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Generalized Minty Vector Variational-Like Inequalities and Vector Optimization Problems

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Abstract In this paper, we study the relationship among the generalized Minty vector variational-like inequality problem, generalized Stampacchia vector variational-like inequality problem and vector optimization problem for nondifferentiable and nonconvex functions. We also consider the weak formulations of the generalized Minty vector variational-like inequality problem and generalized Stampacchia vector variational-like inequality problem and give some relationships between the solutions of these problems and a weak efficient solution of the vector optimization problem.

Keywords Generalized vector variational-like inequalities · Vector optimization problems · Invex functions · Generalized subdifferential

1 Introduction and Formulations

In 1980, Giannessi [1] extended the classical (Stampacchia) variational inequality for vector-valued functions, called vector (Stampacchia) variational inequality, with further applications to alternative theorems. Since then, (Stampacchia) vector variational inequalities (SVVI) and their generalizations have been used as tools to solve vector optimization problems (VOP); See, for example, [2–13] and references therein. For the details on vector variational inequalities (VVI) and their generalizations, we refer

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to [5, 6] and references therein. In most of the papers appeared in the literature on different aspects, namely existence of solutions, applications, etc, of SVVI, the Minty vector variational inequalities are used as via media. Several authors studied VOP by using Stampacchia vector variational-like inequalities. Recently, Ruiz-Garzón et al. [11] established some relationships between Stampacchia vector variational-like inequality and VOP. They studied the weak efficient solution of VOP for differentiable but pseudoinvex functions by using Stampacchia vector variational-like inequality problem. Mishra and Wang [14] established relationships between Stampacchia vector variational-like inequality problems and nonsmooth vector optimization problems under nonsmooth invexity. In 1998, Giannessi [4] first gave a direct application of Minty VVI to establish the necessary and sufficient conditions for a point to be a solution of VOP for differentiable and convex functions are that the point to be a solution of Minty VVI. Later, Yang et al. [12] extended the results of Giannessi [4] for differentiable but pseudoconvex functions. Very recently, Yang and Yang [15] gave some relationships between Minty (VVLIP) and (VOP) for differentiable but pseudoinvex vector-valued functions. In particular, they extended the results of Giannessi [4] and Yang et al. [12] for differentiable but pseudoinvex vector-valued functions. They provided the necessary and sufficient conditions for a point to be a solution of VOP for differentiable but pseudoinvex functions, is that, the point be a solution of a Minty vector variational-like inequality problem. They also considered Stampacchia VVI and proved its equivalence with Minty VVI under continuity assumption.

In this paper, we consider generalized Minty vector variational-like inequality problems, generalized Stampacchia vector variational-like inequality problem and nonsmooth vector optimization problem under nonsmooth invexity. We study the relationship among these problems under nonsmooth invexity. We also consider the weak formulations of generalized Minty vector variational-like inequality problem and generalized Stampacchia vector variational-like inequality problem and give some relationships between the solutions of these problems and a weak efficient solution of vector optimization problem.

2 Preliminaries

In this section, we recall some known definitions and results which will be used in the sequel.

Throughout the paper, unless otherwise specified, we assume that K is a nonempty subset of $\mathbb{R}^n$ and $\eta : K \times K \to \mathbb{R}^n$ is a given map. The interior of K is denoted by $\operatorname{int} K$.

Let $f = (f_1, \ldots, f_\ell) : \mathbb{R}^n \to \mathbb{R}^\ell$ be a vector-valued function. We consider the following *vector optimization problem*:

(VOP) Minimize $f(x) = (f_1(x), \ldots, f_\ell(x))$ subject to $x \in K$.

A point $\bar{x} \in K$ is said to be an *efficient* (or *Pareto*) *solution* (respectively, *weak efficient solution*) of (VOP) if

$$\begin{aligned} f(y) - f(\bar{x}) &= \big(f_1(y) - f_1(\bar{x}), \ldots, f_\ell(y) - f_\ell(\bar{x})\big) \\ &\notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K \end{aligned}$$

(respectively, $f(y) - f(\bar{x}) = (f_1(y) - f_1(\bar{x}), \dots, f_\ell(y) - f_\ell(\bar{x})) \notin -\text{int}\, \mathbb{R}^\ell_+$, for all $y \in K$), where $\mathbb{R}^\ell_+$ is the nonnegative orthant of $\mathbb{R}^\ell$ and $\mathbf{0}$ is the origin of the nonnegative orthant, namely $\mathbb{R}^\ell_+$.

