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Null Controllability with Constraints on the State for the Semilinear Heat Equation

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Abstract We consider a null controllability problem for the semilinear heat equation with finite number of constraints on the state. Interpreting each constraint by means of adjoint state notion, we transform the linearized problem into an equivalent linear problem of null controllability with constraint on the control. Using inequalities of observability adapted to the constraint, we solve the equivalent problem. Then, by a fixed-point method, we prove the main result.

Keywords Systems governed by PDEs · Nonlinear PDEs of parabolic type · Null controllability · Carleman inequalities · Observability inequality

1 Introduction

Let $N, M \in \mathbb{N}^*$ and let Ω be a bounded open subset of $\mathbb{R}^N$ with boundary Γ of class $\mathcal{C}^2$. Let $\omega \subset \Omega$ be an open nonempty subset. For a time $T > 0$, we set $Q = \Omega \times (0, T)$, $\omega_T = \omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$ and we consider the semilinear heat equation

$$\frac{\partial y}{\partial t} - \Delta y + f(y) = v\chi_\omega, \quad \text{in } Q, \tag{1a}$$

$$y = 0, \quad \text{on } \Sigma, \tag{1b}$$

$$y(0) = y^0, \quad \text{in } \Omega, \tag{1c}$$

where $y^0 \in L^2(\Omega)$, the control v belongs to $L^2(Q)$, χ_ω represents the characteristic function of the control set ω and f is a globally Lipschitz function of class $\mathcal{C}^1$ defined

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on $\mathbb{R}$ verifying

$$f(0)=0. \tag{2}$$

The null controllability problem can be stated as follows: *Given* $y^0 \in L^2(\Omega)$, *find* $v \in L^2(Q)$ *such that the solution of* (1) *satisfies* $y(T)=0$ *in* Ω.

Such problems have been widely studied. In [1], Russell proved that the linear heat equation is null controllable in any time T provided the wave equation is exactly controllable for some time T. Later on Lebeau and Robbiano in [2] solved the problem of null boundary controllability in the case $f \equiv 0$ using observability inequalities deriving from Carleman inequalities. The most general result was proved by Imanuvilov and Fursikov [3] using global Carleman inequalities for the evolution operator with variable coefficients and nonzero potentials. They extended their method to the case of some nonlinear heat equations, where they prove that the problem of null boundary controllability holds for sufficiently small initial data. Let us also mention results in [4, 5], where the methods in [3] have been combined with the variational approach to controllability in [6] to prove null controllability results for heat equations with nonlinearities that grow at infinity in a super linear way.

Nakoulima gives in [7] a result of null controllability for the linear heat equation with constraint on a distributed control. His result is based on an observability inequality adapted to the constraint.

In this paper we focus on the null controllability problem with a finite number of constraints on the state that we describe now.

Let $E=\mathrm{Span}(e_1,\dots,e_M)$ be the subspace of $L^2(Q)$ generated by the functions $e_i \in L^2(Q)$, $1\le i\le M$. Assume that the functions e_i, $1\le i\le M$, are such that

$$e_i\chi_\omega,\ 1\le i\le M,\ \text{are linearly independent.} \tag{3}$$

Then the null controllability problem with a finite number of constraints on the state is as follows: *Given* e_i *in* $L^2(Q)$, $1\le i\le M$ *and* $y^0\in L^2(\Omega)$, *find a control* $v\in L^2(Q)$ *such that the solution of* (1) *satisfies*

$$\int_0^T\int_\Omega y e_i\,dx\,dt=0,\quad 1\le i\le M, \tag{4}$$

and

$$y(T)=0,\quad \text{in } \Omega. \tag{5}$$

One may come across with this kind of controllability problem while using Lions's sentinels method [8] to identifying parameters in incomplete data problems. It is in this context, for instance, that the linear case ($f(y)=ay$) of problem (1), (4) and (5) was solved by Massengo Mophou and Nakoulima in [9].

In this paper, we extend the results obtained in [9] to the semilinear case. More precisely, we prove that the null controllability problem with constraints on the state (1), (4) and (5) has a solution. The proof uses a Carleman inequality adapted to the constraints (cf. Sect. 2.2) and a fixed-point method.

The main result of the paper is the following theorem.

Theorem 1.1 *Let Ω be a bounded open subset of $\mathbb{R}^N$ with boundary Γ of class $\mathcal{C}^2$. Let also f be a real function of class $\mathcal{C}^1$, globally Lipschitz verifying* (2)*. Then, for every $e_i \in L^2(Q)$, $1 \le i \le M$, verifying* (3) *and $y^0 \in L^2(\Omega)$, there exists a control $v \in L^2(Q)$ such that the solution $y = y(v)$ of* (1) *satisfies* (4) *and* (5)*. Moreover, the control v can be chosen such that*

$$\|v\|_{L^2(Q)} \le C\|y^0\|_{L^2(\Omega)}, \tag{6}$$

where $C = C(\Omega, \omega, K, T, \sum_{i=1}^M \|e_i\|_{L^2(Q)}) > 0$ and K denotes the Lipschitz constant of the function f.

The rest of the paper is organized as follows. Section 2 is devoted to proving the null controllability problem with constraints on the state for the linearized system. In Sect. 3, we prove Theorem 1.1.

2 Analysis of the Linearized System

Since we use a fixed-point argument to prove the main result, we need to analyze first the controllability of the linearized system.

Let the function a be defined by

$$a(s) = \begin{cases} \frac{f(s)}{s}, & \text{if } s \neq 0, \\ f'(0), & \text{if } s = 0. \end{cases} \tag{7}$$

Since f is a real $\mathcal{C}^1$ function, globally Lipschitz, given any $z \in L^2(Q)$, the function a is such that

$$\|a(z)\|_{L^\infty(Q)} \le K, \tag{8}$$

where K denotes now and in the sequel the Lipschitz constant of the function f.

For every $z \in L^2(Q)$, we consider the linearized system

$$\frac{\partial y}{\partial t} - \Delta y + a(z)y = v\chi_\omega, \quad \text{in } Q, \tag{9a}$$

$$y = 0, \quad \text{on } \Sigma, \tag{9b}$$

$$y(0) = y^0, \quad \text{in } \Omega. \tag{9c}$$

Since $v\chi_\omega \in L^2(Q)$, $a(z) \in L^\infty(Q)$ and $y^0 \in L^2(\Omega)$, problem (9) has a unique solution $y = y(z) \in \mathcal{C}(0, T, L^2(\Omega)) \cap L^2(0, T, H_0^1(\Omega))$.

