1(ii) is based on different arguments than those of Theorem 2.2 in [11].

3 Characterizations of SVVI and WVVI via Scalarization Approach

In this section, we investigate SVVI and WVVI via the scalarization approach of Konnov [7]. We introduce several kinds of SVI and WVI for SVVI and WVVI respectively and establish their equivalence under certain conditions. By means of the scalarization approach, we suggest gap functions for SVVI, WVVI, SVI, WVI respectively and show the relations among these gap functions.

We consider scalar variational inequalities associated with SVVI and WVVI as follows.

Strong Variational Inequalities (SVI^E): Find $x^* \in K$ such that

$$\exists f^* \in E(x^*) : \langle f^*, y - x^*\rangle \ge 0, \quad \text{for all } y \in K.$$

Weak Variational Inequalities (WVI^E): Find $x^* \in K$ such that

$$\forall y \in K, \exists f^* \in E(x^*) : \langle f^*, y - x^* \rangle \geq 0,$$

where $E : K \to 2^{X^*}$.

We denote by S^E_{SVI} and S^E_{WVI}, the solution sets of SVI^E and WVI^E, respectively. Obviously, $S^E_{\text{SVI}} \subseteq S^E_{\text{WVI}}$.

Remark 3.1 In [7], Konnov introduced SVI^{T_0} and WVI^{T_0} as scalar variational inequalities for SVVI and WVVI.

We make particular reference to the mappings $E := T_0$ and $E := G_0$ and establish some equivalence results between SVVI and WVVI.

Theorem 3.1 *The following statements are true*:

(i) $S^{G_0}_{\text{SVI}} = \bigcup_{j=1}^n S^{T_j}_{\text{SVI}} \subseteq S^{T_0}_{\text{SVI}}$; $\bigcup_{j=1}^n S^{T_j}_{\text{WVI}} \subseteq S^{G_0}_{\text{WVI}} \subseteq S^{T_0}_{\text{WVI}}$.

(ii) *If for each $x \in K$, $G_0(x)$ is convex, then $S^{G_0}_{\text{SVI}} = S^{T_0}_{\text{SVI}}$ and $S^{G_0}_{\text{WVI}} = S^{T_0}_{\text{WVI}}$.*

Proof (i) From Lemma 2.1(ii), we have $G_0(x) = \bigcup_{j=1}^n T_j(x) \subseteq T_0(x)$ for any $x \in K$ and so statement (i) is true except $S^{G_0}_{\text{SVI}} = \bigcup_{j=1}^n S^{T_j}_{\text{SVI}}$ and $S^{G_0}_{\text{WVI}} \supseteq \bigcup_{j=1}^n S^{T_j}_{\text{WVI}}$. Let $x^* \in S^{G_0}_{\text{SVI}}$. Then, $x^* \in K$ and there is $f^* \in G_0(x^*) = \bigcup_{j=1}^n T_j(x^*)$ such that

$$\langle f^*, y - x^* \rangle \geq 0, \quad \text{for all } y \in K,$$

which implies that there exists $j^* = 1, 2, \ldots, n$ such that $f^* \in T_{j^*}(x^*)$ and hence $x^* \in S^{T_{j^*}}_{\text{SVI}} \subseteq \bigcup_{j=1}^n S^{T_j}_{\text{SVI}}$. Conversely, let $x^* \in \bigcup_{j=1}^n S^{T_j}_{\text{SVI}}$. Then, $x^* \in K$ and there exists $j^* = 1, 2, \ldots, n$ such that $x^* \in S^{T_{j^*}}_{\text{SVI}}$, that is, there is $f^* \in T_{j^*}(x^*)$ such that

$$\langle f^*, y - x^* \rangle \geq 0, \quad \text{for all } y \in K.$$

Since $T_{j^*}(x^*) \subseteq G_0(x^*)$, then, $x^* \in S^{G_0}_{\text{SVI}}$. Thus, $S^{G_0}_{\text{SVI}} = \bigcup_{j=1}^n S^{T_j}_{\text{SVI}}$ holds. Similarly, $\bigcup_{j=1}^n S^{T_j}_{\text{WVI}} \subseteq S^{G_0}_{\text{WVI}}$ is true.

(ii) If for each $x \in K$, $G_0(x)$ is convex, then we have $G_0(x) = T_0(x)$ for any $x \in K$, and thus statement (ii) follows. □

Lemma 3.1 (Konnov [7]) *Assume that, for each $x \in K$, $T_j(x)$ is nonempty, convex and weakly* compact for $j = 1, \ldots, n$. Then, $S^{T_0}_{\text{SVI}} = S^{T_0}_{\text{WVI}} = S_{\text{SVVI}} = S_{\text{WVVI}}$.*

Corollary 3.1 *Assume that, for each $x \in K$, $T_j(x)$ is nonempty, convex and weakly* compact for $j = 1, \ldots, n$. If for each $x \in K$, $G_0(x)$ is convex, then $S^{G_0}_{\text{SVI}} = \bigcup_{j=1}^n S^{T_j}_{\text{SVI}} = S^{T_0}_{\text{SVI}} = S^{G_0}_{\text{WVI}} = \bigcup_{j=1}^n S^{T_j}_{\text{WVI}} = S^{T_0}_{\text{WVI}} = S_{\text{SVVI}} = S_{\text{WVVI}}$.*