It is clear that every efficient solution is a weak efficient solution.

Let $g : K \to \mathbb{R}$ be a function and let $\eta : K \times K \to \mathbb{R}^n$ be a given map. The following concepts are taken from [16].

Definition 2.1 Let g be locally Lipschitz at a given point $x \in K$. The *Clarke's generalized directional derivative* of g at $x \in K$ in the direction of a vector $v \in K$, denoted by $g^\circ(x; v)$, is defined by

$$g^\circ(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{g(y + tv) - g(y)}{t}.$$

Definition 2.2 Let g be locally Lipschitz at a given point $x \in K$. The *Clarke's generalized subdifferential* of g at $x \in K$, denoted by $\partial^c g(x)$, is defined by

$$\partial^c g(x) = \{\xi \in \mathbb{R}^n : g^\circ(x; v) \geq \langle \xi, v \rangle, \quad \text{for all } v \in \mathbb{R}^n\},$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$.

Definition 2.3 See [14] and [17]. Let g be locally Lipschitz at a given point $x \in K$. Then, g is said to be *invex* wrt η on K if

$$\langle \xi, \eta(y, x) \rangle \leq g(y) - g(x), \quad \text{for all } x, y \in K \text{ and all } \xi \in \partial^c g(x).$$

Definition 2.4 Let x be any arbitrary point of K. The set K is said to be *invex at* x wrt η if, for all $y \in K$,

$$x + \lambda \eta(y, x) \in K, \quad \text{for all } \lambda \in [0, 1].$$

K is said to be an *invex set* wrt η if K is invex at every point $x \in K$ wrt η.

Definition 2.5 See [18]. Let $K \subseteq \mathbb{R}^n$ be an invex set wrt η and let $x, y \in K$ be any arbitrary points in K. A set P_{xz} is said to be a *closed* (respectively, *open*) η*-path* joining the points x and $z = x + \eta(y, x)$ (contained in K) if

$$P_{xz} = \{u = x + \lambda \eta(y, x) : \lambda \in [0, 1]\}$$

(respectively, $P^0_{xz} = \{u = x + \lambda \eta(y, x) : \lambda \in (0, 1)\}$).

Definition 2.6 See [19]. Let $K \subseteq \mathbb{R}^n$ be an invex set wrt η. A function $g : K \to \mathbb{R}$ is said to be *preinvex* wrt η if

$$g(x + \lambda \eta(y, x)) \leq \lambda g(y) + (1 - \lambda) g(x), \quad \text{for all } x, y \in K \text{ and for all } \lambda \in [0, 1].$$

We say that the map η is *skew* if, for all $x, y \in K$,

$$\eta(y, x) + \eta(x, y) = 0.$$

Condition A See [20]. Let $K \subseteq \mathbb{R}^n$ be an invex set wrt η and let $g : K \to \mathbb{R}$ be a function. Then,

$$g(y + \eta(x, y)) \leq g(x), \quad \text{for all } x, y \in K.$$

Condition C Let $K \subseteq \mathbb{R}^n$ be an invex set wrt $\eta : K \times K \to \mathbb{R}^n$. Then, for all $x, y \in K, \lambda_1, \lambda_2, \lambda \in [0, 1]$,

(a) $\eta(x, x + \lambda\eta(y, x)) = -\lambda\eta(y, x)$,
(b) $\eta(y, x + \lambda\eta(y, x)) = (1 - \lambda)\eta(y, x)$,
(c) $\eta(x + \lambda_1\eta(y, x), x + \lambda_2\eta(y, x)) = (\lambda_1 - \lambda_2)\eta(y, x)$.

Obviously, the map $\eta(y, x) = y - x$ satisfies Condition C. Examples of the map η that satisfies Condition C (a) and C (b) are given in [20] and [21]. Now, we give an example of a map η which satisfies Condition C (c).

Example 2.1 Let $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a map defined as

$$\eta(x, y) = \begin{cases} x - y, & x \geq 0, \ y \geq 0, \\ x - y, & x \leq 0, \ y \leq 0, \\ y - x, & x \geq 0, \ y \leq 0, \\ y - x, & x \leq 0, \ y \geq 0. \end{cases}$$

Then, η satisfies Condition C (c). Not only this, η also satisfies Condition C (a).

