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Iterative Algorithms for Mixed Equilibrium Problems, Strict Pseudocontractions and Monotone Mappings

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Abstract In this paper, we introduce some iterative algorithms for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a strict pseudocontraction and the set of solutions of a variational inequality for a monotone, Lipschitz continuous mapping. We obtain both weak and strong convergence theorems for the sequences generated by these processes in Hilbert spaces.

Keywords Mixed equilibrium problem · Extragradient method · Hybrid method · Strict pseudocontraction · Monotone mapping

1 Introduction

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $\|.\|$ and let C be a nonempty closed convex subset of H. Let $\varphi : C \to R$ be a function and F be a bifunction from $C \times C$ to R, where R is the set of real numbers. Ceng and Yao [1] and Bigi, Castellani and Kassay [2] considered the following mixed equilibrium problem:

$$\text{Find} \quad x \in C \quad \text{such that} \quad F(x,y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $\text{MEP}(F,\varphi)$.

If $\varphi = 0$, then the mixed equilibrium problem (1) becomes the following equilibrium problem:

$$\text{Find} \quad x \in C \quad \text{such that} \quad F(x,y) \geq 0, \quad \forall y \in C. \tag{2}$$

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If $\varphi = 0$ and $F(x, y) = \langle Ax, y - x\rangle$ for all $x, y \in C$, where A is a mapping from C into H, then problem (1) becomes the following variational inequality:

$$\text{Find} \quad x \in C \quad \text{such that} \quad \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{3}$$

The set of solutions of problem (3) is denoted by $\mathrm{VI}(C, A)$.

The problem (1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games and others; see for instance, [1–4].

Recall that a mapping $T : C \to C$ is said to be a κ-strict pseudocontraction [5] if there exists $0 \leq \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C,$$

where I denotes the identity operator on C. Clearly, T is nonexpansive if and only if T is a 0-strict pseudocontraction. Note that the class of strict pseudocontraction mappings strictly includes the class of nonexpansive mappings. We denote the set of fixed points of T by $\mathrm{Fix}(T)$.

Ceng and Yao [1] introduced an iterative scheme for finding a common element of the set of solutions of problem (1) and the set of common fixed points of a family of finitely nonexpansive mappings in a Hilbert space and obtained a strong convergence theorem. Some methods have been proposed to solve the problem (2); see, for instance, [3, 4, 6–11] and the references therein. Recently, Combettes and Hirstoaga [6] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. Takahashi and Takahashi [7] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of problem (2) and the set of fixed points of a nonexpansive mapping in a Hilbert space and proved a strong convergence theorem. Su, Shang and Qin [8] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of problem (2) and the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for an α-inverse strongly monotone mapping in a Hilbert space and proved a strong convergence theorem. Tada and Takahashi [9] introduced two iterative schemes for finding a common element of the set of solutions of problem (2) and the set of fixed points of a nonexpansive mapping in a Hilbert space and obtained both strong convergence theorem and weak convergence theorem. Plubtieng and Punpaeng [10] introduced an iterative process based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings, the set of an equilibrium problem and the set of solutions of variational inequality problem for α-inverse strongly monotone mappings. Ceng, AI-Homidan, Ansari and Yao [11] introduced an iterative algorithm for finding a common element of the set of solutions of problem (2) and the set of fixed points of a strict pseudocontraction mapping and obtained a weak convergence theorem.

On the other hand, Browder and Petryshyn [5] showed that, if a κ-strict pseudocontraction T has a fixed point in C, then starting with an initial $x_0 \in C$, the sequence $\{x_n\}$ generated by the recursive formula

$$x_{n+1} = \alpha x_n + (1 - \alpha)Tx_n,$$

where α is a constant such that $\kappa < \alpha < 1$, converges weakly to a fixed point of T. Marino and Xu [12] and Zhou [13] have extended Browder and Petryshyn's above-mentioned result by proving that the sequence $\{x_n\}$ generated by Mann's algorithm,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n,$$

converges weakly to a fixed point of T, provided the control sequence $\{\alpha_n\}$ satisfies some conditions.