In the following of this section, we are interested in the controllability problem with constraints on the state: *Given $a(z) \in L^\infty(Q)$, $y^0 \in L^2(\Omega)$ and e_i in $L^2(Q)$, $1 \le i \le M$, find $v = v(z)$ in $L^2(Q)$ such that the solution of* (9) *satisfies* (4) and (5).

2.1 Equivalence to the Controllability Problem with Constraint on the Control

Proposition 2.1 *Assume that the hypotheses of Theorem 1.1 are satisfied. Then, there exists a positive real weight function θ (a precise definition of θ will be given later on) such that: for every $z \in L^2(Q)$, there exist $\mathcal{U} = \mathcal{U}(z)$ and $\mathcal{U}_\theta = \mathcal{U}_\theta(z)$, two subspaces of $L^2(\omega \times (0,T))$, of finite dimension and $u_0 = u_0(z) \in \mathcal{U}_\theta$ such that the null controllability problem with constraint on the state (9), (4), (5) is equivalent to the null controllability problem with constraint on control: Given $a(z) \in L^\infty(Q)$, $u_0 \in \mathcal{U}_\theta$ and $y^0 \in L^2(\Omega)$, find $u = u(z)$ in $L^2(\omega_T)$ such that*

$$u \in \mathcal{U}^\perp \tag{10}$$

and, if $y = y(x,t,u)$ is solution of

$$\frac{\partial y}{\partial t} - \Delta y + a(z)y = (u_0 + u)\chi_\omega, \quad \text{in } Q, \tag{11a}$$

$$y = 0, \quad \text{on } \Sigma, \tag{11b}$$

$$y(0) = y^0, \quad \text{in } \Omega, \tag{11c}$$

y satisfies

$$y(x,T,u) = 0, \quad \text{in } \Omega. \tag{12}$$

In (10), $\mathcal{U}^\perp$ denotes the orthogonal of $\mathcal{U}$ in $L^2(\omega_T)$.

Proof To obtain the null controllability problem with constraint on the control (10)–(12), we interpret the relations (4) using the adjoint state. More precisely, for each e_i, $1 \le i \le M$, we consider the adjoint system

$$-\frac{\partial p_i}{\partial t} - \Delta p_i + a(z)p_i = e_i, \quad \text{in } Q, \tag{13a}$$

$$p_i = 0, \quad \text{on } \Sigma, \tag{13b}$$

$$p_i(T) = 0, \quad \text{in } \Omega. \tag{13c}$$

Since $a(z) \in L^\infty(Q)$ and $e_i \in L^2(Q)$, problem (13) admits a unique solution $p_i = p_i(z)$ in $\Xi^{1,2}(Q) = L^2(0,T,H^2(\Omega) \cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ (see [10]).

Multiplying both sides of the differential equation in (9) by p_i, which is solution of (13), and integrating in Q, we have

$$\int_0^T \int_\omega v\, p_i\, dx\, dt = \int_0^T \int_\Omega y\, e_i\, dx\, dt - \int_\Omega y^0 p_i(0)\, dx, \quad 1 \le i \le M.$$

Therefore, taking into account the conditions (4), we obtain

$$\int_0^T \int_\omega v\, p_i\, dx\, dt = -\int_\Omega y^0\, p_i(0)\, dx, \quad 1 \le i \le M. \tag{14}$$

We set

$$\mathcal{U} = \mathrm{Span}(p_1\chi_\omega, \ldots, p_M\chi_\omega), \tag{15}$$

the vector subspace of $L^2(\omega_T)$ generated by the M functions $p_i\chi_\omega$, $1 \le i \le M$, which will be proved to be independent (see Lemma 2.1 below) and we denote by $\mathcal{U}^\perp$ the orthogonal of $\mathcal{U}$ in $L^2(\omega_T)$. Then, we consider

$$\mathcal{U}_\theta = \frac{1}{\theta}\mathcal{U}, \tag{16}$$

the vector subspace of $L^2(\omega_T)$ generated by the M functions $\frac{1}{\theta}p_i\chi_\omega$, $1 \le i \le M$, where θ is the positive function precisely defined later on by (25). Clearly, these functions will also be independent.

Since the matrix

$$\left(\int_0^T\int_\omega \frac{1}{\theta} p_i\, p_j\, dx\, dt\right)_{i,j}$$

is symmetric positive definite, there exists a unique $u_0 = u_0(z) \in \mathcal{U}_\theta$ such that

$$-\int_\Omega y^0\, p_i(0)\, dx = \int_0^T\int_\omega u_0\, p_i\, dx\, dt, \quad 1 \le i \le M. \tag{17}$$

Thus, combining (14) with (17), we deduce that

$$\int_0^T\int_\omega (v - u_0) p_i\, dx\, dt = 0, \quad 1 \le i \le M.$$

Consequently,

$$(v - u_0)\chi_\omega \in \mathcal{U}^\perp.$$

We set

$$v\chi_\omega - u_0\chi_\omega = u\chi_\omega \in \mathcal{U}^\perp. \tag{18}$$

Then,

$$v\chi_\omega = (u_0 + u)\chi_\omega. \tag{19}$$

Therefore, replacing $v\chi_\omega$ by $(u_0 + u)\chi_\omega$ in (9), we obtain (11).

Conversely, for every $z \in L^2(Q)$, assume that $a(z) \in L^\infty(Q)$, $y^0 \in L^2(\Omega)$ and $e_i \in L^2(Q)$, $1 \le i \le M$ are given. Assume also that the solution of (11) satisfies (12). Then, solving (13), we obtain the functions p_i, $1 \le i \le M$. Let $\mathcal{U}_\theta$ and $\mathcal{U}$ be respectively defined as in (16) and (15). Let also $\mathcal{U}^\perp$ be the orthogonal of $\mathcal{U}$ in $L^2(\omega \times (0, T))$, $u = u(z)$ belongs to $\mathcal{U}^\perp$ and u_0 verifies (17). Multiplying both sides of the differential equation in (11) by p_i and integrating by parts in Q, we have

$$\int_0^T\int_\omega (u_0 + u)\, p_i\, dx\, dt = \int_0^T\int_\Omega y\, e_i\, dx\, dt - \int_\Omega y^0\, p_i(0)\, dx, \quad 1 \le i \le M.$$

Since $u = u(z)$ belongs to $\mathcal{U}^\perp$ and u_0 verifies (17), this latter identity is reduced to (4). □

Remark 2.1 The function u_0 is such that $\theta u_0 \in L^2(\omega_T)$. The choice of u_0 in $\mathcal{U}_\theta$ will be necessary for the construction of the optimal control for the null controllability problem with constraint on the control (10)–(12) in Sect. 2.3.

