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Lagrange Multipliers for ε-Pareto Solutions in Vector Optimization with Nonsolid Cones in Banach Spaces

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Abstract This paper presents some results concerning the existence of the Lagrange multipliers for vector optimization problems in the case where the ordering cone in the codomain has an empty interior. The main tool for deriving our assertions is a scalarization by means of a functional introduced by Hiriart-Urruty (Math. Oper. Res. 4:79–97, 1979) (the so-called oriented distance function). Moreover, we explain some applications of our results to a vector equilibrium problem, to a vector control-approximation problem and to an unconstrainted vector fractional programming problem.

Keywords Lagrange multipliers · Mordukhovich subdifferential · Proximal subdifferential · Constrained and unconstrained vector optimization

1 Introduction

Vector optimization is at present one of the most interesting areas of optimization theory from the theoretical as well as from the computational point of view. Recently,

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there are some monographs on this subject. For example, the reader can look at the monographs of Luc [2], Göpfert, Riahi, Tammer and Zălinescu [3] and Jahn [4]. Though from the application point of view one may like to concentrate on finite dimensional vector optimization problem it has been shown for example in Jahn [4] how vector optimization problems in infinite dimensions arise in a natural way in many applications. Thus it is important from a theoretical viewpoint to study vector optimization in infinite dimensional spaces. However, keeping in mind the practical applications the best setting for studying vector optimization problems in infinite dimensions is that of Banach spaces. In this paper we study vector optimization problems in such a setting. It is well known that there are two main solution concepts in vector optimization namely the Pareto solution and the weak Pareto solution. It is important to note that in order to define a weak Pareto solution one needs to have an ordering cone with a nonempty interior. The aim of this paper is to present some results concerning the existence of Lagrange multipliers for Pareto optimal solutions of vector optimization problems in the special case where the partial order on the objective (output) space is given by a closed convex cone with empty interior. Note that as mentioned above when the ordering cone has an empty interior then one cannot define the notion of a weak Pareto optimal solution and thus one has to deal only with Pareto optimal solutions. In the case of a cone with non-empty interior the same problem as discussed in this paper was considered for weak Pareto solutions in the recent works of [5] and [6], where a scalarization technique introduced in [7] was used. Since this separating functional has important continuity properties only under the assumption of non-emptiness of the interior of the underlying cone, a natural question is to find another function which can be used outside this framework. On the other hand, it is well known that for many infinite dimensional spaces the usual ordering cones have empty interiors, hence it is important to have a tool to handle these situations as well. In this sense, we use the so-called oriented distance function introduced in [1] in the framework of nonsmooth scalar optimization and which was also used in [8–10] for vector optimization problems. In the non-convex setting the price to pay for the missing interiority condition is that we have to consider a concept of approximate Pareto solutions for the vectorial problems we work on. In the convex case we can deal with Pareto optimal points directly even if the ordering cone has an empty interior. However, in the convex case we need some additional regularity assumptions on the image of the feasible set.

We add here that there has been a huge amount of literature on the optimality conditions for weak Pareto solutions as compared to that of Pareto solutions though Pareto solutions seems to be important from the point of view of applications. Further, to the best of our knowledge there are not many contributions concerning necessary optimality conditions for the case when the ordering cone has an empty interior. However, in a recent paper by Bao and Mordukhovich [11] the case of a (possible) empty interior of the ordering cone is considered as well. In order to stress the fact that we are dealing with the situation where the ordering cone has an empty interior, we have used the term nonsolid in the title of the paper.

Further, it is important to provide some motivation and at least some mathematical justification as to why we need to study vector optimization with an ordering cone having an empty interior. In vector optimization theory one works with convex ordering cones which give order relations on the underlying spaces. The natural framework of the majority of vector optimization problems are the normed vector spaces and each of the usual normed vector spaces has a natural ordering cone. Outside the finite dimensional case there exist only a few examples when the natural ordering cone has a non-empty interior. Consider for example the space l^∞, the space BV of all functions of bounded variation on $\mathbb{R}$ or the space $C(\Omega)$ of all continuous real-valued functions on the compact Hausdorff space Ω. On the other hand the natural ordering cones of the most useful normed vector spaces (for example l^p, L^p, $1 \le p < \infty$) have empty interior. In fact the natural ordering cone for most of the Asplund spaces have an empty interior. In support of our statements, we quote the following from Peressini [12, p. 183]:

> "... the class of ordered topological vector spaces possessing cones with non-empty interiors is not very broad."

Of course, in every normed vector space one can construct a closed convex cone with non-empty interior. For example, simply take the closed conic hull of a ball which does not contain the origin. But, for evident purposes, it is much more convenient to have the possibility to work with the natural ordering cones and hence the above mentioned problem arise. Moreover, in recent papers on the field of mathematical finance coherent risk measures (closely related to scalarizing functionals in vector optimization) are introduced on the space L^2 (compare [13] and [14]). It is well known that the natural ordering cone in L^2 has an empty interior.

Altogether, taking into account these arguments it is very important to go beyond the often used assumption in vector optimization concerning the non-emptiness of the interior of the ordering cone.

The plan of the paper is as following: In Sect. 2, we begin by introducing various notations and also a notion of an ε-Pareto solution slightly different from the usual notion of an approximate Pareto solution present in the literature. We provide an interesting relation between a Pareto optimal solution and the notion of an ε-Pareto solution that we introduce here which justifies the introduction of such a notion. Then we proceed to introduce the results from nonsmooth analysis that are used for deriving our main results. Further, in this section we introduce the scalarizing functional which is a fundamental tool to derive our results. In Sect. 3, we present our main optimality conditions. We begin with a smooth constrained convex vector optimization problem. Then we proceed to present necessary optimality conditions for ε-Pareto solutions for the constrained case in a Banach space setting when the objective function is strictly Lipschitzian and for the unconstrained case in the setting of a Hilbert space when the objective function is just locally Lipschitz. In Sect. 4, we provide some applications of the results derived in Sect. 3 to the vector equilibrium problem, the vector control-approximation problem and unconstrained vector fractional programming problem.