In the present paper, inspired and motivated by the above ideas, we introduce some iterative algorithms based on the extragradient method for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a strict pseudocontraction and the set of the solution sets of a variational inequality for a monotone, Lipschitz continuous mapping. We obtain both weak convergence theorem and strong convergence theorem for the sequences generated by these processes. The results in this paper generalize and improve some well-known results in the literature.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively. In a real Hilbert space H, it is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping from H onto C. It is also known that $P_C(x) \in C$ and

$$\langle x - P_C(x), P_C(x) - y\rangle \geq 0, \tag{4}$$

for all $x \in H$ and $y \in C$.

It is easy to see that (4) is equivalent to

$$\|x-y\|^2 \geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \tag{5}$$

for all $x \in H$ and $y \in C$.

A mapping A of C into H is called monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0,$$

for all $x, y \in C$. A mapping $A : C \to H$ is called k-Lipschitz continuous if there exists a positive real number k such that

$$\|Ax - Ay\| \leq k\|x - y\|,$$

for all $x, y \in C$. Let A be a monotone mapping of C into H. In the context of the variational inequality problem, the characterization of projection (4) implies the following:

$$u \in \mathrm{VI}(C, A) \quad \Rightarrow \quad u = P_C(u - \lambda Au), \quad \lambda > 0,$$

and

$$u = P_C(u - \lambda Au), \quad \text{for some} \quad \lambda > 0 \quad \Rightarrow \quad u \in \mathrm{VI}(C, A).$$

It is also known that H satisfies the Opial condition, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$.

For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction F, φ and the set C:

(A1) $F(x, x) = 0$ for all $x \in C$.
(A2) F is monotone, i.e. $F(x, y) + F(y, x) \le 0$ for any $x, y \in C$.
(A3) For each $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous.
(A4) For each $x \in C$, $y \mapsto F(x, y)$ is convex.
(A5) For each $x \in C$, $y \mapsto F(x, y)$ is lower semicontinuous.
(B1) For each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that, for any $z \in C \setminus D_x$

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0.$$

(B2) C is a bounded set.

Lemma 2.1 [14] *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A4) *and let $\varphi : C \to R$ be a lower semicontinuous and convex function. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows*:

$$T_r(x) = \left\{ z \in C : F(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C \right\},$$

for all $x \in H$. Assume that either (B1) *or* (B2) *holds. Then, the following conclusions hold*:

(i) *For each $x \in H$, $T_r(x) \neq \emptyset$.*
(ii) *T_r is single-valued.*
(iii) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \le \langle T_r(x) - T_r(y), x - y\rangle.$$

(iv) $\mathrm{Fix}(T_r) = \mathrm{MEP}(F, \varphi)$.
(v) $\mathrm{MEP}(F, \varphi)$ *is closed and convex.*

Remark 2.1 We remark that Lemma 2.1 is not a consequence of Lemma 3.1 in [1], because the condition of the sequential continuity from the weak topology to the strong topology for the derivative K' of the function $K: C \to R$ does not cover the case $K(x) = \frac{\|x\|^2}{2}$.

3 Main Results

In this section, we show strong and weak convergence theorems of some iterative algorithms based on extragradient method (and hybrid method) which solves the problem of finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a strict pseudocontraction and the solution set of the variational inequality for a monotone, Lipschitz continuous mapping in a Hilbert space.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A5) *and let $\varphi : C \to R$ be a lower semicontinuous and convex function. Let A be a monotone and k-Lipschitz continuous mapping of C into H. Let $T : C \to C$ be an ε-strict pseudocontraction for some $0 \le \varepsilon < 1$ such that $\Omega = \mathrm{Fix}(T) \cap \mathrm{VI}(C, A) \cap \mathrm{MEP}(F, \varphi) \ne \emptyset$. Assume also that either* (B1) *or* (B2) *holds. Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{z_n\}$ be sequences generated by*

$$
\begin{aligned}
&x_1 = x \in H,\\
&F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C,\\
&y_n = P_C(u_n - \lambda_n A u_n),\\
&t_n = P_C(u_n - \lambda_n A y_n),\\
&z_n = \alpha_n t_n + (1 - \alpha_n) T t_n,\\
&C_n = \{z \in C : \|z_n - z\|^2 \le \|x_n - z\|^2 - (1 - \alpha_n)(\alpha_n - \varepsilon)\|t_n - T t_n\|^2\},\\
&Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \ge 0\},\\
&x_{n+1} = P_{C_n \cap Q_n} x,
\end{aligned}
$$

for every $n = 1, 2, \ldots$. Assume that $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and let $\{r_n\} \subset (0, \infty)$ satisfy $\liminf_{n\to\infty} r_n > 0$. Then, $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{t_n\}$, $\{z_n\}$ converge strongly to $w = P_\Omega(x)$.