Lemma 2.1 *Assume that* (3) *holds. Then, for every* $z \in L^2(Q)$, *the functions* $p_i \chi_\omega$, $1 \le i \le M$, *are linearly independent. Moreover, the functions* $\frac{1}{\theta} p_i \chi_\omega$, $1 \le i \le M$, *are also linearly independent.*

Proof Let $z \in L^2(Q)$. For $\gamma_i \in \mathbb{R}$, $1 \le i \le M$, let $\tilde{k}(z) = \sum_{1=1}^{M} \gamma_i p_i(z)$ on $\Omega \times (0,T)$ be such that $\tilde{k}(z)|_{\omega \times (0,T)} = 0$. Since p_i is solution of (13), we have

$$-\frac{\partial \tilde{k}(z)}{\partial t} - \Delta \tilde{k}(z) + a(z)\tilde{k}(z) = \sum_{i=1}^{M} \gamma_i e_i, \quad \text{in } \Omega \times (0,T),$$
$$\tilde{k} = 0, \quad \text{on } \Sigma.$$

Therefore, $\tilde{k}(z)$ being identically zero on $\omega \times (0,T)$, we deduce that $\tilde{k} = 0$ in $\Omega \times (0,T)$. This means that $\sum_{i=1}^{M} \gamma_i e_i = 0$ in $\Omega \times (0,T)$. Therefore,

$$\sum_{i=1}^{M} \gamma_i e_i = 0, \quad \text{in } \omega \times (0,T),$$

and assumption (3) allows us to conclude that $\gamma_i = 0$ for $1 \le i \le M$.

The second assertion of the lemma follows immediately. □

2.2 Adapted Carleman Inequalities

To solve the null controllability problem with constraint on the control (10)–(12), we use Carleman inequalities adapted to the constraint (10), which themselves are consequence of the adapted Carleman inequality. Thus, we consider an auxiliary function $\psi \in C^2(\overline{\Omega})$ which satisfies the following conditions:

$$\psi(x) > 0, \quad \forall x \in \Omega, \tag{20a}$$
$$\psi(x) = 0, \quad \forall x \in \Gamma, \tag{20b}$$
$$|\nabla \psi(x)| \neq 0, \quad \forall x \in \overline{\Omega - \omega}. \tag{20c}$$

Such a function ψ exists according to Fursikov and Imanuvilov [3]. Then, for any positive parameter value λ, we define the following weight functions:

$$\varphi(x,t) = \frac{e^{\lambda(m\|\psi\|_{L^\infty(\Omega)} + \psi(x))}}{t(T-t)}, \tag{21}$$

$$\eta(x,t) = \frac{e^{2\lambda m\|\psi\|_{L^\infty(\Omega)}} - e^{\lambda(m\|\psi\|_{L^\infty(\Omega)} + \psi(x))}}{t(T-t)}, \tag{22}$$

for $(x,t)\in Q$ and $m>1$, and we adopt the following notations:

$$L=\frac{\partial}{\partial t}-\Delta+a(z)I,$$
$$L^*=-\frac{\partial}{\partial t}-\Delta+a(z)I,$$
$$L_0^*=-\frac{\partial}{\partial t}-\Delta,$$
$$\mathcal{V}=\{\rho\in\mathcal{C}^\infty(\overline{Q})\text{ such that }\rho=0,\ \Sigma\},$$

where the function a defined by (7) satisfies $\|a(z)\|_{L^\infty(Q)}\le K$.

Then, we have the following Carleman inequality [3, 11, 12].

Proposition 2.2 *Let ψ, φ and η be the functions defined respectively by (20a)–(22). Then, there exist $\lambda_0=\lambda_0(\Omega,\omega)>1$ and $s_0=s_0(\Omega,\omega,T)>1$ and there exists some number $C=C(\Omega,\omega)>0$ such that, for any $\lambda\ge\lambda_0$, any $s\ge s_0$, and any $\rho\in\mathcal{V}$, the following estimate holds*:

$$\begin{aligned}&\int_0^T\int_\Omega\frac{e^{-2s\eta}}{s\varphi}\left(\left|\frac{\partial\rho}{\partial t}\right|^2+|\Delta\rho|^2\right)dx\,dt\\&\quad+\int_0^T\int_\Omega s\lambda^2\varphi e^{-2s\eta}|\nabla\rho|^2\,dx\,dt+\int_0^T\int_\Omega s^3\lambda^4\varphi^3e^{-2s\eta}|\rho|^2\,dx\,dt\\&\le C\left(\int_0^T\int_\Omega e^{-2s\eta}|L_0^*\rho|^2\,dx\,dt+\int_0^T\int_\omega s^3\lambda^4\varphi^3e^{-2s\eta}|\rho|^2\,dx\,dt\right).\end{aligned}\tag{23}$$

Proposition 2.3 *Let ψ, φ and η be the functions defined respectively by (20a)–(22). Then, there exist $\lambda_0=\lambda_0(\Omega,\omega,K)>1$ and $s_0=s_0(\Omega,\omega,K,T)>1$ and there exists some number $C=C(\Omega,\omega,K,T)>0$ such that, for any $\lambda\ge\lambda_0$, any $s\ge s_0$, and any $\rho\in\mathcal{V}$,*

$$\begin{aligned}&\int_0^T\int_\Omega\frac{e^{-2s\eta}}{s\varphi}\left(\left|\frac{\partial\rho}{\partial t}\right|^2+|\Delta\rho|^2\right)dx\,dt\\&\quad+\int_0^T\int_\Omega s\lambda^2\varphi e^{-2s\eta}|\nabla\rho|^2\,dx\,dt+\int_0^T\int_\Omega s^3\lambda^4\varphi^3e^{-2s\eta}|\rho|^2\,dx\,dt\\&\le C\left(\int_0^T\int_\Omega e^{-2s\eta}|L^*\rho|^2dx\,dt+\int_0^T\int_\omega s^3\lambda^4\varphi^3e^{-2s\eta}|\rho|^2dx\,dt\right).\end{aligned}\tag{24}$$