Proof Using similar arguments of [7], one can prove that $S^{T_i}_{\text{SVI}} = S^{T_j}_{\text{WVI}}$ for $j = 1, \ldots, n$ and so the conclusion follows directly from Theorem 3.1(ii) and Lemma 3.1. □

Corollary 3.2 *Assume that, for each $x \in K$, $E(x)$ is nonempty, convex and weakly* compact. Then, $S^E_{\mathrm{SVI}} = S^E_{\mathrm{WVI}}$.*

Now, we consider gap functions for strong and weak scalar variational inequalities. Consider the function $p_E : K \to \mathbb{R} \cup \{+\infty\}$ defined by

$$p_E(x) = \inf_{f \in E(x)} \sup_{y \in K} \langle f, x - y \rangle, \quad \forall x \in K. \tag{4}$$

Theorem 3.2 *Assume that, for each $x \in K$, $E(x)$ is nonempty, convex and weakly* compact. Then, the function p_E defined by (4) is a gap function for SVI^E.*

Proof Since $\langle f, y - y \rangle = 0$, then $p_E(x) \geq 0$, $\forall x \in K$. Assume that $p_E(x^*) = 0$, with $x^* \in K$. Then,

$$\inf_{f \in E(x^*)} \sup_{y \in K} \langle f, x^* - y \rangle = 0. \tag{5}$$

Since $E(x^*)$ is weakly* compact, then there exists $f^* \in E(x^*)$ such that

$$\sup_{y \in K} \langle f^*, x^* - y \rangle = 0 \tag{6}$$

and therefore x^* is a solution of SVI^E.

Viceversa, suppose that x^* is a solution of SVI^E. Then, (6) holds for some $f^* \in E(x^*)$ and, since $p_E(x^*) \geq 0$, we obtain (5) and, in turn, $p_E(x^*) = 0$. □

As particular cases, define the functions $p_{T_0}, p_{G_0} : K \to \mathbb{R} \cup \{+\infty\}$ by

$$p_{T_0}(x) = \inf_{f \in T_0(x)} \sup_{y \in K} \langle f, x - y \rangle, \quad \forall x \in K, \tag{7}$$

$$p_{G_0}(x) = \inf_{f \in G_0(x)} \sup_{y \in K} \langle f, x - y \rangle, \quad \forall x \in K, \tag{8}$$

respectively.

Remark 3.2 The function p_{T_0} has been considered by Konnov in [7].

Corollary 3.3 *Suppose that, for each $x \in K$, $T_j(x)$ is nonempty, convex and weakly* compact for $j = 1, 2, \ldots, n$. Then, the following statements are true:*

(i) *The function p_{T_0} defined by (7) is a gap function for SVI^{T_0}.*
(ii) *The function p_{G_0} defined by (8) is a gap function for SVI^{G_0}.*

Remark 3.3 Observe that, from Lemma 2.1(ii), it follows that $p_{T_0}(x) \leq p_{G_0}(x) = \min_{1 \leq j \leq n} p_{T_j}(x)$ for any $x \in K$. Moreover, if for each $x \in K$ $G_0(x)$ is convex, then $T_0(x) = G_0(x)$ and so $p_{T_0}(x) = p_{G_0}(x)$ for any $x \in K$.

We also define the function $q_E : K \to \mathbb{R} \cup \{+\infty\}$ by

$$q_E(x) = \inf_{t \in B^E_x} \sup_{y \in K} \langle t(y), x - y \rangle, \quad \forall x \in K, \tag{9}$$

where

$$B_x^E = \{t | t : K \to E(x)\}.$$

Theorem 3.3 *Suppose that, for each $x \in K$, $E(x)$ is nonempty, convex and weakly* compact. Then, the function q_E given by (9) is a gap function for WVI^E.*

Proof This is analogous to the proof of Theorem 2.1(ii). □

Corollary 3.4 *Suppose that, for each $x \in K$, $T_j(x)$ is nonempty, convex and weakly* compact for $j = 1, 2, \ldots, n$. Then, $q_{T_0}(x) \leq q_{G_0}(x) \leq \min_{1\leq j\leq n} q_{T_j}(x)$ for any $x \in K$ and if, furthermore, for each $x \in K$ $G_0(x)$ is convex, then $q_{T_0}(x) = q_{G_0}(x) \leq \min_{1\leq j\leq n} q_{T_j}(x)$ for any $x \in K$.*

Proof Since for each $x \in K$, $T_j(x)$ is nonempty, convex and weakly* compact for $j = 1, 2, \ldots, n$, from of Lemma 2.1(i), one has $T_0(x)$, $G_0(x)$ are weakly* compact for each $x \in K$.