Proof First observe that C_n is closed and convex by Lemma 1.2 in [12] and Q_n is closed and convex for every $n = 1, 2, \ldots$. By definition of Q_n, $\langle x_n - z, x - x_n\rangle \ge 0$ for all $z \in Q_n$ and by (4), $x_n = P_{Q_n}(x)$. Now we show that $\Omega \subseteq C_n$ for any n. Let $u \in \Omega$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.1. Then, $u = P_C(u - \lambda_n A u) = T_{r_n}(u)$. From $u_n = T_{r_n}(x_n) \in C$, we have

$$\|u_n - u\| = \|T_{r_n}(x_n) - T_{r_n}(u)\| \le \|x_n - u\|. \tag{6}$$

From (5), the monotonicity of A, and $u \in \mathrm{VI}(C, A)$, we have

$$\begin{aligned}
\|t_n - u\|^2 &\le \|u_n - \lambda_n Ay_n - u\|^2 - \|u_n - \lambda_n Ay_n - t_n\|^2 \\
&= \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n \langle Ay_n, u - t_n\rangle \\
&= \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n (\langle Ay_n - Au, u - y_n\rangle \\
&\quad + \langle Au, u - y_n\rangle + \langle Ay_n, y_n - t_n\rangle) \\
&\le \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\lambda_n \langle Ay_n, y_n - t_n\rangle \\
&= \|u_n - u\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - t_n\rangle \\
&\quad - \|y_n - t_n\|^2 + 2\lambda_n \langle Ay_n, y_n - t_n\rangle \\
&= \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle u_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle.
\end{aligned}$$

Further, Since $y_n = P_C(u_n - \lambda_n Au_n)$ and A is k-Lipschitz continuous, we have

$$\begin{aligned}
&\langle u_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle \\
&\quad = \langle u_n - \lambda_n Au_n - y_n, t_n - y_n\rangle + \langle \lambda_n Au_n - \lambda_n Ay_n, t_n - y_n\rangle \\
&\quad \le \langle \lambda_n Au_n - \lambda_n Ay_n, t_n - y_n\rangle \\
&\quad \le \lambda_n k \|u_n - y_n\| \|t_n - y_n\|.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|t_n - u\|^2 &\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&\quad + 2\lambda_n k \|u_n - y_n\| \|t_n - y_n\| \\
&\le \|u_n - u\|^2 - \|u_n - y_n\|^2 - \|y_n - t_n\|^2 + {\lambda_n}^2 k^2 \|u_n - y_n\|^2 \\
&\quad + \|t_n - y_n\|^2 \\
&= \|u_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2. \\
&\le \|u_n - u\|^2. \qquad (7)
\end{aligned}$$

It follows from (6), (7), $z_n = \alpha_n t_n + (1 - \alpha_n)Tt_n$ and $u = Tu$ that

$$\begin{aligned}
\|z_n - u\|^2 &= \alpha_n \|t_n - u\|^2 + (1 - \alpha_n)\|Tt_n - u\|^2 - \alpha_n(1 - \alpha_n)\|t_n - Tt_n\|^2 \\
&\le \alpha_n \|t_n - u\|^2 + (1 - \alpha_n)[\|t_n - u\|^2 + \varepsilon\|t_n - Tt_n\|^2] \\
&\quad - \alpha_n(1 - \alpha_n)\|t_n - Tt_n\|^2 \\
&= \|t_n - u\|^2 + (1 - \alpha_n)(\varepsilon - \alpha_n)\|t_n - Tt_n\|^2 \\
&\le \|u_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2 + (1 - \alpha_n)(\varepsilon - \alpha_n)\|t_n - Tt_n\|^2 \\
&\le \|x_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2 \\
&\le \|x_n - u\|^2, \qquad (8)
\end{aligned}$$

for every $n = 1, 2, \ldots$.