2 Preliminaries

Throughout the paper X, Y are Banach spaces over the real field $\mathbb{R}$. The symbols U_X and S_X denote the closed unit ball and the unit sphere in X, where X is a given

Banach space. For any Banach space X the topological dual of X is denoted by X^*. For a positive ε and for an element $x \in X$, we denote the open ball of radius ε centered in x by $B(x, \varepsilon)$. As usual, for a set $C \subset X$, we denote by I_C the indicator function of C ($I_C(x) = 0$ if $x \in C$ and $I_C(x) = +\infty$ if $x \notin C$) and by d_C the distance function with respect to C, $d_C(x) = d(x, C) := \inf_{c \in C} \|x - c\|$ for every $x \in X$ (by convention, $d(x, \emptyset) = +\infty$). Further, we shall also need the notion of an extended-valued function in order to present the nonsmooth calculus rules. Essentially we will use functions of the form $\varphi : X \to \bar{\mathbb{R}}$, where $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. The set of all points of X where φ takes finite values is denoted as $\operatorname{Dom} \varphi$.

Let us consider a closed convex pointed cone K with **empty interior** in Y, i.e., $\operatorname{int} K = \emptyset$, which induces a partial order relation on Y denoted by $\leq_K$ and given as $y_1 \leq_K y_2$ if and only if $y_2 - y_1 \in K$. We set $K^* := \{y^* \in Y^* \mid y^*(y) \geq 0, \forall y \in K\}$ for the dual cone of K. We recall that for a non-empty set $A \subset Y$, a point $\overline{a} \in A$ is called Pareto minimum of A with respect to K if $(A - \overline{a}) \cap -K = \{0\}$. We denote the set of Pareto minimum points of A w.r.t. K by $\operatorname{Min}(A \mid K)$. If $f : X \to Y$ is a vector-valued function and $S \subset X$ is a non-empty set, a point $\overline{x} \in S$ is said to be Pareto minimizer of f over S with respect to K if $f(\overline{x})$ is a Pareto minimum of $f(S)$ with respect to K. Let us consider a fixed element $e \in K$ with $\|e\| = 1$. For a positive ε, we say that $\overline{a} \in A$ is an (ε, e)-Pareto minimum of A with respect to K if $(A - \overline{a}) \cap (-K - \varepsilon e) = \emptyset$. The set of all these minima is denoted by $(\varepsilon, e) - \operatorname{Min}(A \mid K)$. As above, for a vector-valued function $f : X \to Y$ and a non-empty set $S \subset X$, a point $\overline{x} \in S$ is said to be (ε, e)-Pareto minimizer of f over S with respect to K if $f(\overline{x})$ is an (ε, e)-Pareto minimum of $f(S)$ with respect to K.

Remark 2.1 It is clear that the notion of (ε, e)-Pareto optima that we have defined here is a slightly different version that the standard one found in the literature, i.e., $(A - \overline{a}) \cap ((-K \setminus \{0\}) - \varepsilon e) = \emptyset$. For any $\overline{a} \in A$ with $(A - \overline{a}) \cap (-K - \varepsilon e) = \emptyset$ it follows that $\overline{a}$ is an (ε, e)-Pareto minimum in the standard sense. The reverse is not true. Notice that if a point $\overline{a}$ is an (ε, e)-Pareto minimum for A w.r.t. K in the standard sense, then it is an $(\varepsilon + \delta, e)$-Pareto minimum for A w.r.t. K in our sense for every positive δ (taking into account that K is pointed).

However, we will use our concept of (ε, e)-Pareto minimizers of f over S with respect to K defined above in order to get nontrivial multipliers $y^* \neq 0$ using certain properties of the subdifferential of the distance function (see (1)).

Observe that an interesting part of our definition is the following. Viewing it in a slightly informal manner it is interesting to observe that we in fact want to refer as (ε, e)-Pareto minimum to those points which under very small perturbation will leave the feasible objective set. Points lying very near to the efficient frontier will exhibit such behavior under small perturbations and thus from the practical point of view we are indeed talking about solutions that are very close to the efficient frontier and thus qualify in a better way as an approximate-minimum.

The next proposition justifies the choice we make for the concept of (ε, e)-solution.

Proposition 2.1 *The following relation holds*:

$$\operatorname{Min}(A \mid K) = \bigcap_{e\in K\cap S_Y} \bigcap_{\varepsilon>0} (\varepsilon, e) - \operatorname{Min}(A \mid K).$$

Proof It is clear that, for every positive ε and for every $e \in K \cap S_Y$, the pointedness of the cone K implies $-K - \varepsilon e \subset -K \setminus \{0\}$. We deduce that

$$\operatorname{Min}(A \mid K) \subset \bigcap_{e\in K\cap S_Y} \bigcap_{\varepsilon>0} (\varepsilon, e) - \operatorname{Min}(A \mid K).$$

For the converse inclusion, let us take $\overline{y} \in \bigcap_{e\in K\cap S_Y} \bigcap_{\varepsilon>0} (\varepsilon, e) - \operatorname{Min}(A \mid K)$ and suppose that there exists $y \in A$ s.t. $y - \overline{y} \in -K \setminus \{0\}$. Then, for an $\varepsilon > 0$ small enough,

$$(y - \overline{y}) - \varepsilon \|y - \overline{y}\|^{-1}(y - \overline{y}) \in -K \setminus \{0\}.$$

Consequently,

$$y - \overline{y} \in -K - \varepsilon \|y - \overline{y}\|^{-1}(\overline{y} - y),$$

whence $\overline{y} \notin (\varepsilon, \|y - \overline{y}\|^{-1}(\overline{y} - y)) - \operatorname{Min}(A \mid K)$. Since we arrive at a contradiction, the proof is complete. □

In general, for a nonempty set $A \subset Y$, $A \neq Y$, the oriented distance function $\Delta_A : Y \to \mathbb{R}$ is given as $\Delta_A(y) = d_A(y) - d_{Y\setminus A}(y)$ (cf. Hiriart-Urruty [1]). It is well known that this function has very good general properties (see [10]) and we list below for the reader's convenience the properties that we shall use in the sequel.

Proposition 2.2 ([10, Proposition 3.2])

(i) Δ_A *is Lipschitzian of rank* 1.
(ii) *If A is convex*, *then Δ_A is convex and*, *if A is a cone*, *then Δ_A is positively homogeneous*.
(iii) *If A is a closed convex cone and $y_1, y_2 \in Y$ with $y_1 - y_2 \in A$*, *then* $\Delta_A(y_1) \leq \Delta_A(y_2)$.