Proof It is consequence of (23). Indeed, if we write $L_0^*\rho=L^*\rho-a(z)\rho$, the inequality (23) holds for all $z\in L^2(Q)$, for fixed $\lambda\ge\lambda_0(\Omega,\omega)>1$ and $s\ge s_0(\Omega,\omega,T)>1$. Therefore, observing that

$$\int_0^T\int_\Omega e^{-2s\eta}|L_0^*\rho|^2\,dx\,dt$$
$$\le 2\left[\int_0^T\int_\Omega e^{-2s\eta}|L^*\rho|^2\,dx\,dt + K^2\int_0^T\int_\Omega e^{-2s\eta}|\rho|^2\,dx\,dt\right],$$

since $\|a(z)\|_{L^\infty(Q)} \le K$, and choosing s and λ sufficiently large depending on K, we absorb the term $2K^2\int_0^T\int_\Omega e^{-2s\eta}|\rho|^2\,dx\,dt$ in the left-hand side and we deduce from (23), the estimate (24). □

Since φ does not vanish on Q, we set

$$\theta = \varphi^{-3/2}e^{s\eta}. \tag{25}$$

Then according to the definition of φ and η given respectively by (21) and (22), the function θ is positive and $\frac{1}{\theta}$ is bounded. Thus, replacing $\varphi^{-3/2}e^{s\eta}$ by θ in (24), the following inequality holds:

$$\int_0^T\int_\Omega \frac{1}{\theta^2}|\rho|^2\,dx\,dt \le C\left(\int_0^T\int_\Omega \frac{1}{\theta^2\varphi^3 s^3\lambda^4}|L^*\rho|^2\,dx\,dt + \int_0^T\int_\omega \frac{1}{\theta^2}|\rho|^2\,dx\,dt\right).$$

Hence, since the functions $\frac{1}{\theta}$ and $\frac{1}{\varphi}$ are bounded, $s \ge s_0 > 1$ and $\lambda \ge \lambda_0 > 1$, we get the next observability inequality for any $\rho \in \mathcal{V}$,

$$\int_0^T\int_\Omega \frac{1}{\theta^2}|\rho|^2\,dx\,dt \le C\left(\int_0^T\int_\Omega |L^*\rho|^2\,dx\,dt + \int_0^T\int_\omega |\rho|^2\,dx\,dt\right). \tag{26}$$

Corollary 2.1 *Let θ be defined by (25). Then, there exist $\lambda_0 = \lambda_0(\Omega,\omega,K) > 1$ and $s_0 = s_0(\Omega,\omega,K,T) > 1$ and there exists some number $C = C(\Omega,\omega,K,T) > 0$ such that, for fixed $\lambda \ge \lambda_0$ and $s \ge s_0$ and for any $\rho \in \mathcal{V}$,*

$$\int_\Omega |\rho(0)|^2\,dx + \int_0^T\int_\Omega \frac{1}{\theta^2}|\rho|^2\,dx\,dt$$
$$\le C\left(\int_0^T\int_\Omega |L^*\rho|^2\,dx\,dt + \int_0^T\int_\omega |\rho|^2\,dx\,dt\right). \tag{27}$$

Proof We refer to [13, 14]. □

Remark 2.2 When $\mathcal{U}^\perp = L^2(\omega_T)$, the null controllability problem (10)–(12) has no constraint on the control. Therefore, using the observability inequality (27), one can prove that the null controllability problem without constraint on the control holds (see for example [11]). Since the control belongs to $\mathcal{U}^\perp \ne L^2(\omega_T)$, we need an observability inequality adapted to this constraint.

We denote

- $P = P(z)$ the orthogonal projection operator from $L^2(\omega_T)$ into $\mathcal{U}$,
- $P\rho$ the orthogonal projection of $\rho\chi_\omega$, for $\rho \in L^2(Q)$.

Proposition 2.4 (Adapted Carleman Inequality) *Assume that* (3) *holds. Let* θ *be defined by* (25). *Then, there exist* $\lambda_0 = \lambda_0(\Omega, \omega, K) > 1$ *and* $s_0 = s_0(\Omega, \omega, K, T) > 1$ *and there exists some number* $C = C(\Omega, \omega, K, T) > 0$ *such that, for any* $z \in L^2(Q)$, *for fixed* $\lambda \geq \lambda_0$ *and* $s \geq s_0$ *and for any* $\rho \in \mathcal{V}$,

$$\int_\Omega |\rho(0)|^2\,dx + \int_0^T\int_\Omega \frac{1}{\theta^2}|\rho|^2\,dx\,dt \leq C\left(\int_0^T\int_\Omega |L^*\rho|^2\,dx\,dt + \int_0^T\int_\omega |\rho - P\rho|^2\,dx\,dt\right). \tag{28}$$

The proof of this proposition requires the following lemmas:

Lemma 2.2 *Assume that* (3) *holds. Let* $\gamma \in L^\infty(Q)$ *and let* q_i *be the solution of*

$$-\frac{\partial q_i}{\partial t} - \Delta q_i + \gamma q_i = e_i, \quad \text{in } Q, \tag{29a}$$

$$q_i = 0, \quad \text{on } \Sigma, \tag{29b}$$

$$q_i(T) = 0, \quad \text{in } \Omega. \tag{29c}$$

We set $\mathcal{U}_\gamma = \mathrm{Span}(q_1\chi_\omega, \ldots, q_M\chi_\omega)$, *the vector subspace of* $L^2(\omega_T)$ *generated by the* M *independent functions* $q_i\chi_\omega$, $1 \leq i \leq M$. *Then, any function* ρ *verifying* $-\frac{\partial\rho}{\partial t} - \Delta\rho + \gamma\rho = 0$ *in* $\omega \times (0, T)$ *and* $\rho_{|_\omega} \in \mathcal{U}_\gamma$ *is identically zero in* $\omega \times (0, T)$.