From (6)–(8), we obtain that

$$\|z_n - u\|^2 \le \|x_n - u\|^2 + (1-\alpha_n)(\varepsilon - \alpha_n)\|t_n - Tt_n\|^2, \tag{9}$$

for every $n = 1, 2, \dots$, and hence $u \in C_n$. So, $\Omega \subset C_n$ for every $n = 1, 2, \dots$. Next, let us show by mathematical induction that $\{x_n\}$ is well defined and $\Omega \subset C_n \cap Q_n$ for every $n = 1, 2, \dots$. For $n = 1$ we have $x_1 = x \in H$ and $Q_1 = H$. Hence we obtain $\Omega \subset C_1 \cap Q_1$. Suppose that x_k is given and $\Omega \subset C_k \cap Q_k$ for some positive integer k. Since Ω is nonempty, $C_k \cap Q_k$ is a nonempty closed convex subset of H. So, there exists a unique element $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k}(x)$. It is also obvious that there holds $\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0$ for every $z \in C_k \cap Q_k$. Since $\Omega \subset C_k \cap Q_k$, we have $\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0$ for every $z \in \Omega$ and hence $\Omega \subset Q_{k+1}$. Therefore, we obtain $\Omega \subset C_{k+1} \cap Q_{k+1}$.

Let $l_0 = P_\Omega x$. From $x_{n+1} = P_{C_n \cap Q_n} x$ and $l_0 \in \Omega \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \le \|l_0 - x\|, \tag{10}$$

for every $n = 1, 2, \dots$. Therefore, $\{x_n\}$ is bounded. From (6)–(8), we also obtain that $\{t_n\}$, $\{z_n\}$ and $\{u_n\}$ are bounded. Since $x_{n+1} \in C_n \cap Q_n \subset C_n$ and $x_n = P_{Q_n}(x)$, we have

$$\|x_n - x\| \le \|x_{n+1} - x\|,$$

for every $n = 1, 2, \dots$. Therefore, $\lim_{n\to\infty} \|x_n - x\|$ exists.

Since $x_n = P_{Q_n}(x)$ and $x_{n+1} \in Q_n$, using (5) we have

$$\|x_{n+1} - x_n\|^2 \le \|x_{n+1} - x\|^2 - \|x_n - x\|^2,$$

for every $n = 1, 2, \dots$. This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 - (1-\alpha_n)(\alpha_n - \varepsilon)\|t_n - Tt_n\|^2 \le \|x_n - x_{n+1}\|^2$$

and hence

$$\|x_n - z_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\| \le 2\|x_n - x_{n+1}\|,$$

for every $n = 1, 2, \dots$. From $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, we have $\|x_n - z_n\| \to 0$.

For $u \in \Omega$, from (8) we obtain

$$\|z_n - u\|^2 \le \|x_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2.$$

Thus, we have

$$\|u_n - y_n\|^2 \le \frac{1}{1-\lambda_n{}^2k^2}\Big(\|x_n - u\|^2 - \|z_n - u\|^2\Big)$$
$$\le \frac{1}{1-b^2k^2}(\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|.$$

It follows from $\|x_n - z_n\| \to 0$, $\{x_n\}$ and $\{z_n\}$ bounded that $\|u_n - y_n\| \to 0$. From the definition of t_n and y_n, we have

$$\begin{aligned}\|t_n - y_n\| &= \|P_C(u_n - \lambda_n Ay_n) - P_C(u_n - \lambda_n Au_n)\|\\ &\le \|(u_n - \lambda_n Ay_n) - (u_n - \lambda_n Au_n)\|\\ &\le \lambda_n k\|y_n - u_n\|. \end{aligned}\tag{11}$$

This implies that $\lim_{n\to\infty}\|t_n - y_n\| = 0$. From $\|u_n - t_n\| \le \|u_n - y_n\| + \|y_n - t_n\|$ we also have $\|u_n - t_n\| \to 0$. As A is k-Lipschitz continuous, we have $\|Ay_n - At_n\| \to 0$.