Note that the properties described above for the oriented distance function are similar to those required for risk measures used in mathematical finance [13, 14].

Let us consider the functional Δ_{-K} that is a convex, positively homogeneous and a 1-Lipschitzian function following the above proposition. In fact, in our particular situation, all these facts come from the simpler form of the functional. Indeed, the emptiness of the interior of K implies that the closure of $Y \setminus (-K)$ is Y itself, so the second distance function in the expression of Δ_{-K} reduces to 0. Hence, in fact $\Delta_{-K} = d_{-K}$.

Recall that, for a convex closed subset A of Y, the normal cone at a point $\overline{a} \in A$ is given as

$$N_A(\overline{a}) = \{y^* \in Y^* \mid y^*(a - \overline{a}) \leq 0,\ \forall a \in A\}.$$

For the convex continuous function d_A, the classical Fenchel subdifferential is given by the following formula (see e.g. [15]):

$$\partial d_A(y) = \begin{cases} S_{X^*} \cap N_{A_y}(y), & \text{if } y \notin A, \\ U_{X^*} \cap N_A(y), & \text{if } y \in A, \end{cases} \tag{1}$$

where $A_y := A + d_A(y)U_Y$.

Furthermore, for the special case of the convex functional Δ_{-K} it holds for every $y \in Y$, $\partial\Delta_{-K}(y) \subset K^*$. Indeed, for $y^* \in \partial\Delta_{-K}(y)$, it holds that

$$y^*(z - y) \leq \Delta_{-K}(z) - \Delta_{-K}(y), \quad \forall z \in Y. \tag{2}$$

From Proposition 2.2(iii), it follows that $\Delta_{-K}(u + y) \leq \Delta_{-K}(y)$, for every $u \in -K$, and whence $y^*(u) \leq 0$ with (2). This implies that, for every $y \in Y$,

$$\partial\Delta_{-K}(y) \subset K^*.$$

Remark 2.2 In Theorems 3.2, 3.3, 3.4 and in Sect. 4, we consider the case $\operatorname{int} K = \emptyset$. In this case, the interior of $-K - \varepsilon e$ is empty too (being a subset of $-K$), whence $\Delta_{-K-\varepsilon e}(y) = d_{-K-\varepsilon e}(y)$. In order to show that $\partial\Delta_{-K-\varepsilon e}(y) \subset K^*$ for every $y \in Y$, we take $y^* \in \partial\Delta_{-K-\varepsilon e}(y) = \partial d_{-K-\varepsilon e}(y)$ for a fixed $y \in Y$. Then, for every $k \in K$, one has

$$\begin{aligned} y^*(-k - \varepsilon e - y) &\leq d_{-K-\varepsilon e}(-k - \varepsilon e) - d_{-K-\varepsilon e}(y) \\ &= -d_{-K-\varepsilon e}(y) \leq 0. \end{aligned}$$

This yields $y^*(k) \geq -\varepsilon y^*(e) - y^*(y)$. Because $y \in Y$ (the reference point) is the same for every $k \in K$ and $y^* \in \partial d_{-K-\varepsilon e}(y)$ in this relation, we obtain $y^* \in K^*$: Indeed, if there exist $k \in K$ s.t. $y^*(k) < 0$, then $y^*(nk) \to -\infty$ as $n \to \infty$ and we get a contradiction with the above inequality since, obviously, $nk \in K$ for every natural n and $-\varepsilon y^*(e) - y^*(y)$ is a constant once we have chosen y^* from $\partial d_{-K-\varepsilon e}(y)$. So we get, for every $y \in Y$,

$$\partial\Delta_{-K-\varepsilon e}(y) \subset K^*.$$

The basic result linking the concept of Pareto minima with the scalarizing functional is the following.

Theorem 2.1 ([10, Theorem 4.3]) *A point $\overline{y} \in A \subset Y$ is a Pareto minimum of A with respect to K if and only if $\overline{y}$ is the unique global solution of the problem* $\min_{y \in A} \Delta_{-K}(y - \overline{y})$.

In a similar way, one has the following result corresponding to ε-solutions.

Theorem 2.2 *Assume that* $\varepsilon > 0$, $e \in K$, $\|e\| = 1$. *If a point* $\overline{y} \in A \subset Y$ *is an* (ε, e)-*Pareto minimum of* A *with respect to* K, *then* $\overline{y}$ *is an* ε-*solution of the problem*

$$\min_{y \in A} \Delta_{-K-\varepsilon e}(y - \overline{y}).$$

Proof The proof is based on the obvious inequality $d_{-K-\varepsilon e}(0) \leq \varepsilon$, since for every $y \in A$ we have

$$d_{-K-\varepsilon e}(0) \leq \varepsilon < d_{-K-\varepsilon e}(y - \overline{y}) + \varepsilon,$$

whence $\overline{y}$ is an ε-solution over A for the scalar problem $\min_{y \in A} \Delta_{-K-\varepsilon e}(y - \overline{y})$. □

In order to present our results concerning the existence of Lagrange multipliers, we work mainly with some concrete subdifferentials, cf. Mordukhovich [16, 17]: The subdifferential of Mordukhovich (∂_M), which satisfies exact calculus rules on Asplund spaces and furthermore, the proximal subdifferential (∂_P), which satisfies fuzzy calculus rules on Hilbert spaces. In this way, under the assumptions that X is an Asplund space, $f_1, f_2 : X \to \overline{\mathbb{R}}$, $\overline{x} \in \operatorname{Dom} f_1 \cap \operatorname{Dom} f_2$ and f_1 is Lipschitz around $\overline{x}$ and f_2 is l.s.c. around $\overline{x}$, it holds that [16, 17, Theorem 2.36]:

$$\partial_M(f_1 + f_2)(\overline{x}) \subset \partial_M f_1(\overline{x}) + \partial_M f_2(\overline{x}).$$