Lemma 2.1 *See* [17]. *Let $K \subseteq \mathbb{R}^n$ be an invex set wrt $\eta : K \times K \to \mathbb{R}^n$ and let $g : K \to \mathbb{R}$ be locally Lipschitz on K. If g is invex wrt η, then $\partial^c g$ is η-monotone, that is, for all $x, y \in K$,*

$$\langle \xi, \eta(y, x)\rangle + \langle \zeta, \eta(x, y)\rangle \leq 0, \quad \textit{for all } \xi \in \partial^c g(x) \textit{ and } \zeta \in \partial^c g(y).$$

Lemma 2.2 *See* [17]. *Let $K \subseteq \mathbb{R}^n$ be an invex set wrt $\eta : K \times K \to \mathbb{R}^n$ and let $g : K \to \mathbb{R}$ be locally Lipschitz on K such that g and η satisfy Condition* A *and Condition* C (a) *and* C (b), *respectively. If g is invex wrt η on K, then it is preinvex wrt the same η on K.*

We shall use the following mean-value theorem for invex functions to prove one of the main results of this paper.

Theorem 2.1 *See* [18] *and* [22]. *Let $K \subseteq \mathbb{R}^n$ be an invex set wrt $\eta : K \times K \to \mathbb{R}^n$, let $x, z \in K$ be any arbitrary points in K and let P_{xz} be an arbitrary η-path contained in* int K. *Let $g : K \to \mathbb{R}$ be locally Lipschitz on K. Then, for any $y = x + \eta(z, x) \in K$, there exist $w \in P^0_{xy}$ and $\xi \in \partial^c g(w)$ such that*

$$\langle \xi, \eta(z, x)\rangle = g(y) - g(x).$$

In other words, for any $y = x + \eta(z, x) \in K$, there exist $\hat{\lambda} \in (0, 1)$, $\hat{w} := x + \hat{\lambda}\eta(z, x)$ and $\xi \in \partial^c g(w)$ such that

$$\langle \xi, \eta(z, x)\rangle = g(y) - g(x),$$

that is,

$$\langle \xi, \eta(z, x)\rangle = g(x + \eta(z, x)) - g(x).$$

3 Generalized Minty Vector Variational-Like Inequalities

Let K be a nonempty subset of $\mathbb{R}^n$ and let $\eta : K \times K \to \mathbb{R}^n$ be a given map. Let $f = (f_1, \dots, f_\ell) : \mathbb{R}^n \to \mathbb{R}^\ell$ be a vector-valued function. We consider the following *generalized Minty vector variational-like inequality problem*:

(GMVVLIP) Find $\bar{x} \in K$ such that, for all $y \in K$ and all $\xi_i \in \partial^c f_i(y)$, $i \in \mathscr{I} = \{1, \dots, \ell\}$,

$$\langle \xi, \eta(y, \bar{x})\rangle_\ell = \big(\langle \xi_1, \eta(y, \bar{x})\rangle, \dots, \langle \xi_\ell, \eta(y, \bar{x})\rangle\big) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}.$$

When $\eta(y, x) = y - x$, then (GMVVLIP) reduces to the generalized Minty vector variational inequality problem considered and studied in [8].

We also consider the following *generalized Stampacchia vector variational-like inequality problem*:

(GSVVLIP) Find $\bar{x} \in K$ such that, for all $y \in K$, there exists $\zeta_i \in \partial^c f_i(\bar{x})$, $i \in \mathscr{I} = \{1, \dots, \ell\}$ such that

$$\langle \zeta, \eta(y, \bar{x})\rangle_\ell = \big(\langle \zeta_1, \eta(y, \bar{x})\rangle, \dots, \langle \zeta_\ell, \eta(y, \bar{x})\rangle\big) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}.$$

We present necessary and sufficient conditions for an efficient solution of (VOP).

Theorem 3.1 *Let $K \subseteq \mathbb{R}^n$ be invex wrt $\eta : K \times K \to \mathbb{R}^n$ such that any η-path is contained in* int K *and η is skew and satisfies Condition C. For each $i \in \mathscr{I} = \{1, \dots, \ell\}$, let $f_i : K \to \mathbb{R}$ be locally Lipschitz, invex wrt η on K and satisfy Condition A. Then, $\bar{x} \in K$ is an efficient solution of* (*VOP*) *if and only if it is a solution of* (*GMVVLIP*).