Proof Let ρ be such that $\rho_{|_\omega} \in \mathcal{U}_\gamma$ and $-\frac{\partial\rho}{\partial t} - \Delta\rho + \gamma\rho = 0$ in $\omega \times (0, T)$. Then one can find $\alpha_i \in \mathbb{R}$, $1 \leq i \leq M$, such that

$$\rho = \sum_{i=1}^M \alpha_i\, q_i\, \chi_\omega.$$

Therefore, for any ω', open subset of ω, we have

$$-\frac{\partial\rho}{\partial t} - \Delta\rho + \gamma\rho = \sum_{i=1}^M \alpha_i\left(-\frac{\partial q_i}{\partial t} - \Delta q_i + \gamma q_i\right), \quad \text{in } \omega' \times (0, T),$$

which in view of (29) gives

$$-\frac{\partial\rho}{\partial t} - \Delta\rho + \gamma\rho = \sum_{i=1}^M \alpha_i e_i, \quad \text{in } \omega' \times (0, T). \tag{30}$$

Consequently, using (30) and the fact that $-\frac{\partial\rho}{\partial t} - \Delta\rho + \gamma\rho = 0$ in $\omega \times (0, T)$, it follows that

$$\sum_{i=1}^M \alpha_i\, e_i = 0, \quad \text{in } \omega \times (0, T).$$

Then, thanks to (3), for all $1 \leq i \leq M$, we have $\alpha_i = 0$. Hence, $\rho = 0$ in $\omega \times (0, T)$. □

Lemma 2.3 *Let* $(H, \|\cdot\|_H)$ *be a Hilbert space. For* $n \in \mathbb{N}^*$, *let* $\{p_i^n, 1 \le i \le M\}$ *be a family of linearly independent functions and let* $h^n \in \mathrm{Span}(p_1^n, \dots, p_M^n)$. *Assume that there exists a family of linear independent functions* $\{q_i, 1 \le i \le M\}$ *such that*

$$p_i^n \to q_i, \quad \textit{strongly in } H,\ 1 \le i \le M. \tag{31}$$

Assume also that there exists $C > 0$, *independent of* n, *such that* $\|h^n\|_H \le C$. *Then, there exists a subsequence of* (h^n) *still denoted by* (h^n) *such that*

$$h^n \to h \in \mathrm{Span}(q_1, \dots, q_M), \quad \textit{strongly in } H.$$

Proof Denote by $(.,.)_H$, the scalar product in H. Consider $\{\hat{p}_i^n, 1 \le i \le M\}$, the orthonormal basis obtained by applying the Gram-Schmidt algorithm to the family of functions $\{p_i^n, 1 \le i \le M\}$. Then,

$$\hat{p}_i^n = \frac{w_i^n}{\|w_i^n\|_H}, \quad 1 \le i \le M, \tag{32}$$

where

$$w_1^n = p_1^n, \tag{33a}$$

$$w_i^n = p_i^n - \sum_{k=1}^{i-1} (p_i^n, \hat{p}_k^n)_H \hat{p}_k^n, \quad 2 \le i \le M. \tag{33b}$$

As $\|\hat{p}_i^n\|_H = 1$, $1 \le i \le M$, we can extract a subsequence of $(\hat{p}_i^n)$ still denoted $(\hat{p}_i^n)$ such that

$$\hat{p}_i^n \rightharpoonup \hat{q}_i, \quad \text{weakly in } H. \tag{34}$$

Let us show by induction that, for $1 \le i \le M$,

$$w_i^n \to w_i = q_i - \sum_{k=1}^{i-1} (q_i, \hat{q}_k)_H \hat{q}_k, \quad \text{strongly in } H. \tag{35}$$

In view of (33a) and (31), we have

$$w_1^n \to w_1 = q_1, \quad \text{strongly in } H.$$

Thus, relation (35) is true for $i = 1$. Moreover,

$$\hat{p}_1^n = \frac{w_1^n}{\|w_1^n\|_H} \to \hat{q}_1 = \frac{w_1}{\|w_1\|_H}, \quad \text{strongly in } H.$$

Now, assume that

$$w_j^n \to w_j = q_j - \sum_{k=1}^{j-1} (q_j, \hat{q}_k)_H \hat{q}_k, \quad \text{strongly in } H \text{ for } 1 \le j \le i-1,\ 2 \le i \le M.$$

Then,

$$\hat{p}_j^n = \frac{w_j^n}{\|w_j^n\|_H} \to \hat{q}_j = \frac{w_j}{\|w_j\|_H}, \quad \text{strongly in } H,\ \forall j \le i-1,\ 2 \le i \le M. \tag{36}$$

Hence, using (33b), (31) and (36), we get

$$w_i^n \to w_i = q_i - \sum_{k=1}^{i-1} (q_i, \hat{q}_k)_H \hat{q}_k, \quad \text{strongly in } H.$$

Thus, relation (35) is true for $1 \le i \le M$. In addition,

$$\hat{p}_i^n = \frac{w_i^n}{\|w_i^n\|_H} \to \hat{q}_i = \frac{w_i}{\|w_i\|_H} \quad \text{strongly in } H,\ 2 \le i \le M. \tag{37}$$

Therefore, passing to the limit in (32) and (33), we obtain

$$\hat{q}_i = \frac{w_i}{\|w_i\|_H}, \quad 1 \le i \le M,$$

where

$$w_1 = q_1,$$
$$w_i = q_i - \sum_{k=1}^{i-1} (q_i, \hat{q}_k)_H \hat{q}_k, \quad 2 \le i \le M.$$

This means that the functions $\hat{q}_i$, $1 \le i \le M$ are deducted from q_i, $1 \le i \le M$, by the Gram-Schmidt algorithm. Consequently, $\{\hat{q}_i,\ 1 \le i \le M\}$ is an orthonormal basis, since the family $\{q_i,\ 1 \le i \le M\}$ is linearly independent.