One says [16, 17, Definition 3.25] that a function $f : X \to Y$ is strictly Lipschitzian at $\overline{x}$ if it is locally Lipschitzian around this point and there exists a neighborhood V of the origin in X s.t. the sequence $(t_k^{-1}(f(x_k + t_k v) - f(x_k)))_{k \in \mathbb{N}}$ contains a norm convergent subsequence whenever $v \in V$, $x_k \to \overline{x}$, $t_k \downarrow 0$. It is clear that this notion reduces to local Lipschitz continuity if Y is finite dimensional. For more details regarding this class of mappings with values in infinite dimensional spaces see Sect. 3.1.3 in [16, 17]. Now, the following chain rule holds [16, 17, Theorem 3.43]: If X and Y are Asplund spaces, $f : X \to Y$ is strictly Lipschitzian at $\overline{x}$ and $\varphi : Y \to \mathbb{R}$ is Lipschitz around $f(\overline{x})$, then

$$\partial_M(\varphi \circ f)(\overline{x}) \subset \bigcup\nolimits_{y^* \in \partial_M \varphi(f(\overline{x}))} \partial_M(y^* \circ f)(\overline{x}).$$

For the fuzzy sum rules, we use the following notations:

- $u \xrightarrow{f} x$ means that $u \to x$ and $f(u) \to f(x)$; note that, if f is continuous, then $u \xrightarrow{f} x$ is equivalent to $u \to x$.
- $x^* \in \|\cdot\|^* - \limsup_{u \to x} \partial f(u)$ means that, for every $\varepsilon > 0$, there exist x_ε and x_ε^* such that $x_\varepsilon^* \in \partial f(x_\varepsilon)$ and $\|x_\varepsilon - x\| < \varepsilon$, $\|x_\varepsilon^* - x^*\| < \varepsilon$; the notation $x^* \in \|\cdot\|^* - \limsup_{u \xrightarrow{f} x} \partial f(u)$ has a similar interpretation and it is equivalent to $x^* \in \|\cdot\|^* - \limsup_{u \to x} \partial f(u)$ provided that f is continuous.

Let X, Y be Hilbert spaces. For the proximal subdifferential, we have (see [18, Theorems 8.3, 9.1]): If $f_1, f_2 : X \to \overline{\mathbb{R}}$, $\overline{x} \in \operatorname{Dom} f_1 \cap \operatorname{Dom} f_2$ s.t. one of the functions is Lipschitz around $\overline{x}$, then

$$\partial_P(f_1 + f_2)(x) \subset \|\cdot\|^* - \limsup_{y \xrightarrow{f_1} x,\, z \xrightarrow{f_2} x} (\partial_P f_1(y) + \partial_P f_2(z)),$$

and if $f : X \to Y$ is locally Lipschitz and $\varphi : Y \to \mathbb{R}$ is Lipschitz around $f(\bar{x})$, then

$$\partial_P(\varphi \circ f)(x) \subset \|\cdot\|^* - \limsup_{u \xrightarrow{f} x, v \to f(x)} \bigcup_{u^* \in \partial_P \varphi(v)} \partial_P(u^* \circ f)(u).$$

As usual, for a closed set $S \subset X$ the set $\partial_* I_S(x)$ is denoted by $N_{\partial_*}(S, x)$ and is called the set of normal directions to S at $x \in S$ with respect to ∂_*, where $* \in \{M, P\}$.

3 Main Results

In this section, we intend to study the necessary optimality conditions for the minimization of a function $f : X \to Y$ over a closed set $S \subset X$, where X and Y are Banach spaces and the closed convex pointed ordering cone K in Y has an empty interior. First, we derive a necessary condition for Pareto minimizers when the function f is K-convex and S is a closed convex subset of X. Let us recall the definition of a K-convex function: For any $x, y \in X$ and $\lambda \in [0, 1]$, it holds that

$$\lambda f(y) + (1 - \lambda) f(x) - f(\lambda y + (1 - \lambda)x) \in K.$$

It is important to note that, even if f is K-convex and the set S is a closed convex set, the image set $f(S)$ needs not to be convex. However, the set $f(S) + K$ is a convex set under the convexity hypothesis on f and S. Further, it is not difficult to observe that

$$\min(f(S)|K) = \min(f(S) + K|K). \tag{3}$$

Theorem 3.1 *Let us consider a K-convex function $f : X \to Y$ and let S be a closed convex subset of X. Assume that f is a continuously Frechet differentiable function. Further, assume that the set $f(S)$ has a nonempty interior. Let $\bar{x}$ be a Pareto minimizer of f over S with respect to the ordering cone K which has an empty interior. Then, there exists $v \in K^* \setminus \{0\}$ such that*

$$0 \in f'(\bar{x})^* v + N(S, \bar{x}), \tag{4}$$

where $f'(\bar{x})$ is the Frechet derivative of f at $\bar{x}$ and $f'(\bar{x})^*$ is the adjoint of the Frechet derivative of f at $\bar{x}$ and $N(S, \bar{x})$ denotes the normal cone to the closed convex set S at the point $\bar{x}$.

Proof Since $\bar{x}$ is a Pareto minimizer of f over S, we have using (3)

$$(f(S) + K) \cap (f(\bar{x}) - K) = \{f(\bar{x})\}.$$

Since the interior of $f(S)$ is nonempty, the interior of $f(S) + K$ is also nonempty. Noting the fact that the Pareto minimum point $f(\bar{x})$ lies on the boundary of $f(S) + K$, we get

$$\text{int}(f(S) + K) \cap (f(\bar{x}) - K) = \emptyset,$$

taking into account the above expression.

Thus, by applying a standard separation theorem from convex analysis (see e.g. Theorem 3.16 in [4]) there exists $v \in Y^*$ with $v \neq 0$ such that

$$v(z) \geq v(w), \quad \forall z \in f(S) + K,\ \forall w \in (f(\bar{x}) - K). \tag{5}$$

Now, for any given arbitrary $x \in S$ and $k \in K$, from (5) we have

$$v(f(x) + k) \geq v(f(\bar{x}) - k). \tag{6}$$

By setting $k = 0$, we see that $v(f(x)) \geq v(f(\bar{x}))$ for all $x \in S$. We will now show that $v \in K^*$. On the contrary assume that there exists $k \in K$ such that $v(k) < 0$. From (6), we have

$$v(f(x)) \geq v(f(\bar{x})) - v(k), \quad \forall x \in S.$$

However, since K is a cone, the right-hand side of the above expression can be made arbitrarily large so as to exceed $v(f(x))$ for any given $x \in S$. This leads to a contradiction and thus $v \in K^*$. Thus, we have proved that $\bar{x}$ is a minimum of the convex function $v(f(x))$ over S. Thus, from the well known optimality condition in convex optimization (see for example Zălinescu [19]), we get

$$0 \in (v \circ f)'(\bar{x}) + N(S, \bar{x}).$$

The result now follows by applying the standard chain rule of differentiation. □

Remark 3.1 It is important to note that even in the convex case the above expression is only a necessary condition for the existence of a Pareto minimum and not a sufficient condition. Further, observe that the loss of interiority condition of the ordering cone had to be compensated by the interiority assumption on the image set $f(S)$.