Proof Let $\bar{x} \in K$ be a solution of (GMVVLIP) but not an efficient solution of (VOP). Then, there exists $z \in K$ such that

$$\big(f_1(z) - f_1(\bar{x}), \dots, f_\ell(z) - f_\ell(\bar{x})\big) \in -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}. \quad (1)$$

Set $z(\lambda) := \bar{x} + \lambda\eta(z, \bar{x})$, for all $\lambda \in [0, 1]$. Since K is invex, $z(\lambda) \in K$, for all $\lambda \in [0, 1]$. By Lemma 2.2, each f_i is preinvex wrt η and, therefore,

$$f_i(z(\lambda)) = f_i(\bar{x} + \lambda\eta(z, \bar{x})) \le \lambda f_i(z) + (1 - \lambda) f_i(\bar{x}), \quad \text{for each } i = 1, 2, \dots, \ell,$$

that is,

$$f_i(\bar{x} + \lambda\eta(z, \bar{x})) - f_i(\bar{x}) \le \lambda[f_i(z) - f_i(\bar{x})],$$

for all $\lambda \in [0, 1]$ and for each $i = 1, \dots, \ell$. In particular, for $\lambda = 1$, we have

$$f_i(\bar{x} + \eta(z, \bar{x})) - f_i(\bar{x}) \le f_i(z) - f_i(\bar{x}), \quad \text{for each } i = 1, \dots, \ell. \quad (2)$$

By the mean-value Theorem 2.1, there exist $\lambda_i \in (0,1)$ and $\xi_i \in \partial^c f_i(z(\lambda_i))$, where $z(\lambda_i) := \bar{x} + \lambda_i \eta(z,\bar{x})$, such that

$$\langle \xi_i, \eta(z,\bar{x})\rangle = f_i(\bar{x} + \eta(z,\bar{x})) - f_i(\bar{x}), \quad \text{for each } i = 1,\dots,\ell. \tag{3}$$

By combining (2)–(3), we obtain

$$\langle \xi_i, \eta(z,\bar{x})\rangle \le f_i(z) - f_i(\bar{x}), \quad \text{for each } i = 1,\dots,\ell. \tag{4}$$

Suppose that $\lambda_1,\dots,\lambda_\ell$ are all equal. Then, by Condition C (a), for all $i = 1,\dots,\ell$, we have

$$\begin{aligned}\langle \xi_i, \eta(\bar{x}, z(\lambda_i))\rangle &= -\lambda_i \langle \xi_i, \eta(z,\bar{x})\rangle \\ &\ge -\lambda_i [f_i(z) - f_i(\bar{x})] \quad \text{by (4)} \\ &= \lambda_i [f_i(\bar{x}) - f_i(z)].\end{aligned}$$

Since η is skew, we have

$$\langle \xi_i, \eta(z(\lambda_i),\bar{x})\rangle \le \lambda_i [f_i(z) - f_i(\bar{x})], \quad \text{for all } \lambda_i \in (0,1) \text{ and for each } i \in \mathscr{I}. \tag{5}$$

By combining (1) and (5), it follows that $\bar{x}$ is not a solution of (GMVVLIP). This contradicts the fact the $\bar{x}$ is a solution of (GMVVLIP).

Consider the case when $\lambda_1, \lambda_2, \dots, \lambda_\ell$ are not all equal. Let $\lambda_1 \ne \lambda_2$. Then, from (4), we have

$$\langle \xi_1, \eta(z,\bar{x})\rangle \le f_1(z) - f_1(\bar{x}) \tag{6}$$

and

$$\langle \xi_2, \eta(z,\bar{x})\rangle \le f_2(z) - f_2(\bar{x}). \tag{7}$$

Since f_i and f_2 are invex wrt η, by Lemma 2.1 $\partial^c f_1$ and $\partial^c f_2$ are η-monotone, that is,

$$\langle \xi_1 - \xi_2^*, \eta(z(\lambda_1), z(\lambda_2))\rangle \ge 0, \quad \text{for all } \xi_2^* \in \partial^c f_1(z(\lambda_2)) \tag{8}$$

and

$$\langle \xi_1^* - \xi_2, \eta(z(\lambda_1), z(\lambda_2))\rangle \ge 0, \quad \text{for all } \xi_1^* \in \partial^c f_2(z(\lambda_1)). \tag{9}$$