Next, as $h^n \in \mathrm{Span}(p_1^n, \dots, p_M^n)$, there exists $\beta_i^n \in \mathbb{R}$, $1 \le i \le M$, such that $h^n = \sum_{i=1}^M \beta_i^n \hat{p}_i^n$. Consequently $\|h^n\|_H \le C$ if and only if $\sum_{i=1}^M |\beta_i^n|^2 \le C^2$. Thus, we can extract subsequence of (β_i^n) still denoted (β_i^n) such that $\beta_i^n \to \beta_i$ in $\mathbb{R}$, $1 \le i \le M$. Hence, $h^n \to h = \sum_{i=1}^M \beta_i \hat{q}_i$ strongly in H, since (37) holds. Thus, $h \in \mathrm{Span}(\hat{q}_1, \dots, \hat{q}_M) = \mathrm{Span}(q_1, \dots, q_M)$. □

Proof of Proposition 2.4 We proceed by contradiction. Suppose that (28) does not hold. Then, $\forall n \in N^*$, $\exists z_n \in L^2(Q)$, $\exists \rho_n \in \mathcal{V}$, such that

$$\int_\Omega |\rho_n(0)|^2\, dx + \int_0^T \int_\Omega \frac{1}{\theta^2} |\rho_n|^2\, dx\, dt = 1, \tag{38a}$$

$$\int_0^T \int_\Omega |L_n^* \rho_n|^2\, dx\, dt \le \frac{1}{n}, \tag{38b}$$

$$\int_0^T \int_\omega |\rho_n - P_n \rho_n|^2\, dx\, dt \le \frac{1}{n}, \tag{38c}$$

where $L_n^*\rho_n = -\frac{\partial\rho_n}{\partial t} - \Delta\rho_n + a(z_n)\rho_n$ and $P_n = P(z_n)$ is the orthogonal projection operator from $L^2(\omega_T)$ into $\mathcal{U}(z_n) = \mathrm{Span}(p_1(z_n)\chi_\omega, \dots, p_M(z_n)\chi_\omega)$.

Now, the rest of the proof consists in showing that (38) yields a contradiction. We do it in three steps.

Step 1. We have

$$\int_0^T\int_\omega \frac{1}{\theta^2}|P_n\rho_n|^2\,dx\,dt \le 2\int_0^T\int_\omega \frac{1}{\theta^2}|\rho_n|^2\,dx\,dt + 2\int_0^T\int_\omega \frac{1}{\theta^2}|\rho_n - P_n\rho_n|^2\,dx\,dt.$$

Since $1/\theta^2$ is bounded, it follows from (38) that

$$\int_0^T\int_\omega \frac{1}{\theta^2}|P_n\rho_n|^2\,dx\,dt \le C.$$

Since $P_n\rho_n$ belongs to $\mathcal{U}(z_n)$, which is of finite dimension,

$$\|P_n\rho_n\|_{L^2(\omega_T)} \le C. \tag{39}$$

Hence, using again (38c), we deduce that

$$\|\rho_n\|_{L^2(\omega_T)} \le C. \tag{40}$$

Step 2. Let us define $L^2(\frac{1}{\theta}, X) = \{\rho \in L^2(X), \int_X \frac{1}{\theta^2}|\rho|^2\,dX < \infty\}$. Then, in view of (38a), there exists a subsequence of (ρ_n) still denoted by (ρ_n) such that

$$\rho_n \rightharpoonup \rho \quad \text{weakly in } L^2\left(\frac{1}{\theta}, Q\right).$$

If we refer to the definition of (21)–(22) and the definition of $\frac{1}{\theta}$ given by (25), we can see that (ρ_n) is bounded in $L^2(]\beta, T-\beta[; L^2(\Omega))$, $\forall\beta > 0$. Then, we have in particular, for every $\beta > 0$,

$$\rho_n \rightharpoonup \rho, \quad \text{weakly in } L^2(]\beta, T-\beta[\times\Omega).$$

This implies that

$$\rho_n \rightharpoonup \rho, \quad \text{weakly in } D'(Q).$$

Therefore, using (40), we have

$$\rho_n\chi_\omega \rightharpoonup \rho\chi_\omega, \quad \text{weakly in } L^2(\omega_T). \tag{41}$$

According to the definition of a given by (7), we have $\|a(z_n)\|_{L^\infty(Q)} \le K$. Since the embedding of $L^\infty(Q)$ into $L^2(Q)$ is continuous, there exists a positive constant C such that $\|a(z_n)\|_{L^2(Q)} \le C$. Consequently, we can extract a subsequence of $(a(z_n))$ (still called $a(z_n)$) such that

$$a(z_n) \overset{*}{\rightharpoonup} \gamma, \quad \text{weakly star in } L^\infty(Q), \tag{42}$$

$$a(z_n) \rightharpoonup \gamma, \quad \text{weakly in } L^2(Q). \tag{43}$$

Now, since $p_i(z_n)$ is solution of (13), we deduce on the one hand that

$$\|p_i(z_n)\|_{L^2(0,T;H_0^1(\Omega))} \leq C(\Omega, T, K)\|e_i\|_{L^2(Q)}, \tag{44}$$

and on the other hand that $p_i(z_n)$ verifies

$$\begin{aligned} -\frac{\partial p_i(z_n)}{\partial t} - \Delta p_i(z_n) &= c_n, \quad \text{in } Q,\\ p_i(z_n) &= 0, \quad \text{on } \Sigma,\\ p_i(z_n)(T) &= 0, \quad \text{in } \Omega, \end{aligned}$$

where $c_n = e_i - a(z_n)p_i(z_n)$ is uniformly bounded in $L^2(Q)$ according to (44) and (8). Then, from the regularizing effect of the heat equation, $p_i(z_n)$ is bounded in $\Xi^{1,2}(Q) = (L^2(0,T;H^2(\Omega)\cap H_0^1(\Omega)))\cap H^1(0,T,L^2(\Omega))$. Therefore, we can extract a subsequence of $(p_i(z_n))$ (still called $p_i(z_n)$) such that

$$p_i(z_n) \rightharpoonup q_i, \quad \text{weakly in } \Xi^{1,2}(Q). \tag{45}$$

Hence, using the compactness embedding of $\Xi^{1,2}(Q)$ into $L^2(0,T,H_0^1(\Omega))$, we have

$$p_i(z_n) \to q_i, \quad \text{strongly in } L^2(0,T,H_0^1(\Omega)),\ 1 \leq i \leq M. \tag{46}$$