We will now present our results for the nonconvex case. Here we study both the constrained and the unconstrained case. In the first assertion, we work on Asplund spaces and with strictly Lipschitzian functions in order to apply the exact calculus rules of the Mordukhovich subdifferential. In the latter case, we work on Hilbert spaces with locally Lipschitzian functions such that it is possible to apply the fuzzy calculus rules for the proximal subdifferential.

Theorem 3.2 *Let X, Y be Asplund spaces, let K be a closed convex pointed cone in Y with empty interior, let S be a closed subset of X and let $f : X \to Y$ be a strictly Lipschitzian function on S. Assume that $\varepsilon > 0$ and $e \in K$, $\|e\| = 1$. If $\overline{x}$ is an (ε, e)-Pareto minimizer of f over S with respect to K, then there exist $x \in B(\overline{x}, \sqrt{\varepsilon}) \cap S$ and $y^* \in S_{Y^*} \cap K^*$ s.t.*

$$0 \in \partial_M(y^* \circ f)(x) + \sqrt{\varepsilon} U_{X^*} + N_{\partial_M}(S, x).$$

Proof We consider the function $\varphi : X \to Y$ given by $\varphi(x) = f(x) - f(\overline{x})$. Following Theorem 2.2, $\overline{x}$ is an ε-minimum point over S for the functional $z : X \to \mathbb{R}$ defined by $z(x) = (\Delta_{-K-\varepsilon e} \circ \varphi)(x)$. Whence, from the Ekeland variational principle applied for z on S (as a complete metric space), we get an element $x \in B(\overline{x}, \sqrt{\varepsilon}) \cap S$ which is a minimum point on S for the perturbed function $z(\cdot) + \sqrt{\varepsilon}\|\cdot - x\|$. Applying the exact calculus rules of Mordukhovich subdifferential, we have

$$
\begin{aligned}
0 &\in \partial_M(z(\cdot) + \sqrt{\varepsilon}\|\cdot - x\| + I_S(\cdot))(x) \\
&\subset \partial_M(\Delta_{-K-\varepsilon e} \circ \varphi)(x) + \sqrt{\varepsilon} U_{X^*} + \partial_M I_S(x) \\
&\subset \bigcup_{y^* \in \partial_M \Delta_{-K-\varepsilon e}(f(x) - f(\overline{x}))} \partial_M(y^* \circ f)(x) + \sqrt{\varepsilon} U_{X^*} + N_{\partial_M}(S, x).
\end{aligned}
$$

Therefore, we get that there exists $y^* \in \partial_M \Delta_{-K-\varepsilon e}(f(x) - f(\overline{x}))$ s.t. $0 \in \partial_M(y^* \circ f)(x) + \sqrt{\varepsilon} U_{X^*} + N_{\partial_M}(S, x_0)$. Since $\Delta_{-K-\varepsilon e} = d_{-K-\varepsilon e}$ is a convex function (cf. Proposition 2.2), its subdifferential in the sense of Mordukhovich coincides with the subdifferential in the sense of convex analysis. Taking into account (1) and the fact that $f(x) - f(\overline{x}) \notin -K - \varepsilon e$ (from the definition of (ε, e)-minimum), we get together with Remark 2.2 the assertion $y^* \in S_{Y^*} \cap K^*$. □

Remark 3.2 Under convexity assumptions with respect to f and S, we have shown in Theorem 3.1 using a standard separation theorem from convex analysis that there is a nontrivial multiplier $v \in K^* \setminus \{0\}$ such that (4) holds for a Pareto minimizer of f over S. However, in the nonconvex case (even in the case that the ordering cone has a nonempty interior) we were not successful to show for Pareto minimizers that the corresponding multiplier is nontrivial, i.e., $y^* \neq 0$. That leads us to consider in Theorem 3.2 the notion of an approximate optimal solution for which some necessary nontrivial optimality conditions could be obtained by using a variational principle for a perturbed objective function. For the case that K has an empty interior, we get the property $y^* \neq 0$ from (1) taking into account $f(x) - f(\overline{x}) \notin -K - \varepsilon e$. Moreover, from Remark 2.2, we get $y^* \in K^*$. However, observe that no assumption on the image set $f(S)$ is required once we consider approximate solutions.

Theorem 3.3 *Let X, Y be Hilbert spaces, let K be a closed convex pointed cone in Y with empty interior and let $f : X \to Y$ be a locally Lipschitzian function. Assume $\varepsilon > 0$ and $e \in K$, $\|e\| = 1$. If $\overline{x} \in X$ is an (ε, e)-Pareto minimizer of f over X with respect to K, then there exist $x \in B(\overline{x}, \frac{2}{3}\sqrt{\varepsilon})$ and $y^* \in S_{Y^*} \cap K^*$ s.t.*

$$
0 \in \partial_P(y^* \circ f)(x) + \frac{5}{3}\sqrt{\varepsilon} U_{X^*}.
$$

Proof As above, $\overline{x}$ is an unconstrained ε-minimum point for the functional $z : X \to \mathbb{R}$ defined by $z(x) = (\Delta_{-K-\varepsilon e} \circ f)(x)$. Once again, from the Ekeland variational principle we get an element $x^1 \in B(\overline{x}, \sqrt{\varepsilon})$ which is a minimum point on X for the perturbed function $z(\cdot) + \sqrt{\varepsilon}\|\cdot - x^1\|$. Whence, applying the fuzzy calculus rules of the proximal subdifferential, we can find $x^2 \in B(x^1, 3^{-1}\sqrt{\varepsilon})$, $x^3 \in B(x^1, 3^{-1}\sqrt{\varepsilon})$ s.t.

$$
0 \in \partial_P z(x^2) + \partial_P(\sqrt{\varepsilon}\|\cdot - x^1\|)(x^3) + 3^{-1}\sqrt{\varepsilon} U_{X^*}.
$$