If $\lambda_1 > \lambda_2$, then by using Condition C (c) and (8), we obtain

$$0 \le \langle \xi_1 - \xi_2^*, \eta(z(\lambda_1), z(\lambda_2))\rangle = (\lambda_1 - \lambda_2)\langle \xi_1 - \xi_2^*, \eta(z,\bar{x})\rangle$$

and so

$$\langle \xi_1 - \xi_2^*, \eta(z,\bar{x})\rangle \ge 0 \;\Leftrightarrow\; \langle \xi_1, \eta(z,\bar{x})\rangle \ge \langle \xi_2^*, \eta(z,\bar{x})\rangle.$$

From (6), we have

$$\langle \xi_2^*, \eta(z,\bar{x})\rangle \le f_1(z) - f_1(\bar{x}), \quad \text{for any } \xi_2^* \in \partial^c f_1(z(\lambda_2)).$$

If $\lambda_1 < \lambda_2$, then by using Condition C (c) and (9), we have

$$0 \le \langle \xi_1^* - \xi_2, \eta(z(\lambda_1), z(\lambda_2))\rangle = (\lambda_1 - \lambda_2)\langle \xi_1^* - \xi_2, \eta(z, \bar{x})\rangle$$

and so

$$\langle \xi_1^* - \xi_2, \eta(z, \bar{x})\rangle \le 0 \ \Leftrightarrow \ \langle \xi_1^*, \eta(z, \bar{x})\rangle \le \langle \xi_2, \eta(z, \bar{x})\rangle.$$

From (7), we obtain

$$\langle \xi_1^*, \eta(z, \bar{x})\rangle \le f_2(z) - f_2(\bar{x}), \quad \text{for any } \xi_1^* \in \partial^c f_2(z(\lambda_1)).$$

Therefore, for the case $\lambda_1 \neq \lambda_2$, let $\bar{\lambda} = \min\{\lambda_1, \lambda_2\}$. Then, we can find $\bar{\xi}_i \in \partial^c f_i(z(\bar{\lambda}))$ such that

$$\langle \bar{\xi}_i, \eta(z, \bar{x})\rangle \le f_i(z) - f_i(\bar{x}), \quad \text{for any } i = 1, 2.$$

By continuing this process, we can find $\lambda^* \in (0, 1)$ and $\xi_i^* \in \partial^c f_i(z(\lambda^*))$ such that $\lambda^* = \min\{\lambda_1, \lambda_2, \ldots, \lambda_\ell\}$ and

$$\langle \xi_i^*, \eta(z, \bar{x})\rangle \le f_i(z) - f_i(\bar{x}), \quad \text{for each } i = 1, 2, \ldots, \ell. \tag{10}$$

From (1) and (10), we have $\xi_i^* \in \partial^c f_i(z(\lambda^*))$, $i = 1, 2, \ldots, \ell$, and

$$\big(\langle \xi_1^*, \eta(z, \bar{x})\rangle, \ldots, \langle \xi_\ell^*, \eta(z, \bar{x})\rangle\big) \in -\mathbb{R}_+^\ell \setminus \{\mathbf{0}\}.$$

By multiplying the above inclusion by $-\lambda^*$ and using Condition C (a) and the skewness of η, we obtain

$$\big(\langle \xi_1^*, \eta(z(\lambda^*), \bar{x})\rangle, \ldots, \langle \xi_\ell^*, \eta(z(\lambda^*), \bar{x})\rangle\big) \in -\mathbb{R}_+^\ell \setminus \{\mathbf{0}\}.$$

which contradicts our supposition that $\bar{x}$ is a solution of (GMVVLIP).