Since $\bigcup_{1\leq j\leq n} B_x^{T_j} \subseteq B_x^{G_0} \subseteq B_x^{T_0}$ for any $x \in K$, we have $q_{T_0}(x) \leq q_{G_0}(x)$ for any $x \in K$. We conclude that $q_{G_0}(x) \leq \min_{1\leq j\leq n} q_{T_j}(x)$ for any $x \in K$. In fact,

$$\begin{aligned}
q_{G_0}(x) &= \inf_{t\in B_x^{G_0}} \sup_{y\in K} \langle t(y), x - y\rangle \\
&\leq \inf_{t\in \bigcup_{j=1}^n B_x^{T_j}} \sup_{y\in K} \langle t(y), x - y\rangle \\
&= \min_{1\leq j\leq n} \inf_{t\in B_x^{T_j}} \sup_{y\in K} \langle t(y), x - y\rangle \\
&= \min_{1\leq j\leq n} q_{T_j}(x).
\end{aligned}$$

Furthermore, if for each $x \in K$ $G_0(x)$ is convex, then from of Lemma 2.1(i), we have that $q_{T_0}(x) = q_{G_0}(x)$, for any $x \in K$. □

Corollary 3.5 *Suppose that, for each $x \in K$, $T_j(x)$ is nonempty, convex and weakly* compact for $j = 1, 2, \ldots, n$. Then, every function $p_{T_0}, q_{T_0}, \varphi_T, \phi_T$ is a gap function for SVVI, WVVI, SVI^{T_0}, WVI^{T_0}.*

Proof The desired conclusion follows directly from Lemma 3.1 and Theorems 3.1–3.3. □

Since every function of $p_{T_0}, q_{T_0}, \varphi_T, \phi_T$ can be suggested as a gap function for SVVI, WVVI, SVI^{T_0}, WVI^{T_0} under certain conditions, it is interesting to analyze their relations.

Theorem 3.4 *Suppose that, for each $x \in K$, $T_j(x)$ is nonempty, convex and weakly* compact for $j = 1, 2, \ldots, n$. Then, the following conclusions hold:*

(i) *$p_{T_0}(x) \geq \varphi_T(x)$ for any $x \in K$.*
(ii) *$q_{T_0}(x) \geq \phi_T(x)$ for any $x \in K$.*

Proof Since for each $x \in K$ $T_j(x)$ is nonempty, convex and weakly* compact for $j = 1, 2, \ldots, n$, from Lemma 2.1(i), $T(x)$ and $T_0(x)$ are weakly* compact.

(i) Let $x \in K$ and $f \in T_0(x)$. Then, there exist $t_j \in T_j(x)$, $j = 1, 2, \ldots, n$, or equivalently, $t = (t_1, t_2, \ldots, t_n) \in T(x)$ and $\lambda_j \geq 0$, $j = 1, 2, \ldots, n$ with $\sum_{j=1}^n \lambda_j = 1$ such that $f = \sum_{j=1}^n \lambda_j t_j$. Then, for any $y \in K$,

$$\min_{1\leq j\leq n} \langle t_j, x-y\rangle \leq \sum_{j=1}^n \lambda_j \langle t_j, x-y\rangle$$

$$= \left\langle \sum_{j=1}^n \lambda_j t_j, x-y \right\rangle = \langle f, x-y\rangle,$$

and so

$$\sup_{y\in K} \min_{1\leq j\leq n} \langle t_j, x-y\rangle \leq \sup_{y\in K} \langle f, x-y\rangle.$$

It follows that

$$\varphi_T(x) = \min_{t\in T(x)} \sup_{y\in K} \min_{1\leq j\leq n} \langle t_j, x-y\rangle$$

$$\leq \min_{f\in T_0(x)} \sup_{y\in K} \langle f, x-y\rangle = p_{T_0}(x), \quad \forall x \in K.$$

(ii) Let $x \in K$ and $t \in B_x^{T_0} = \{t | t : K \to T_0(x)\}$. Then, for any $y \in K$, there exist $t_j(y) \in T_j(x)$, $j = 1, 2, \ldots, n$, and $\lambda_j \geq 0$, $j = 1, 2, \ldots, n$ with $\sum_{j=1}^n \lambda_j = 1$ such that $t(y) = \sum_{j=1}^n \lambda_j t_j(y)$.

Thus, an operator $t_j : K \to T_j(x)$, $j = 1, 2, \ldots, n$, or equivalently $(t_1, t_2, \ldots, t_n) \in B_x^T = \{t | t : K \to T(x)\}$ has been defined. Then, for any $y \in K$,

$$\min_{1\leq j\leq n} \langle t_j(y), x-y\rangle \leq \sum_{j=1}^n \lambda_j \langle t_j(y), x-y\rangle$$

$$= \left\langle \sum_{j=1}^n \lambda_j t_j(y), x-y \right\rangle = \langle t(y), x-y\rangle,$$

which implies that

$$\sup_{y\in K} \min_{1\leq j\leq n} \langle t_j(y), x-y\rangle \leq \sup_{y\in K} \langle t(y), x-y\rangle.$$

It follows that

$$\phi_T(x) = \inf_{t\in B_x^T} \sup_{y\in K} \min_{1\leq j\leq n} \langle t_j(y), x-y\rangle$$

$$\leq \inf_{t\in B_x^{T_0}} \sup_{y\in K} \langle t(y), x-y\rangle = q_{T_0}(x), \quad \forall x \in K.$$

The proof is complete. □

4 Applications to Error Bounds for Gap Functions of SVVI and WVVI

In this section, we apply the results obtained in Sects. 2 and 3 to error bounds. We obtain some existence theorems of global error bounds for gap functions under strong monotonicity and derive some characterizations of global (respectively, local) error bounds for gap functions.