Since z is a composite function, we can apply the fuzzy calculus for its subdifferential to get $x^4 \in B(x^2, 3^{-1}\sqrt{\varepsilon})$ and $y^* \in \partial_P \Delta_{-K-\varepsilon e}(f(x^4) - f(\overline{x}))$ with

$$0 \in \partial_P(y^* \circ f)(x^4) + \partial_P(\sqrt{\varepsilon}\| \cdot -x^1\|)(x^3) + \frac{2}{3}\sqrt{\varepsilon}U_{X^*}.$$

Since $\sqrt{\varepsilon}\| \cdot -x^1\|$ is a convex function, the proximal subdifferential of this function coincides with the usual subdifferential of a convex function. Further, we also know that

$$\partial(\sqrt{\varepsilon}\| \cdot -x^1\|)(x^3) \subset \sqrt{\varepsilon}U_{X^*},$$

where ∂ denotes the subdifferential of a convex function. Hence, we conclude that

$$0 \in \partial_P(y^* \circ f)(x^4) + \frac{5}{3}\sqrt{\varepsilon}U_{X^*}.$$

Of course, taking into account the above estimations and the fact that $f(x^4) - f(\overline{x}) \notin -K - \varepsilon e$, one has with (1) and Remark 2.2

$$\|x^4 - \overline{x}\| \le \frac{2}{3}\sqrt{\varepsilon} \quad \text{and} \quad y^* \in S_{Y^*} \cap K^*.$$

The proof is complete taking $x = x^4$. □

In order to complete our theoretical tour let us finally consider the set-valued case. In the sequel of this section we denote by F a set valued map acting between Banach spaces X and Y. As usual, we make the common assumption that the graph of F, denoted by $\operatorname{Gr} F = \{(x, y) \mid y \in F(x)\}$, is closed. We also recall the very useful concept of the Mordukhovich coderivative associated to F at a point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ as the set-valued map $D_M^* F(\overline{x}, \overline{y}) : Y^* \rightrightarrows X^*$ given by

$$D_M^* F(\overline{x}, \overline{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N_M(\operatorname{Gr} F, (\overline{x}, \overline{y}))\}.$$

We work with the following solution concept: Let $\varepsilon > 0$ and $e \in K$, $\|e\| = 1$. A point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is called an (ε, e)-Pareto minimum of F with respect to K if $\overline{y} \in (\varepsilon, e) - \operatorname{Min}(F(X) \mid K)$. Using the above notation, we have the following result.

Theorem 3.4 *Let X, Y be Asplund spaces, let K be a closed convex pointed cone in Y with empty interior, $\varepsilon > 0$ and $e \in K$, $\|e\| = 1$. If $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is an (ε, e)-Pareto minimum of F with respect to K, then there exist $(x, y) \in B((\overline{x}, \overline{y}), \sqrt{\varepsilon})$ and $y^* \in S_{Y^*} \cap K^*$ s.t.*

$$0 \in D_M^* F(x, y)(y^* + \sqrt{\varepsilon}U_Y^*) + \sqrt{\varepsilon}U_X^*.$$

Proof Let us consider the function $g : X \times Y \to \mathbb{R}$ given by $g(x, y) = \Delta_{-K-\varepsilon e}(y - \overline{y})$. Then, $(\overline{x}, \overline{y})$ is an ε-minimum point over $\operatorname{Gr} F$ for the functional g. Using the same technique as in Theorem 3.2, we get a pair of elements $(x, y) \in B((\overline{x}, \overline{y}), \sqrt{\varepsilon}) \cap \operatorname{Gr} F$ which is a minimum point on $\operatorname{Gr} F$ for the perturbed function

$g(\cdot,\cdot)+\sqrt{\varepsilon}\|(\cdot,\cdot)-(x,y)\|$. We apply again the exact calculus rules of the Mordukhovich subdifferential and we can write

$$\begin{aligned}(0,0) &\in \partial_M(g(\cdot,\cdot)+\sqrt{\varepsilon}\|(\cdot,\cdot)-(x,y)\|+I_{\operatorname{Gr}F}(\cdot,\cdot))(x,y)\\ &\subset \{0\}\times\partial_M\Delta_{-K-\varepsilon e}(y-\overline{y})+\sqrt{\varepsilon}(U_{X^*}\times U_{Y^*})+\partial_M I_{\operatorname{Gr}F}(x,y).\end{aligned}$$

Using the definition of the coderivative and invoking the same arguments as in the proof of Theorem 3.2, we obtain the conclusion. □

4 Applications

In this section we try to apply the concepts and results to particular vector optimization problems where we consider the situation that the natural ordering cone of the range space has an empty interior.

A first application of our previous results concerns an *abstract generalized equilibrium problem* given by means of a (bi)function $f: X\times X\to Y$ from a Banach space X into another Banach space Y ordered by a closed convex cone K. If $T\subset X$ is a nonempty set and $f(x,x)=0$ for every $x\in T$, the problem $(\mathcal{P})$ is to find an element $\overline{x}\in T$ s.t.

$$f(\overline{x},y)\notin -K\setminus\{0\},\quad \forall y\in T.$$

In the case when $\operatorname{int}K$ is nonempty, one can consider the weak form $(\mathcal{PW})$ of this problem: Find an element $\overline{x}\in T$ s.t.

$$f(\overline{x},y)\notin -\operatorname{int}K,\quad \forall y\in T.$$

For this latter problem there exists a large literature concerning existence conditions for its solutions [3, Sect. 4.2 and references therein]. In contrast, for the former form of the problem, to our knowledge, there do not exist many results. On the other hand, in the settings of this paper, one cannot speak about the weak form. However, it is clear that $\overline{x}$ is a solution of the weak problem if and only if 0 is a weak minimum of the set $f(\overline{x},T)$. This observation gives us the idea of defining the concept of (ε,e)-solution for the general problem (we kept the above notations for e and ε). So, we shall say that $\overline{x}$ is an (ε,e)-solution of $(\mathcal{P})$ if 0 is an (ε,e)-Pareto minimum of $f(\overline{x},T)$, i.e.,

$$f(\overline{x},y)\notin -K-\varepsilon e,\quad \forall y\in T.$$

The next result shows how the equilibrium problem $(\mathcal{P})$ fits into our context.