From $\varepsilon < c \le \alpha_n \le d < 1$ and (9), we have

$$\begin{aligned}(1-d)(c-\varepsilon)\|t_n - Tt_n\|^2 &\le (1-\alpha_n)(\alpha_n - \varepsilon)\|t_n - Tt_n\|^2\\ &\le \|x_n - u\|^2 - \|z_n - u\|^2\\ &\le (\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|.\end{aligned}$$

This implies that

$$\lim_{n\to\infty}\|t_n - Tt_n\| = 0. \tag{12}$$

For $u \in \Omega$, we have from Lemma 2.1,

$$\begin{aligned}\|u_n - u\|^2 &= \|T_{r_n}x_n - T_{r_n}u\|^2\\ &\le \langle T_{r_n}x_n - T_{r_n}u, x_n - u\rangle\\ &= \frac{1}{2}\{\|u_n - u\|^2 + \|x_n - u\|^2 - \|x_n - u_n\|^2\}.\end{aligned}$$

Hence,

$$\|u_n - u\|^2 \le \|x_n - u\|^2 - \|x_n - u_n\|^2. \tag{13}$$

It follows from (8) and (13) that

$$\|z_n - u\|^2 \le \|u_n - u\|^2 \le \|x_n - u\|^2 - \|x_n - u_n\|^2.$$

Hence,

$$\begin{aligned}\|x_n - u_n\|^2 &\le \|x_n - u\|^2 - \|z_n - u\|^2\\ &\le (\|x_n - u\| + \|z_n - u\|)\|x_n - z_n\|.\end{aligned}$$

Since $\|x_n - z_n\| \to 0$, $\{x_n\}$ and $\{z_n\}$ are bounded, we obtain $\|x_n - u_n\| \to 0$. From $\|t_n - x_n\| \le \|t_n - u_n\| + \|x_n - u_n\|$, we also have $\|t_n - x_n\| \to 0$.

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{ni}\}$ of $\{x_n\}$ such that $x_{ni} \rightharpoonup w$. From $\|x_n - u_n\| \to 0$ and $\|t_n - x_n\| \to 0$, we obtain that $u_{ni} \rightharpoonup w$ and $t_{ni} \rightharpoonup w$. Since $\{u_{ni}\} \subset C$ and C is closed and convex, we obtain $w \in C$. It follows from (12) that

$$\|t_{n_i} - Tt_{n_i}\| \to 0.$$

So by the demiclosedness principle (Proposition 2.1(ii) in [12]), it follows that $w \in \mathrm{Fix}(T)$. In order to show that $w \in \Omega$, we need show $w \in \mathrm{MEP}(F, \varphi)$. By $u_n = T_{r_n} x_n$, we know that

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

It follows from (A2) that

$$\varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n), \quad \forall y \in C.$$

Hence,

$$\varphi(y) - \varphi(u_{n_i}) + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq F(y, u_{n_i}), \quad \forall y \in C.$$

It follows from (A4), (A5), and the weakly lower semicontinuity of φ, $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$ and $u_{n_i} \rightharpoonup w$ that

$$F(y, w) + \varphi(w) - \varphi(y) \leq 0, \quad \forall y \in C.$$

For $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we obtain $y_t \in C$ and hence $F(y_t, w) + \varphi(w) - \varphi(y_t) \leq 0$. So by (A4) and the convexity of φ, we have

$$\begin{aligned} 0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq tF(y_t, y) + (1-t)F(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\ &\leq t[F(y_t, y) + \varphi(y) - \varphi(y_t)]. \end{aligned}$$

Dividing by t, we get

$$F(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0.$$

Letting $t \to 0$, it follows from (A3) and the weakly lower semicontinuity of φ that

$$F(w, y) + \varphi(y) - \varphi(w) \geq 0,$$

for all $y \in C$ and hence $w \in \mathrm{MEP}(F, \varphi)$.

Exactly as in the proof of Theorem 3.1 in [14], we can prove that $w \in \mathrm{VI}(C, A)$. This implies $w \in \Omega$.