Definition 4.1 Let $h : K \to \mathbb{R} \cup \{+\infty\}$ be a proper function (that is, $\operatorname{dom} h = \{x \in K : h(x) < +\infty\}$ is nonempty). Suppose that $S = \{x \in K : h(x) \le 0\}$ is nonempty. Let $h_+(x) = \max\{h(x), 0\}$ and $d(x, S) = \inf_{s \in S} d(x, s)$. We say that:

(i) h has a global error bound if there exists $\mu > 0$ such that:

$$d(x, S) \le \mu h_+(x), \quad \forall x \in K.$$

(ii) h has a local error bound if, for some $0 < \epsilon < +\infty$, there exists $\mu(\epsilon) > 0$ such that

$$d(x, S) \le \mu(\epsilon) h_+(x), \quad \forall x \in K \text{ with } h(x) < \epsilon.$$

It is clear that, if h has a global error bound, then it has a local error bound.

Definition 4.2 A set-mapping $E : K \to 2^{X^*}$ is said to be strongly monotone with modulus $\alpha > 0$ on K if, for any $x, y \in K$,

$$\langle e_x - e_y, x - y\rangle \ge \alpha \|x - y\|^2, \quad \forall e_x \in E(x),\ e_y \in E(y).$$

E is said to be monotone if the above inequality holds with $\alpha = 0$. E is said to be strictly monotone if it is monotone and the strict relation in the above inequality holds when $x \ne y$.

Remark 4.1 Let $E_1, E_2 : K \to 2^{X^*}$ be two set-valued mappings with $E_1(x) \subseteq E_2(x)$ for any $x \in K$. We observe that, by the definition of strong monotonicity (respectively, monotonicity and strict monotonicity), if E_2 is strongly monotone with modulus $\alpha > 0$ (respectively, monotone and strictly monotone) on K, then, E_1 is also strongly monotone with modulus $\alpha > 0$ (respectively, monotone and strictly monotone) on K.

Remark 4.2 It is easy to see that strong monotonicity implies strict monotonicity and that strict monotonicity implies monotonicity, but the converse is not true.

Proposition 4.1 *If T_0 is strongly monotone with modulus $\alpha > 0$ (respectively, monotone and strictly monotone) on K, then G_0 and T_j, $j = 1, 2, \ldots, n$, are all strongly monotone with modulus $\alpha > 0$ (respectively monotone and strictly monotone) on K.*

Proof From Lemma 2.1(ii), we obtain $G_0(x) = \bigcup_{j=1}^n T_j(x) \subseteq T_0(x)$, for any $x \in K$. Thus, Remark 4.1 yields the desired conclusion. □

Throughout this section, we always suppose that $S_{\text{SVI}}^{G_0}$, $S_{\text{SVI}}^{T_j}$, $S_{\text{SVI}}^{T_0}$, $S_{\text{WVI}}^{G_0}$, $S_{\text{WVI}}^{T_j}$, $S_{\text{WVI}}^{T_0}$, S_{SVVI}, S_{WVVI} are nonempty.

Remark 4.3 Under only monotonicity assumption, the existence of solutions of a VI cannot be assured. In order to derive the nonemptiness of the solutions set of a VI, we should assume either monotonicity and some generalized continuity (see for example [22]) or some coercivity conditions (see for example [23]). But here, we study the existence of global (local) error bounds. As is well known, a global (local) error bound is always investigated in the case of nonemptiness of solutions set.

We prove some existence results of global error bounds for gap functions under strong monotonicity.

Theorem 4.1 *Suppose that for each $x \in K$, $T_j(x)$ is nonempty, convex and weakly* compact for $j = 1, 2, \dots, n$. Then, the following assertions are true*:

- (i) *If G_0 is strongly monotone with modulus $\alpha_{G_0} > 0$ on K, then the function $\sqrt{\varphi_T}$ has a global error bound.*
- (ii) *If T_0 is strongly monotone with modulus $\alpha_{T_0} > 0$ on K, then the function $\sqrt{p_{T_0}}$ has a global error bound.*
- (iii) *If G_0 is strongly monotone with modulus $\alpha_{G_0} > 0$ on K, then the function $\sqrt{p_{G_0}}$ has a global error bound.*
- (iv) *If T_j is strongly monotone with modulus $\alpha_{T_j} > 0$ on K, then the function $\sqrt{p_{T_j}}$ has a global error bound.*