Theorem 4.1 *Let X,Y be Asplund spaces, let K be a closed convex pointed cone in Y with empty interior, let T be a closed subset of X and let $f: X\times X\to Y$ be a strictly Lipschitzian function at $\overline{x}$ in the second variable. Assume $\varepsilon>0$ and $e\in K$, $\|e\|=1$. If $\overline{x}$ is an (ε,e)-Pareto minimizer of $(\mathcal{P})$, then there exist $u_\varepsilon\in B(\overline{x},\sqrt{\varepsilon})$ and $y^*\in S_{Y^*}\cap K^*$ s.t.*

$$0\in\partial_M(y^*\circ f(\overline{x},\cdot))(u_\varepsilon)+\sqrt{\varepsilon}U_{X^*}+N_{\partial_M}(T,u_\varepsilon).$$

Proof Let us consider the function $g : X \to Y$, given by $g(z) = f(\overline{x}, z)$ for every $z \in X$. It is obvious that $\overline{x}$ is an (ε, e)-solution of $(\mathcal{P})$ if and only if $\overline{x}$ is an (ε, e)-Pareto minimum for g over T (take into account that $f(\overline{x}, \overline{x}) = 0$). Now, it is enough to apply Theorem 3.2 to get the conclusion. □

As a second example of studying approximate Pareto optimality under the condition when the range space of the objective does not have an ordering cone with non-empty interior, we consider the *vector control approximation problem* as given in [3]. In order to formulate the control approximation problem, one needs to consider what is known as a vector-valued norm. Consider the Banach space Z and consider K to be a closed convex and pointed cone in a Banach space Y.

The function $|||\cdot||| : Z \to K$ is called a vector-valued norm if, for all z_1 and z_2 in Z and $\lambda \in \mathbb{R}$, we have:

(i) $|||z||| = 0$ if and only if $z = 0$.
(ii) $|||\lambda z||| = |\lambda| |||z|||$.
(iii) $|||z_1 + z_2||| \in |||z_1||| + |||z_2||| - K$.

It is important to note that the vector norm defined above is a K-convex function. We shall now introduce the vector control approximation problem as stated in [3]: Consider the vector valued function $f : X \to Y$ defined by

$$f(x) = f_1(x) + \sum_{i=1}^{n} \alpha_i |||A_i(x) - a^i|||,$$

where X is a Banach space and Y is already given above to be a Banach space. In the above expression $f_1 : X \to Y$ is locally Lipschitz and each $A_i : X \to Z$ is a linear map with $\alpha_i \geq 0$ for all $i = 1, \dots, n$ and $a^i \in Z$ for each $i = 1, \dots, n$. In [3], the function f_1 was considered to be a linear map between X and Y, but in general there is no such need. The vector control approximation problem as stated in [3] is to find the set of efficient solutions of the set $f(X)$ with respect to the ordering cone K. Since in our setting the ordering cone K has an empty interior, it is more convenient to compute the set $(\varepsilon, e) - \mathrm{Min}(f(X) \mid K)$. Thus, we will now characterize the (ε, e)-minimum points with the help of the results derived in Sect. 3. In the sequel, we will consider a more simplified situation in which all the spaces X, Y, Z are Hilbert spaces. Borrowing the notation of the subdifferential of a K-convex function from [3] we mention beforehand that the subdifferential of the vector norm at a point $z \in Z$ is given as $\partial^{\leq} |||z|||$. Further, we denote by A_i^* the adjoint operator associated the linear mappings A_i.

Now, we have the following necessary optimality condition.

Theorem 4.2 *Consider the vector control approximation problem as given above. Let $\varepsilon > 0$ and $e \in K$, $\|e\| = 1$. Assume that all the associated spaces are Hilbert spaces and that the closed convex pointed ordering cone K has an empty interior. Further, suppose that the function f_1 is locally Lipschitz and the vector norm is also locally Lipschitz. Let us consider $\bar{x} \in X$ to be an (ε, e)-Pareto minimizer of f over*

X with respect to K for the vector control approximation problem. Then, there exist $x_1, x_2 \in B(\bar{x}, \sqrt{\varepsilon})$ *and* $y^* \in S_{Y^*} \cap K^*$ *s.t.*

$$0 \in \partial_P(y^* \circ f_1)(x_1) + \left(\sum_{i=1}^{n} \alpha_i \partial(y^*(|||A_i - a^i|||))(x_2)\right) + 2\sqrt{\varepsilon} U_{X^*}. \tag{7}$$

Proof First of all, for the given $\varepsilon > 0$, by using Theorem 3.3 we conclude that there exist $x \in B(\bar{x}, \frac{2}{3}\sqrt{\varepsilon})$ and $y^* \in S_{Y^*} \cap K^*$ such that

$$0 \in \partial_P(y^* \circ f)(x) + \frac{5}{3}\sqrt{\varepsilon} U_{X^*}.$$

Now, by applying the fuzzy sum rule, we can show that there exist $x_1, x_2 \in B(x, \frac{1}{3}\sqrt{\varepsilon})$ such that

$$0 \in \partial_P(y^* \circ f_1)(x_1) + \partial_P\left(\sum_{i=1}^{n} \alpha_i y^*(|||A_i - a^i|||)(x_2)\right) + \frac{5}{3}\sqrt{\varepsilon} U_{X^*} + \frac{1}{3}\sqrt{\varepsilon} U_{X^*}.$$

Noting that the function $(\sum_{i=1}^{n} \alpha_i y^*(|||A_i - a^i|||))$ is a real-valued convex function and that

$$\frac{5}{3}\sqrt{\varepsilon} U_{X^*} + \frac{1}{3}\sqrt{\varepsilon} U_{X^*} = 2\sqrt{\varepsilon} U_{X^*},$$

we can write the above expression as

$$0 \in \partial_P(y^* \circ f_1)(x_1) + \left(\sum_{i=1}^{n} \alpha_i \partial(y^*(|||A_i - a^i|||))(x_2)\right) + 2\sqrt{\varepsilon} U_{X^*},$$

where ∂ represents the subdifferential of a real-valued convex function. Further, it is simple to observe that $x_1, x_2 \in B(\bar{x}, \sqrt{\varepsilon})$. This completes the proof. □

Further, if the cone K has a weakly compact base, then by using a result of Valadier (see for example Theorem 2.4.8 in [3]) the expression (7) can be reduced to

$$0 \in \partial_P(y^* \circ f_1)(x_1) + \sum_{i=1}^{n} \alpha_i A_i^* y^* \partial^{\leq} |||\cdot|||(A_i(x_2) - a^i) + 2\sqrt{\varepsilon} U_{X^*}.$$

Our third example refers to a *mixed fractional programming problem*. Let us consider again two Hilbert spaces X and Y as above, let $f : X \to Y$ be a vector-valued function and let $g : X \to (0, +\infty)$ be a real-valued function with positive values. The objective function of the fractional program that we envisage is given as $h : X \to Y$,

$$h(x) := g^{-1}(x) f(x).$$

We have the next result.