Conversely, suppose that $\bar{x} \in K$ is an efficient solution of (VOP). Then, we have

$$\big(f_1(y) - f_1(\bar{x}), \ldots, f_\ell(y) - f_\ell(\bar{x})\big) \notin -\mathbb{R}_+^\ell \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K. \tag{11}$$

Since each f_i is invex wrt η, we deduce that

$$\langle \xi_i, \eta(\bar{x}, y)\rangle \le f_i(\bar{x}) - f_i(y), \quad \text{for all } y \in K, \xi_i \in \partial^c f_i(y) \text{ and } i \in \mathscr{I}.$$

Also, since η is skew, we obtain

$$\langle \xi_i, \eta(y, \bar{x})\rangle \ge f_i(y) - f_i(\bar{x}), \quad \text{for all } y \in K, \xi_i \in \partial^c f_i(y) \text{ and } i \in \mathscr{I}. \tag{12}$$

From (11) and (12), it follows that $\bar{x}$ is a solution of (GMVVLIP). □

Remark 3.1 Theorem 3.1 extends Proposition 1 in [4], Theorem 2.1 in [8] and Theorem 3.1 in [15].

Theorem 3.2 *Let $K \subseteq \mathbb{R}^n$ be invex wrt $\eta : K \times K \to \mathbb{R}^n$. For each $i \in \mathscr{I} = \{1, \ldots, \ell\}$, let $f_i : K \to \mathbb{R}$ be locally Lipschitz and invex wrt η on K. If $\bar{x} \in K$ is a solution (GSVVLIP), then it is an efficient solution of (VOP). Furthermore, if for each $i \in \mathscr{I}$, f_i satisfies Condition A and η is skew and satisfies Condition C, then $\bar{x} \in K$ a solution of (GMVVLIP).*

Proof Since $\bar{x} \in X$ is a solution of (GSVVLIP), for any $y \in K$, there exist $\zeta_i \in \partial^c f_i(\bar{x})$, $i = 1, \ldots, \ell$, such that

$$\big(\langle \zeta_1, \eta(y, \bar{x})\rangle, \ldots, \langle \zeta_\ell, \eta(y, \bar{x})\rangle\big) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}. \tag{13}$$

Since each f_i is invex wrt η, we have

$$\langle \zeta_1, \eta(y, \bar{x})\rangle \le f_i(y) - f_i(\bar{x}), \quad \text{for any } y \in K \text{ and for all } i \in \mathscr{I}. \tag{14}$$

By combining (13) and (14), we obtain

$$\big(f_1(y) - f_1(\bar{x}), \ldots, f_\ell(y) - f_\ell(\bar{x})\big) \notin -\mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}, \quad \text{for all } y \in K.$$

Thus, $\bar{x} \in K$ is an efficient solution of (VOP). □

Remark 3.2 Theorem 3.2 extends Theorem 2.2 in [8] to the nonconvex setting.

4 Generalized Week Vector Variational-like Inequalities

We consider the following weak forms of the generalized Minty vector variational-like inequality problem and generalized Stampacchia vector variational-like inequality problem.

(WGMVVLIP) Find $\bar{x} \in K$ such that, for all $y \in K$ and all $\xi_i \in \partial^c f_i(y)$, $i \in \mathscr{I} = \{1, \ldots, \ell\}$,

$$\langle \xi, \eta(y, \bar{x})\rangle_\ell = \big(\langle \xi_1, \eta(y, \bar{x})\rangle, \ldots, \langle \xi_\ell, \eta(y, \bar{x})\rangle\big) \notin -\operatorname{int} \mathbb{R}^\ell_+.$$

(WGSVVLIP) Find $\bar{x} \in K$ such that, for all $y \in K$, there exists $\zeta_i \in \partial^c f_i(\bar{x})$, $i \in \mathscr{I} = \{1, \ldots, \ell\}$, such that

$$\langle \zeta, \eta(y, \bar{x})\rangle_\ell = \big(\langle \zeta_1, \eta(y, \bar{x})\rangle, \ldots, \langle \zeta_\ell, \eta(y, \bar{x})\rangle\big) \notin -\operatorname{int} \mathbb{R}^\ell_+.$$

Of course, when $\eta(y, x) = y = x$, then (WGMVVLIP) and (WGSVVLIP) reduce to weak forms of the generalized Minty vector variational inequality problem and generalized Stampacchia vector variational inequality problem, respectively, considered and studied in [8].

Now, we present some results which show the relationship among the solutions of (WGMVVLIP), (WGSVVLIP) and a weak efficient solution of (VOP).