Proof It suffices to show that assertions (i) and (ii) hold. Proofs of (iii) and (iv) follow that of (ii). Since for each $x \in K$ $T_j(x)$ is nonempty, convex and weakly* compact for $j = 1, 2, \dots, n$, Lemma 2.1(i) implies that, for each $x \in K$, $T(x)$ and $T_0(x)$ are nonempty, convex and weakly* compact. From the definition of gap function and Theorem 2.1(i), we have $S_{\text{SVVI}} = \{x \in K : \sqrt{\varphi_T(x)} = 0\}$ and $S_{\text{SVI}}^{T_0} = \{x \in K : \sqrt{p_{T_0}(x)} = 0\}$. We first show that assertion (i) holds. Let $x^* \in S_{\text{SVVI}}$. Then, $x^* \in K$ and

$$\exists t^* \in T(x^*) : \langle t^*, y - x^* \rangle \not< 0, \quad \text{for all } y \in K,$$

which implies that, for any $y \in K$, there exists $j_y \in \{1, 2, \dots, n\}$ such that $t_{j_y}^* \in T_{j_y}(x^*) \subseteq G_0(x^*)$ and

$$\langle t_{j_y}^*, y - x^* \rangle \geq 0. \tag{10}$$

From the definition of φ_T and recalling that, by Lemma 2.1, $T(x)$ is weakly* compact, we have

$$\varphi_T(x) = \min_{t \in T(x)} \sup_{y \in K} \min_{1 \leq j \leq n} \langle t_j, x - y \rangle, \quad \forall x \in K.$$

Then, for any $x \in K$, there exists $t^x \in T(x)$ such that

$$\varphi_T(x) = \sup_{y \in K} \min_{1 \leq j \leq n} \langle t_j^x, x - y \rangle \geq \min_{1 \leq j \leq n} \langle t_j^x, x - y \rangle, \quad \forall y \in K. \tag{11}$$

We remark that $t_j^x \in T_j(x) \subseteq G_0(x)$, for each $j = 1, 2, \ldots, n$. If G_0 is strongly monotone with modulus $\alpha_{G_0} > 0$ on K, then by combining (10) and (11), we have

$$\begin{aligned}\varphi_T(x) &\geq \min_{1\leq j\leq n} \langle t_j^x, x - x^*\rangle \\ &= \min_{1\leq j\leq n} \langle t_j^x - t_{j_x}^*, x - x^*\rangle + \langle t_{j_x}^*, x - x^*\rangle \\ &\geq \alpha_{G_0}\|x - x^*\|^2 + \langle t_{j_x}^*, x - x^*\rangle \\ &\geq \alpha_{G_0}\|x - x^*\|^2 \\ &\geq \alpha_{G_0} d^2(x, S_{\mathrm{SVVI}}).\end{aligned}$$

Set $\mu_T = \sqrt{\frac{1}{\alpha_{G_0}}}$. Then, $\mu_T\sqrt{\varphi_T(x)} \geq d(x, S_{\mathrm{SVVI}})$, which shows that assertion (i) is true.

We now prove assertion (ii). Letting $x^* \in S_{\mathrm{SVI}}^{T_0}$ yields that $x^* \in K$ and there exists $f_{x^*} \in T_0(x^*)$ such that

$$\langle f_{x^*}, y - x^*\rangle \geq 0, \quad \forall y \in K. \tag{12}$$

From (4), recalling that, by Lemma 2.1, $T_0(x)$ is weakly* compact, we have

$$p_{T_0}(x) = \min_{f\in T_0(x)} \sup_{y\in K}\langle f, x - y\rangle, \quad \forall x \in K.$$

Thus, for any $x \in K$, there exists $f_x \in T_0(x)$ such that

$$p_{T_0}(x) = \sup_{y\in K}\langle f_x, x - y\rangle \geq \langle f_x, x - y\rangle, \quad \forall y \in K. \tag{13}$$

If T_0 is strongly monotone with modulus $\alpha_{T_0} > 0$ on K, then it follows from (12) and (13) that

$$\begin{aligned}p_{T_0}(x) &\geq \langle f_x, x - x^*\rangle \\ &= \langle f_x - f_{x^*}, x - x^*\rangle + \langle f_{x^*}, x - x^*\rangle \\ &\geq \alpha_{T_0}\|x - x^*\|^2 + \langle f_{x^*}, x - x^*\rangle \\ &\geq \alpha_{T_0}\|x - x^*\|^2 \\ &\geq \alpha_{T_0} d^2(x, S_{\mathrm{SVI}}^{T_0}).\end{aligned}$$

Set $\mu_{T_0} = \sqrt{\frac{1}{\alpha_{T_0}}}$. It follows that $\mu_{T_0}\sqrt{p_{T_0}(x)} \geq d(x, S_{\mathrm{SVI}}^{T_0})$, $\forall x \in K$.