Therefore, it can be shown using (43)–(46) that q_i is solution of (29),

$$-\frac{\partial q_i}{\partial t} - \Delta q_i + \gamma q_i = e_i, \quad \text{in } Q, \tag{47a}$$

$$q_i = 0, \quad \text{on } \Sigma, \tag{47b}$$

$$q_i(T) = 0, \quad \text{in } \Omega. \tag{47c}$$

Since $P_n\rho_n$ belongs to $\mathcal{U}(z_n) = \text{Span}(p_1(z_n)\chi_\omega, \dots, p_M(z_n)\chi_\omega)$ and verifies (39), it suffices to apply Lemma 2.3 with $H = L^2(\omega_T)$, $p_i^n = p_i(z_n)$ and $h^n = P_n\rho_n$ to obtain that there exists $g \in \mathcal{U}_\gamma = \text{Span}(q_1\chi_\omega, \dots, q_M\chi_\omega)$ such that

$$P_n\rho_n \to g, \quad \text{strongly in } L^2(\omega_T). \tag{48}$$

As in view of (38c),

$$\rho_n - P\rho_n \to 0, \quad \text{strongly in } L^2(\omega_T), \tag{49}$$

combining (49) with (48), we obtain

$$\rho_n \to g, \quad \text{strongly in } L^2(\omega_T).$$

Hence, from (41), we deduce on the one hand that

$$\rho_n\chi_\omega \to \rho\chi_\omega, \quad \text{strongly in } L^2(\omega_T), \tag{50}$$

and on the other hand that $\rho\chi_\omega = g$. This means that $\rho\chi_\omega \in \mathcal{U}_\gamma$.

Next, using (50) and (43), we have

$$L_n^*\rho_n \rightharpoonup -\frac{\partial\rho}{\partial t} - \Delta\rho + \gamma\rho, \quad \text{weakly in } D'(\omega\times(0,T)),$$

which according to (38b) implies that

$$-\frac{\partial\rho}{\partial t} - \Delta\rho + \gamma\rho = 0, \quad \text{in } \omega\times(0,T)$$

since

$$L_n^*\rho_n \to 0, \quad \text{strongly in } L^2(Q). \tag{51}$$

In short, we proved that ρ is such that

$$-\frac{\partial\rho}{\partial t} - \Delta\rho + \gamma\rho = 0, \quad \text{in } \omega\times(0,T),$$
$$\rho\chi_\omega \in \mathcal{U}_\gamma.$$

Therefore, Lemma 2.2 allows us to conclude that $\rho = 0$ on $\omega\times(0,T)$ and (50) becomes

$$\rho_n \to 0, \quad \text{strongly in } L^2(\omega_T). \tag{52}$$

Step 3. Since $\rho_n \in \mathcal{V}$, it follows from the inequality (27) that

$$\int_\Omega |\rho_n(0)|^2\,dx + \int_0^T\int_\Omega \frac{1}{\theta^2}|\rho_n|^2\,dx\,dt$$
$$\le C\left(\int_0^T\int_\Omega |L^*\rho_n|^2 + \int_0^T\int_\omega |\rho_n|^2\,dx\,dt\right).$$

Then, in view of (51) and (52), we deduce that

$$\int_\Omega |\rho_n(0)|^2\,dx + \int_0^T\int_\Omega \frac{1}{\theta^2}|\rho_n|^2\,dx\,dt \to 0,$$

when $n\to+\infty$. The contradiction occurs thanks to (38a). The proof of (28) is complete. □

We also need the following estimates to prove that problem (10), (11), (5) has a solution.

Proposition 2.5 *Let θ be defined by (25). Let p_i and u_0 be respectively defined by (13) and (17). Then, there exists $C = C(\Omega, K, T, \sum_{i=1}^M \|e_i\|_{L^2(Q)}) > 0$ such that, for any $z\in L^2(Q)$,*

$$\|\theta u_0(z)\|_{L^2(\omega_T)} \le C\|y^0\|_{L^2(\Omega)}, \tag{53a}$$

$$\|u_0(z)\|_{L^2(\omega_T)} \le C\|y^0\|_{L^2(\Omega)}. \tag{53b}$$

To prove Proposition 2.5, we need the following results:

Lemma 2.4 *Let p_i and θ be respectively defined by* (13) *and* (25). *Let also* $A_\theta(z) = (\int_0^T\int_\omega \frac{1}{\theta} p_i(z)p_j(z)\,dx\,dt)_{ij}$, $1 \le i, j \le M$. *Then, there exists* $\delta > 0$ *such that, for all* $z \in L^2(Q)$,

$$(A_\theta(z)X(z), X(z))_{\mathbb{R}^M} \ge \delta \|X(z)\|_{\mathbb{R}^M},$$

where

$$(A_\theta(z)X(z), X(z))_{\mathbb{R}^M} = \int_0^T\int_\omega \frac{1}{\theta}\left(\sum_{i=1}^M X_i(z)p_i(z)\right)\left(\sum_{j=1}^M X_j(z)p_j(z)\right)dx\,dt$$

and

$$X(z) = (X_1(z), \dots, X_M(z)) \in \mathbb{R}^M.$$

Proof We proceed by contradiction. Assume that, $\forall n \in \mathbb{N}^*$, $\exists z_n \in L^2(Q)$, $\exists X(z_n) = (X_1(z_n), \dots, X_M(z_n)) \in \mathbb{R}^M$ such that

$$(A_\theta(z_n)X(z_n), X(z_n))_{\mathbb{R}^M} \le \frac{1}{n}\|X(z_n)\|_{\mathbb{R}^M}.$$

Set $\tilde{X}(z_n) = \frac{X(z_n)}{\|X(z_n)\|_{\mathbb{R}^M}}$. Then,

$$\|\tilde{X}(z_n)\|_{\mathbb{R}^M} = \sqrt{\sum_{i=1}^M |\tilde{X}_i(z_n)|^2} = 1,$$

$$\big(A_\theta(z_n)\tilde{X}(z_n), \tilde{X}(z_n)\big)_{\mathbb{R}^M} \le \frac{1}{n}.$$

Hence, we can extract subsequence of $(\tilde{X}_i(z_n))$, $1 \le i \le M$, still called $(\tilde{X}_i(z_n))$, $1 \le i \le M$, such that

$$\tilde{X}_i(z_n) \to \tilde{X}_i, \quad \text{in } \mathbb{R},\ 1 \le i \le M.$$

Moreover, $\sum_{i=1}^M |\tilde{X}_i|^2 = 1$.