From $l_0 = P_\Omega(x)$, $w \in \Omega$ and (10), we have

$$\|l_0 - x\| \leq \|w - x\| \leq \liminf_{i\to\infty} \|x_{n_i} - x\| \leq \limsup_{i\to\infty} \|x_{n_i} - x\| \leq \|l_0 - x\|.$$

So, we obtain

$$\lim_{i\to\infty} \|x_{n_i} - x\| = \|w - x\|.$$

From $x_{n_i} - x \rightharpoonup w - x$, we have $x_{n_i} - x \to w - x$ and hence $x_{n_i} \to w$. Since $x_n = P_{Q_n}(x)$ and $l_0 \in \Omega \subset C_n \cap Q_n \subset Q_n$, we have

$$-\|l_0 - x_{n_i}\|^2 = \langle l_0 - x_{n_i}, x_{n_i} - x\rangle + \langle l_0 - x_{n_i}, x - l_0\rangle \geq \langle l_0 - x_{n_i}, x - l_0\rangle.$$

As $i \to \infty$, we obtain $-\|l_0 - w\|^2 \geq \langle l_0 - w, x - l_0\rangle \geq 0$ by $l_0 = P_\Omega(x)$ and $w \in \Omega$. Hence we have $w = l_0$. This implies that $x_n \to l_0$. It is easy to see $u_n \to l_0$, $y_n \to l_0$, $t_n \to l_0$, and $z_n \to l_0$. The proof is now complete. □

Remark 3.1 Since the nonexpansive mapping has been replaced by a strict pseudocontraction or the inverse strongly monotonicity of the mapping A has been weakened by the monotonicity of A, Theorem 3.1 improves or extends the main results in [1, 7–10].

Theorem 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A5) *and let $\varphi : C \to R$ be a lower semicontinuous and convex function. Let A be a monotone and k-Lipschitz continuous mapping of C into H. Let $T : C \to C$ be an ε-strict pseudocontraction for some $0 \leq \varepsilon < 1$ such that $\Omega = \mathrm{Fix}(T) \cap \mathrm{VI}(C, A) \cap \mathrm{MEP}(F, \varphi) \neq \emptyset$. Assume that either* (B1) *or* (B2) *holds. Let $\{x_n\}$, $\{u_n\}$, $\{t_n\}$, $\{y_n\}$ be sequences generated by*

$$\begin{aligned}
&x_1 = x \in H,\\
&F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,\\
&y_n = P_C(u_n - \lambda_n A u_n),\\
&t_n = P_C(u_n - \lambda_n A y_n),\\
&x_{n+1} = \alpha_n t_n + (1 - \alpha_n) T t_n,
\end{aligned}$$

for every $n = 1, 2, \ldots$. Assume that $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (\varepsilon, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfy $\liminf_{n\to\infty} r_n > 0$. Then, $\{x_n\}$, $\{u_n\}$, $\{t_n\}$, $\{y_n\}$ converge weakly to $w \in \Omega$, where $w = \lim_{n\to\infty} P_\Omega x_n$.

Proof Let $u \in \Omega$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.1. Then $u = P_C(u - \lambda_n Au) = T_{r_n}(u)$. As in the proof of Theorem 3.1, we know that (6), (7), (11) and (13) still hold.

It follows from (6), (7), $x_{n+1} = \alpha_n t_n + (1 - \alpha_n) T t_n$ and $u = Tu$ that

$$\begin{aligned}
\|x_{n+1} - u\|^2 &= \|\alpha_n t_n + (1 - \alpha_n) T t_n - u\|^2\\
&\leq \|t_n - u\|^2 + (1 - \alpha_n)(\varepsilon - \alpha_n)\|t_n - T t_n\|^2\\
&\leq \|u_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2 + (1 - \alpha_n)(\varepsilon - \alpha_n)\|t_n - T t_n\|^2
\end{aligned}$$

$$\leq \|x_n - u\|^2 + ({\lambda_n}^2 k^2 - 1)\|u_n - y_n\|^2$$
$$\leq \|x_n - u\|^2, \tag{14}$$

for every $n = 1, 2, \ldots$. Therefore, there exists $\theta_u = \lim_{n\to\infty} \|x_n - u\|$ for any $u \in \Omega$ and $\{x_n\}$ is bounded. From (6) and (7), we also obtain that $\{t_n\}$ and $\{u_n\}$ are bounded.