The proof is complete. □

Remark 4.4 Suppose that $E : K \to 2^{X^*}$ is strictly monotone, where E denotes T_0, G_0, T_j and S_{SVI}^E is nonempty. Then, S_{SVI}^E consists of one point. In fact, let x_1^*, x_2^* be two points in S_{SVI}^E. Then,

$$\exists f_1^* \in E(x_1^*) : \langle f_1^*, x_2^* - x_1^*\rangle \geq 0$$

and

$$\exists f_2^* \in E(x_2^*) : \langle f_2^*, x_1^* - x_2^* \rangle \geq 0.$$

Suppose on the contrary that $x_1^* \neq x_2^*$. Since E is strictly monotone, one has

$$\langle f_2^* - f_1^*, x_2^* - x_1^* \rangle > 0.$$

It follows from the first and the third inequality above that

$$\langle f_2^*, x_1^* - x_2^* \rangle < 0,$$

which contradicts the second one. Since strong monotonicity implies strict monotonicity, the strong monotonicity of G_0 in Theorem 4.1(i) and (iii) implies that $S_{\mathrm{SVI}}^{G_0}$ and $S_{\mathrm{SVI}}^{T_j}$ (by Proposition 4.1) consist of one point; the strong monotonicity of T_0 in Theorem 4.1(ii) implies that $S_{\mathrm{SVI}}^{T_0}(= S_{\mathrm{WVI}}^{T_0} = S_{\mathrm{SVVI}} = S_{\mathrm{WVVI}})$ (by Lemma 3.1), $S_{\mathrm{SVI}}^{G_0}$ and $S_{\mathrm{SVI}}^{T_i}$ (by Proposition 4.1) consist of one point; the strong monotonicity of T_j in Theorem 4.1(iv), implies that $S_{\mathrm{SVI}}^{T_j}$ consists of one point.

We now show several relations concerning global and local error bounds for gap functions.

Theorem 4.2 *Suppose that, for each $x \in K$, $T_j(x)$ is nonempty, convex and weakly* compact for $j = 1, 2, \ldots, n$. Then, the following statements are true*:

(i) *If for each $j = 1, 2, \ldots, n$, the function p_{T_j} has a global error bound, then the function p_{G_0} has a global error bound.*

(ii) *Assume that $S_{\mathrm{SVI}}^{G_0} \equiv S_{\mathrm{SVI}}^{T_j}$, for each $j = 1, 2, \ldots, n$. If p_{G_0} has a global error bound, then for each $j = 1, 2, \ldots, n$, the function p_{T_j} has a global error bound.*

(iii) *Assume that $S_{\mathrm{WVI}}^{G_0} \equiv S_{\mathrm{WVI}}^{T_j}$, for each $j = 1, 2, \ldots, n$. If q_{G_0} has a global error bound, then for each $j = 1, 2, \ldots, n$, the function q_{T_j} has a global error bound.*

(iv) *If the function φ_T has a global error bound, then the function p_{T_0} has a global error bound. Furthermore, suppose that, for each $x \in K$, $G_0(x)$ is convex. If φ_T has a global error bound, then p_{G_0} has a global error bound.*

(v) *If the function ϕ_T has a global error bound, then the function q_{T_0} has a global error bound. Furthermore, suppose that, for each $x \in K$, $G_0(x)$ is convex. If ϕ_T has a global error bound, then q_{G_0} has a global error bound.*

Proof From the definition of gap function, Corollary 3.2 and Theorem 2.1, we have that $S_{\mathrm{SVI}}^E = \{x \in K : p_E(x) = 0\}$, $S_{\mathrm{WVI}}^E = \{x \in K : q_E(x) = 0\}$, $S_{\mathrm{SVVI}} = \{x \in K : \varphi_T(x) = 0\}$ and $S_{\mathrm{WVVI}} = \{x \in K : \phi_T(x) = 0\}$, where E denotes G_0, T_j, T_0, respectively.

(i) Suppose that, for each $j = 1, 2, \ldots, n$, the function p_{T_j} has a global error bound. Then, for each $j = 1, 2, \ldots, n$, there exists $\mu_j > 0$ such that

$$d(x, S_{\mathrm{SVI}}^{T_j}) \leq \mu_j p_{T_j}(x), \quad \forall x \in K.$$

Set $\mu = \max_{1\le j\le n} \mu_j$. By Theorem 3.1(i), one has $S_{\mathrm{SVI}}^{G_0} = \bigcup_{j=1}^{n} S_{\mathrm{SVI}}^{T_j}$. Observing that

$$p_{G_0}(x) = \min_{1\le j\le n} p_{T_j}(x), \quad \forall x \in K, \tag{14}$$

it follows that

$$\begin{aligned} d(x, S_{SVI}^{G_0}) &= \min_{1\le j\le n} d(x, S_{\mathrm{SVI}}^{T_j}) \\ &\le \mu \min_{1\le j\le n} p_{T_j}(x) \\ &= \mu p_{G_0}(x), \quad \forall x \in K, \end{aligned}$$

which implies that the function p_{G_0} has a global error bound.