Proposition 4.1 *Let $K \subseteq \mathbb{R}^n$ be invex wrt $\eta : K \times K \to \mathbb{R}^n$ such that η is skew. For each $i \in \mathscr{I} = \{1, \ldots, \ell\}$, let $f_i : K \to \mathbb{R}$ be locally Lipschitz and invex wrt η on K. If $\bar{x} \in K$ is a solution (WGSVVLIP), then it is a solution of (GMVVLIP).*

Proof Let $\bar{x} \in X$ be a solution of (WGSVVLIP). Then, for any $y \in K$, there exist $\zeta_i \in \partial^c f_i(\bar{x})$, $i = 1, \dots, \ell$, such that

$$\big(\langle \zeta_1, \eta(y, \bar{x})\rangle, \dots, \langle \zeta_\ell, \eta(y, \bar{x})\rangle\big) \notin -\mathrm{int}\,\mathbb{R}^\ell_+. \tag{15}$$

Since each f_i is invex wrt η, by Lemma 2.1, each $\partial^c f_i$ ($i \in \mathscr{I}$) is η-monotone and therefore we have

$$\langle \xi_i - \zeta_i, \eta(y, \bar{x})\rangle \geq 0, \quad \text{for all } y \in K, \xi_i \in \partial^c f_i(y) \text{ and for each } i \in \mathscr{I}. \tag{16}$$

From (15) and (16), it follows that, for any $y \in K$ and any $\xi_i \in \partial^c f_i(y)$, $i \in \mathscr{I}$,

$$\big(\langle \xi_1, \eta(y, \bar{x})\rangle, \dots, \langle \xi_\ell, \eta(y, \bar{x})\rangle\big) \notin -\mathrm{int}\,\mathbb{R}^\ell_+.$$

Thus, $\bar{x} \in K$ is a solution of (WGMVVLIP). □

Proposition 4.2 *Let $K \subseteq \mathbb{R}^n$ be invex wrt $\eta : K \times K \to \mathbb{R}^n$. For each $i \in \mathscr{I} = \{1, \dots, \ell\}$, let $f_i : K \to \mathbb{R}$ be locally Lipschitz and invex wrt η on K. If $\bar{x} \in K$ is a solution* (*WGMVVLIP*), *then it is a solution of* (*GSVVLIP*).

Proof Let $\bar{x} \in K$ be a solution of (WGMVVLIP). Consider any $y \in K$ and any sequence $\{\lambda_m\} \searrow 0$ with $\alpha_m \in (0, 1]$. Since K is invex,

$$y_m := \bar{x} + \alpha_m \eta(y, \bar{x}) \in K.$$

Since $\bar{x} \in K$ is a solution of (WGMVVLIP), there exist $\xi_i^m \in \partial^c f_i(y_m)$, $i \in \mathscr{I}$, such that

$$\big(\langle \xi_1^m, \eta(y_m, \bar{x})\rangle, \dots, \langle \xi_\ell^m, \eta(y_m, \bar{x})\rangle\big) \notin -\mathrm{int}\,\mathbb{R}^\ell_+.$$

Since each f_i is locally Lipschitz, there exists $k > 0$ such that, for sufficiently large m and for all $i \in \mathscr{I}$, $\|\xi_i^m\| \leq k$. So, we can assume that the sequence $\{\xi_i^m\}$ converges to ζ_i for each $i \in \mathscr{I}$. Since the multifunction $y \mapsto \partial^c f_i(y)$ is closed (see [16, p. 29]), $\xi_i^m \in \partial^c f_i(y_m)$ and $y_m \to \bar{x}$ as $m \to \infty$, we have that $\zeta_i \in \partial^c f_i(\bar{x})$ for each $i \in \mathscr{I}$. Therefore, for any $y \in K$, there exist $\zeta_i \in \partial^c f_i(\bar{x})$, $i \in \mathscr{I}$, such that

$$\big(\langle \zeta_1, \eta(y, \bar{x})\rangle, \dots, \langle \zeta_\ell, \eta(y, \bar{x})\rangle\big) \notin -\mathrm{int}\,\mathbb{R}^\ell_+.$$

Hence, $\bar{x} \in K$ is a solution of (WGSVVLIP). □

From Propositions 4.1 and 4.2, we have the following result.