By (14), we have

$$\|u_n - y_n\|^2 \leq \frac{1}{1 - {\lambda_n}^2 k^2}\Big(\|x_n - u\|^2 - \|x_{n+1} - u\|^2\Big).$$

Hence, $\|u_n - y_n\| \to 0$. It follows from (11) that $\lim_{n\to\infty} \|t_n - y_n\| = 0$. From $\|u_n - t_n\| \leq \|u_n - y_n\| + \|y_n - t_n\|$ we also have $\|u_n - t_n\| \to 0$. As A is k-Lipschitz continuous, we have $\|Ay_n - At_n\| \to 0$.

From (14) and (6), we also have

$$\|x_{n+1} - u\|^2 \leq \|x_n - u\|^2 + (1 - \alpha_n)(\varepsilon - \alpha_n)\|t_n - Tt_n\|^2, \tag{15}$$

for every $n = 1, 2, \ldots$.

From $\varepsilon < c \leq \alpha_n \leq d < 1$ and (15), we have

$$(1 - d)(c - \varepsilon)\|t_n - Tt_n\|^2 \leq (1 - \alpha_n)(\alpha_n - \varepsilon)\|t_n - Tt_n\|^2$$
$$\leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2.$$

This implies that

$$\lim_{n\to\infty} \|t_n - Tt_n\| = 0. \tag{16}$$

Then, by (14) and (13), we have

$$\|x_{n+1} - u\|^2 \leq \|u_n - u\|^2$$
$$\leq \|x_n - u\|^2 - \|x_n - u_n\|^2.$$

Hence,

$$\|x_n - u_n\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2.$$

Thus, we obtain $\|x_n - u_n\| \to 0$. From $\|t_n - x_n\| \leq \|t_n - u_n\| + \|x_n - u_n\|$, we also have $\|t_n - x_n\| \to 0$.

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{ni}\}$ of $\{x_n\}$ such that $x_{ni} \rightharpoonup w$. From $\|x_n - u_n\| \to 0$ and $\|t_n - x_n\| \to 0$, we obtain that $u_{ni} \rightharpoonup w$ and $t_{ni} \rightharpoonup w$. Since $\{u_{ni}\} \subset C$ and C is closed and convex, we obtain $w \in C$. Exactly as in the proof of Theorem 3.1, we can obtain that $w \in \Omega$. Indeed, for this we do not use the definition of x_n that here is different from that in Theorem 3.1. We use only the fact that $u_n = T_{r_n}(x_n)$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z$. Then $z \in \Omega$. Let us show $w = z$. Assume that $w \neq z$. From the Opial condition, we have

$$\lim_{n\to\infty} \|x_n - w\| = \liminf_{i\to\infty} \|x_{n_i} - w\| < \liminf_{i\to\infty} \|x_{n_i} - z\|$$

$$= \lim_{n\to\infty} \|x_n - z\| = \liminf_{j\to\infty} \|x_{n_j} - z\|$$
$$< \liminf_{j\to\infty} \|x_{n_j} - w\| = \lim_{n\to\infty} \|x_n - w\|.$$

This is a contradiction. Thus, we have $w = z$. This implies that $x_n \rightharpoonup w \in \Omega$. Since $\|x_n - u_n\| \to 0$, we have $u_n \rightharpoonup w \in \Omega$. Since $\|y_n - u_n\| \to 0$, we have also $y_n \rightharpoonup w \in \Omega$.

Now put $w_n = P_\Omega(x_n)$. We show that $w = \lim_{n\to\infty} w_n$.

From $w_n = P_\Omega(x_n)$ and $w \in \Omega$, we have

$$\langle w - w_n, w_n - x_n\rangle \geq 0.$$

From (14) and Lemma 3.2 in [15], we know that $\{w_n\}$ converges strongly to some $w_0 \in \Omega$. Then, we have

$$\langle w - w_0, w_0 - w\rangle \geq 0$$

and hence $w = w_0$. The proof is now complete. □

Remark 3.2

(i) Theorem 3.2 extends and improves Theorem 4.1 in [9].
(ii) Let $A = 0$ and $\varphi = 0$, by Theorem 3.2, we recover Theorem 3.1 in [11] with some modified conditions.

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