(ii) Suppose that $S_{\mathrm{SVI}}^{G_0} \equiv S_{\mathrm{SVI}}^{T_j}$, for each $j = 1, 2, \ldots, n$, and that p_{G_0} has a global error bound. Then, there exists $\mu > 0$ such that

$$\begin{aligned} d(x, S_{\mathrm{SVI}}^{T_j}) &= d(x, S_{\mathrm{SVI}}^{G_0}) \\ &\le \mu p_{G_0}(x) \\ &= \mu \min_{1\le j\le n} p_{T_j}(x) \\ &\le \mu p_{T_j}(x), \quad \forall j = 1, 2, \ldots, n,\ x \in K. \end{aligned}$$

The above inequality yields that, for each $j = 1, 2, \ldots, n$, the function p_{T_j} has a global error bound.

(iii) The proof is similar to that of (ii) and so we omit it.

(iv) From the assumptions, Theorem 3.1 and Lemma 3.1 imply that $S_{\mathrm{SVI}}^{G_0} \subseteq S_{\mathrm{SVI}}^{T_0} = S_{\mathrm{SVVI}}$, and Theorem 3.4 allows us to prove that $\varphi_T(x) \le p_{T_0}(x) \le p_{G_0}(x)$, for any $x \in K$ (if for each $x \in K$, $G_0(x)$ is convex, then $S_{\mathrm{SVI}}^{G_0} = S_{\mathrm{SVI}}^{T_0} = S_{\mathrm{SVVI}}$, and $\varphi_T(x) \le p_{T_0}(x) = p_{G_0}(x)$). If the function φ_T has a global error bound, then there exists $\mu > 0$ such that

$$d(x, S_{\mathrm{SVVI}}) \le \mu \varphi_T(x), \quad \forall x \in K.$$

It follows that the desired conclusion holds.

(v) The proof is similar to that of (iv) and so we omit it. The proof is complete. □

Corollary 4.1 *Suppose that, for each $x \in K$, $T_j(x)$ is nonempty, convex and weakly* compact, for $j = 1, 2, \ldots, n$. Suppose that G_0 is strongly monotone with modulus $\alpha_{G_0} > 0$ on K. Then, the function p_{T_0} has a global error bound. Furthermore, if for each $x \in K$ $G_0(x)$ is convex, then p_{G_0} has a global error bound.*

Proof The desired conclusion follows directly from Theorem 4.1(i) and Theorem 4.2(iv). □

Similarly, one has the following characterizations of local error bounds for gap functions of SVVI and WVVI.

Theorem 4.3 *Suppose that for each $x \in K$, $T_j(x)$ is nonempty, convex and weakly* compact, for $j = 1, 2, \dots, n$. Then, the following assertions are true*:

(i) *Assume that $S_{\mathrm{SVI}}^{G_0} \equiv S_{\mathrm{SVI}}^{T_j}$, for $j = 1, \dots, n$. If p_{G_0} has a local error bound, then, for each $j = 1, 2, \dots, n$, the function p_{T_j} has a local error bound.*

(ii) *Assume that $S_{\mathrm{WVI}}^{G_0} \equiv S_{\mathrm{WVI}}^{T_j}$, for $j = 1, \dots, n$. If q_{G_0} has a local error bound, then for each $j = 1, 2, \dots, n$, the function q_{T_j} has a local error bound.*

(iii) *If the function φ_T has a local error bound, then the function p_{T_0} has a local error bound. Furthermore, suppose that, for each $x \in K$, $G_0(x)$ is convex. If φ_T has a local error bound, then p_{G_0} has a local error bound.*

(iv) *If the function ϕ_T has a local error bound, then the function q_{T_0} has a local error bound. Furthermore, suppose that, for each $x \in K$, $G_0(x)$ is convex. If ϕ_T has a local error bound, then q_{G_0} has a local error bound.*

Proof (i) Suppose that $S_{\mathrm{SVI}}^{G_0} \equiv S_{\mathrm{SVI}}^{T_j}$, for each $j = 1, 2, \dots, n$ and that p_{G_0} has a local error bound. Then, for some $0 < \epsilon < +\infty$, there exists $\mu(\epsilon) > 0$ such that

$$d(x, S_{\mathrm{SVI}}^{G_0}) \le \mu(\epsilon) p_{G_0}(x), \quad \forall x \in K \text{ with } p_{G_0}(x) < \epsilon.$$

By Theorem 3.1(i), we have that $S_{\mathrm{SVI}}^{G_0} = \bigcup_{j=1}^{n} S_{\mathrm{SVI}}^{T_j}$. Since (14) holds, it follows that

$$\begin{aligned} d(x, S_{\mathrm{SVI}}^{T_j}) &= d(x, S_{\mathrm{SVI}}^{G_0}) \\ &\le \mu(\epsilon) p_{G_0}(x) \\ &= \mu(\epsilon) \min_{1 \le j \le n} p_{T_j}(x) \\ &\le \mu(\epsilon) p_{T_j}(x), \quad \forall x \in K \text{ with } p_{T_j}(x) < \epsilon, \end{aligned}$$

for each $j = 1, 2, \dots, n$, which yields the desired conclusion.

(ii) The proof is similar to that of (i) and so we omit it.