Theorem 4.3 *Assume that X, Y are Hilbert spaces, K is a closed convex pointed cone in Y with empty interior and $h : X \to Y$, $h(x) := g^{-1}(x)f(x)$. Let $\varepsilon > 0$ and $e \in K$, $\|e\| = 1$ and let $\bar{x} \in X$ be an (ε, e)-Pareto minimizer of h over X with respect to K. If the objective function h is locally Lipschitz, then there exist $x_1, x_2 \in B(\overline{x}, \sqrt{\varepsilon})$ and $y^* \in S_{Y^*} \cap K^*$ such that*

$$0 \in \partial_P(y^* \circ f)(x_1) + \partial_P(y^* \circ (\varepsilon e - (g^{-1}(\overline{x})f(\overline{x}))g(\cdot)))(x_2) + 2\sqrt{\varepsilon}U_{X^*}. \qquad (8)$$

Proof First, we observe that, if $\bar{x} \in X$ is an (ε, e)-Pareto minimum for the function h, then it is an $(g(\overline{x})\varepsilon, e)$-Pareto minimum for the function $\varphi : X \to Y$, $\varphi(x) = f(x) - (g^{-1}(\overline{x})f(\overline{x}) - \varepsilon e)g(x)$. Now, we can use Theorem 3.3 to obtain that there exist $x \in B(\bar{x}, \frac{2}{3}\sqrt{\varepsilon})$ and $y^* \in S_{Y^*} \cap K^*$ such that

$$0 \in \partial_P(y^* \circ \varphi)(x) + \frac{5}{3}\sqrt{\varepsilon}U_{X^*}.$$

Now, by applying the fuzzy sum rule, we can show that there exist $x_1, x_2 \in B(x, \frac{1}{3}\sqrt{\varepsilon})$ and hence $x_1, x_2 \in B(\bar{x}, \sqrt{\varepsilon})$ such that

$$0 \in \partial_P(y^* \circ f)(x_1) + \partial_P(y^* \circ (\varepsilon e - (g^{-1}(\overline{x})f(\overline{x}))g(\cdot)))(x_2) + 2\sqrt{\varepsilon}U_{X^*}$$

and this is the conclusion. □

Note that, if in the above theorem one has $y^*(\varepsilon e - g^{-1}(\overline{x})f(\overline{x})) > 0$, then the conclusion can be written as [18, Exercise 2.10]

$$0 \in \partial_P(y^* \circ f)(x_1) + y^*(\varepsilon e - g^{-1}(\overline{x})f(\overline{x}))\partial_P g(x_2) + 2\sqrt{\varepsilon}U_{X^*}.$$

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References

1. Hiriart-Urruty, J.-B.: Tangent cones, generalized gradients and mathematical programming in Banach spaces. Math. Oper. Res. **4**, 79–97 (1979)
2. Luc, D.T.: Theory of Vector Optimization. Lecture Notes in Economics and Mathematical Systems, vol. 319. Springer, Berlin (1989)
3. Göpfert, A., Riahi, H., Tammer, C., Zălinescu, C.: Variational Methods in Partially Ordered Spaces. Springer, New York (2003)
4. Jahn, J.: Vector Optimization: Theory, Applications and Extensions. Springer, Berlin (2004)
5. Dutta, J., Tammer, C.: Lagrangian conditions for vector optimization in Banach spaces. Math. Methods Oper. Res. **64**, 521–541 (2006)
6. Durea, M., Tammer, C.: Fuzzy necessary optimality conditions for vector optimization problems. Reports of the Institute for Mathematics, Martin-Luther-University Halle-Wittenberg, Report 08. Optimization **58**, 449–467 (2009)
7. Gerth (Tammer), C., Weidner, P.: Nonconvex separation theorems and some applications in vector optimization. J. Optim. Theory Appl. **67**, 297–320 (1990)
8. Amahroq, T., Taa, A.: On Lagrange-Kuhn-Tucker multipliers for multiobjective optimization problems. Optimization **41**, 159–172 (1997)
9. Taa, A.: ε-subdifferentials of set-valued maps and ε-weak Pareto optimality for multiobjective optimization. Math. Methods Oper. Res. **62**, 187–209 (2005)

10. Zaffaroni, A.: Degrees of efficiency and degrees of minimality. SIAM J. Control Optim. **42**, 1071–1086 (2003)
11. Bao, T.Q., Mordukhovich, B.S.: Variational principles for set-valued mappings with applications to multiobjective optimization. Control Cybern. **36**, 1–33 (2007)
12. Peressini, A.L.: Ordered Topological Vector Spaces. Harper and Row, New York (1967)
13. Rockafellar, R.T., Uryasev, S., Zabarankin, M.: Deviation measures in risk analysis and optimization. Preprint 7, University of Florida (2002)
14. Heyde, F.: Coherent risk measures and vector optimization. In: Multicriteria Decision Making and Fuzzy Systems, Theory, Methods and Applications, pp. 3–12. Shaker, Aachen (2006)
15. Burke, J.V., Ferris, M.C., Qian, M.: On the Clarke subdifferential of the distance function of a closed set. J. Math. Anal. Appl. **166**, 199–213 (1992)
16. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, Vol. I: Basic Theory. Grundlehren der mathematischen Wissenschaften (A Series of Comprehensive Studies in Mathematics), vol. 330. Springer, Berlin (2006)
17. Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, Vol. II: Applications. Grundlehren der mathematischen Wissenschaften (A Series of Comprehensive Studies in Mathematics), vol. 331. Springer, Berlin (2006)
18. Clarke, F.H., Ledyaev, Y.S., Stern, R.J., Wolenski, P.R.: Nonsmooth Analysis and Control Theory. Springer, New York (1998)
19. Zălinescu, C.: Convex Analysis in General Vector Spaces. World Scientific, Singapore (2003)