Theorem 4.1 *Let $K \subseteq \mathbb{R}^n$ be invex wrt $\eta : K \times K \to \mathbb{R}^n$ such that η is skew. For each $i \in \mathscr{I} = \{1, \dots, \ell\}$, let $f_i : K \to \mathbb{R}$ be locally Lipschitz and invex wrt η on K. Then, $\bar{x} \in K$ is a solution* (*WGSVVLIP*) *if and only if it is a solution of* (*GMVVLIP*).

Remark 4.1 Theorem 4.1 extends Theorem 3.1 in [8] to nonconvex settings. We also remark that the result similar to Theorem 4.1 is established in [3] for subinvex functions.

Now, we present the equivalence between the solution of (WGSVVLIP) and a weak efficient solution of (VOP).

Proposition 4.3 *Let $K \subseteq \mathbb{R}^n$ be invex wrt $\eta : K \times K \to \mathbb{R}^n$ such that η is skew. For each $i \in \mathscr{I} = \{1, \ldots, \ell\}$, let $f_i : K \to \mathbb{R}$ be locally Lipschitz and invex wrt η on K. If $\bar{x} \in K$ is a solution of (WGSVVLIP), then it is a weak efficient solution of (VOP).*

Proof Suppose that $\bar{x}$ is a solution of (WGSVVLIP), but not a weak efficient solution of (VOP). Then, there exists $y \in K$ such that

$$\big(f_1(y) - f_1(\bar{x}), \ldots, f_\ell(y) - f_\ell(\bar{x})\big) \in -\operatorname{int} \mathbb{R}^\ell_+. \tag{17}$$

Since each f_i, $i \in \mathscr{I}$, is invex wrt η, we have

$$\langle \zeta_i, \eta(y, \bar{x})\rangle \le f_i(y) - f_i(\bar{x}), \quad \text{for all } \zeta_i \in \partial^c f_i(\bar{x}). \tag{18}$$

Combining (17) and (18), we obtain

$$\big(\langle \zeta_1, \eta(y, \bar{x})\rangle, \ldots, \langle \zeta_\ell, \eta(y, \bar{x})\rangle\big) \in -\operatorname{int} \mathbb{R}^\ell_+, \quad \text{for all } \zeta_i \in \partial^c f_i(\bar{x})$$

which contradicts our supposition that $\bar{x}$ is a solution of (WGSVVLIP). This completes the proof. □

Proposition 4.4 *Let $K \subseteq \mathbb{R}^n$ be invex wrt $\eta : K \times K \to \mathbb{R}^n$ such that η is skew. For each $i \in \mathscr{I} = \{1, \ldots, \ell\}$, let $f_i : K \to \mathbb{R}$ be locally Lipschitz and invex wrt η on K. If $\bar{x} \in K$ is a weak efficient solution of (VOP), then it is a solution of (WGMVVLIP).*

Proof Assume that $\bar{x} \in K$ is a weak efficient solution of (VOP) but not a solution of (WGMVVLIP). Then, there exist $y \in K$ and $\xi_i \in \partial^c f_i(y)$, $i \in \mathscr{I}$, such that

$$\big(\langle \xi_1, \eta(y, \bar{x}), \ldots, \langle \xi_\ell, \eta(y, \bar{x})\rangle\big) \in -\operatorname{int} \mathbb{R}^\ell_+. \tag{19}$$

By invexity of f_i, $i \in \mathscr{I}$, wrt η, we have

$$\langle \xi_i, \eta(y, \bar{x})\rangle \le f_i(y) - f_i(\bar{x}). \tag{20}$$

From (19) and (20), we then have

$$\big(f_1(y) - f_1(\bar{x}), \ldots, f_\ell(y) - f_\ell(\bar{x})\big) \in -\operatorname{int} \mathbb{R}^\ell_+,$$

which contradicts our assumption that $\bar{x}$ is a weak efficient solution of (VOP). Hence, the result is proved. □

From Theorem 4.1 and Propositions 4.3 and 4.4, we have the following result.

Theorem 4.2 *Let $K \subseteq \mathbb{R}^n$ be invex wrt $\eta : K \times K \to \mathbb{R}^n$ such that η is skew. For each $i \in \mathscr{I} = \{1, \ldots, \ell\}$, let $f_i : K \to \mathbb{R}$ be locally Lipschitz and invex wrt η on K. Then, $\bar{x} \in K$ is a solution (WGSVVLIP) if and only if it is a weak efficient solution of (VOP).*

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