(iii) Note that $S_{\mathrm{SVI}}^{G_0} \subseteq S_{\mathrm{SVI}}^{T_0} = S_{\mathrm{SVVI}}$ (by Theorem 3.1 and Lemma 3.1) and $\varphi_T(x) \le p_{T_0}(x) \le p_{G_0}(x)$, for any $x \in K$; (if for each $x \in K$, $G_0(x)$ is convex, then $S_{\mathrm{SVI}}^{G_0} = S_{\mathrm{SVI}}^{T_0} = S_{\mathrm{SVVI}}$ and $\varphi_T(x) \le p_{T_0}(x) = p_{G_0}(x)$). If the function φ_T has a local error bound, then for some $0 < \epsilon < +\infty$, there exists $\mu(\epsilon) > 0$ such that

$$d(x, S_{\mathrm{SVVI}}) \le \mu(\epsilon) \varphi_T(x), \quad \forall x \in K \text{ with } \varphi_T(x) < \epsilon.$$

It is clear that the desired conclusion holds.

(iv) The proof is similar to that of (iii) and so we omit it. This completes the proof. □

References

1. Giannessi, F.: Theorems of the alternative, quadratic programs, and complementarity problems. In: Cottle, R.W., Giannessi, F., Lions, J.L. (eds.), Variational Inequality and Complementarity Problems, pp. 151–186. Wiley, Chichester (1980)

2. Chen, G.Y.: Existence of solutions for a vector variational inequality: an extension of Hartmann-Stampacchia theorem. J. Optim. Theory Appl. **74**, 445–456 (1992)
3. Chen, G.Y., Goh, C.J., Yang, X.Q.: On gap functions for vector variational inequalities. In: Giannessi, F. (ed.) Vector Variational Inequalities and Vector Equilibria. Mathematical Theories, pp. 55–72. Kluwer Academic, Dordrecht (2000)
4. Chen, G.Y., Huang, X.X., Yang, X.Q.: Vector Optimization: Set-valued and Variational Analysis. Lecture Notes in Economics and Mathematical Systems, vol. 541. Springer, Berlin (2005)
5. Giannessi, F. (ed.): Vector Variational Inequalities and Vector Equilibria. Mathematical Theories. Kluwer Academic, Dordrecht (2000)
6. Hadjisavvas, N., Schaible, S.: Quasimonotonicity and pesudomonotonicity in variational inequalities and equilibrium problems. In: Crouzeix, J.P., Martinez-Legaz, J.E., Volle, M. (eds.) Generalized Convexity, Generalized Monotonicity, pp. 257–275. Kluwer Academic, Dordrecht (1998)
7. Konnov, I.V.: A scalarization approach for vector variational inequalities with applications. J. Global Optim. **32**, 517–527 (2005)
8. Konnov, I.V., Yao, J.C.: Existence of solutions for generalized vector equilibrium problems. J. Math. Anal. Appl. **233**, 328–335 (1999)
9. Li, J., He, Z.Q.: Gap functions and existence of solutions to generalized vector variational inequalities. Appl. Math. Lett. **18**, 989–1000 (2005)
10. Luc, D.T.: Theory of Vector Optimization. Springer, Berlin (1989)
11. Yang, X.Q., Yao, J.C.: Gap functions and existence of solutions to set-valued vector variational inequalities. J. Optim. Theory Appl. **115**, 407–417 (2002)
12. Yang, X.Q.: On the gap functions of prevariational inequalities. J. Optim. Theory Appl. **116**, 437–452 (2003)
13. Huang, N.J., Li, J., Yao, J.C.: Gap functions and existence of solutions to a system of vector equilibrium problems. J. Optim. Theory Appl. **133**, 201–212 (2007)
14. Li, J., Huang, N.J.: Implicit vector equilibrium problems via nonlinear scalarisation. Bull. Austral. Math. Soc. **72**, 161–172 (2005)
15. Li, J., Huang, N.J.: An extension of gap functions for a system of vector equilibrium problems with applications to optimization problems. J. Global Optim. **39**, 247–260 (2007)
16. Li, S.J., Teo, K.L., Yang, X.Q., Wu, S.Y.: Gap functions and existence of solutions to generalized vector quasi-equilibrium problems. J. Global Optim. **34**, 427–440 (2006)
17. Mastroeni, G.: Gap functions for equilibrium problems. J. Global Optim. **27**, 411–426 (2003)
18. Luo, X.D., Luo, Z.Q.: Extension of Hoffman's error bound to polynomial systems. SIAM J. Optim. **4**, 383–392 (1994)
19. Ng, K.F., Zheng, X.Y.: Error bounds for lower semicontinuous functions in normal spaces. SIAM J. Optim. **12**, 1–17 (2002)
20. Pang, J.S.: Error bounds in mathematical programming. Math. Program. Ser. A **79**, 299–332 (1997)
21. Wu, Z.L., Ye, J.J.: On error bounds for lower semicontinuous functions. Math. Program. Ser. A **92**, 301–314 (2002)
22. Konnov, I.V., Yao, J.C.: On the generalized vector variational inequality problem. J. Math. Anal. Appl. **206**, 42–58 (1997)
23. Daniilidis, A., Hadjisavvas, N.: Coercivity conditions and variational inequalities. Math. Program. Ser. A **86**, 433–438 (1999)