Let

$$\tilde{u}^n = \sum_{i=1}^M \tilde{X}_i(z_n)p_i(z_n).$$

Then, in view of (46),

$$\tilde{u}^n \to \tilde{u} = \sum_{i=1}^M \tilde{X}_i q_i, \quad \text{strongly in } L^2(Q).$$

And since

$$\int_0^T\int_\omega \frac{1}{\theta}|\tilde{u}^n|^2\,dx\,dt = \big(A_\theta(z_n)\tilde{X}(z_n), \tilde{X}(z_n)\big)_{\mathbb{R}^M} \leq \frac{1}{n},$$

we deduce that $\int_0^T\int_\omega \frac{1}{\theta}|\tilde{u}|^2 dx\,dt = 0$. Consequently, $\tilde{u} = 0$ in $\omega \times (0, T)$.

As q_i verifies (47), we have

$$-\frac{\partial \tilde{u}}{\partial t} - \Delta\tilde{u} + \gamma\tilde{u} = \sum_{i=1}^M \tilde{X}_i e_i, \quad \text{in } Q,$$
$$\tilde{u} = 0, \quad \text{on } \Sigma,$$
$$\tilde{u}(T) = 0, \quad \text{in } \Omega,$$

which combined with the fact that $\tilde{u} = 0$ in $\omega \times (0, T)$ gives $\tilde{u} = 0$ in $\Omega \times (0, T)$. Thus

$$\sum_{i=1}^M \tilde{X}_i e_i = 0, \quad \text{in } \Omega \times (0, T).$$

Hence,

$$\sum_{i=1}^M \tilde{X}_i e_i = 0, \quad \text{in } \omega \times (0, T)$$

and from assumption (3), we deduce that $\tilde{X}_i = 0$, $1 \leq i \leq M$. This is impossible because

$$\sum_{i=1}^M |\tilde{X}_i|^2 = 1.$$

□

Proof of Proposition 2.5 In view of (17), we have

$$\int_0^T\int_\omega u_0(z)p_i(z)\,dx\,dt = -\int_\Omega y^0 p_i(z)(0)\,dx, \quad 1 \leq i \leq M. \tag{54}$$

Since $u_0(z) \in \text{Span}(\frac{1}{\theta}p_1(z)\chi_\omega, \ldots, \frac{1}{\theta}p_M(z)\chi_\omega)$, there exists

$$\alpha(z) = (\alpha_1(z), \ldots, \alpha_M(z)) \in \mathbb{R}^M$$

such that

$$u_0(z) = \sum_{j=1}^M \alpha_i(z)\frac{1}{\theta}p_j(z)\chi_\omega.$$

Therefore, replacing $u_0(z)$ by $\sum_{j=1}^{M} \alpha_j(z)\frac{1}{\theta} p_j(z)\chi_\omega$ in (54), we obtain

$$\int_0^T\int_\omega \sum_{i=1}^{M}\alpha_j(z)\frac{1}{\theta}p_j(z)p_i(z)\,dx\,dt = -\int_\Omega y^0 p_i(z)(0)\,dx, \quad 1\le i\le M,$$

from which we deduce that

$$\int_0^T\int_\omega \frac{1}{\theta}\left(\sum_{i=1}^{M}\alpha_i(z)p_i(z)\right)\left(\sum_{j=1}^{M}\alpha_j(z)p_j(z)\right)dx\,dt$$
$$= -\int_\Omega y^0\sum_{i=1}^{M}\alpha_i(z)p_i(z)(0)\,dx.$$

Therefore, applying to this latter identity Lemma 2.4 with $X(z)=\alpha(z)$ to the left-hand side and to the right-hand side, the Cauchy-Schwartz inequality, we get

$$\delta\|\alpha(z)\|^2_{\mathbb{R}^M} \le \|y^0\|_{L^2(\Omega)}\sum_{i=1}^{M}|\alpha_i(z)|\|p_i(z)(0)\|_{L^2(\Omega)}. \tag{55}$$

From the energy inequality for $p_i(z)$, solution of (13), it follows that

$$\|p_i(z)(0)\|_{L^2(\Omega)} \le C(\Omega,K,T)\|e_i\|_{L^2(Q)}, \quad 1\le i\le M,$$

which combined with (55) and the fact that $\delta>0$ gives

$$\|\alpha(z)\|^2_{\mathbb{R}^M} \le \delta^{-1}C(\Omega,K,T)\|y^0\|_{L^2(\Omega)}\|\alpha(z)\|_{\mathbb{R}^M}\sqrt{\sum_{i=1}^{M}\|e_i\|^2_{L^2(Q)}}. \tag{56}$$

Finally, as from $u_0(z)=\sum_{j=1}^{M}\alpha_i(z)\frac{1}{\theta}p_j(z)\chi_\omega$, we have

$$\|u_0(z)\|_{L^2(\omega_T)} \le \sum_{j=1}^{M}|\alpha_i(z)|\frac{1}{\theta}\|p_j(z)\|_{L^2(\omega_T)},$$
$$\|\theta u_0(z)\|_{L^2(\omega_T)} \le \sum_{j=1}^{M}|\alpha_i(z)|\|p_j(z)\|_{L^2(\omega_T)},$$

using (56), the fact that $\frac{1}{\theta}$ and p_j are respectively bounded in $L^\infty(Q)$ and $L^2(Q)$, and setting

$$C\left(\Omega,K,T,\sqrt{\sum_{i=1}^{M}\|e_i\|^2_{L^2(Q)}}\right) = \delta^{-1}C(\Omega,K,T)\sqrt{\sum_{i=1}^{M}\|e_i\|^2_{L^2(Q)}}$$

we deduce that (53a) and (53b) hold. □

2.3 Linear Null Controllability Problem with Constraint on the Control

We consider the following symmetric bilinear form

$$a(\rho,\hat{\rho}) = \int_0^T\int_\Omega L^*\rho\, L^*\hat{\rho}\,dx\,dt + \int_0^T\int_\omega (\rho - P\rho)(\hat{\rho} - P\hat{\rho})\,dx\,dt. \tag{57}$$

According to Proposition 2.4, this symmetric bilinear form is a scalar product on $\mathcal{V}$. Let V be the completion of $\mathcal{V}$ with respect to the norm

$$\rho \mapsto \|\rho\|_V = \sqrt{a(\rho,\rho)}. \tag{58}$$

The closure of $\mathcal{V}$ is the Hilbert space V.

Let θ and u_0 be respectively defined as in (25) and (17). Then, thanks to the Cauchy-Schwartz inequality, (28) and (53a), the linear form defined on V by

$$\rho \mapsto \int_0^T\int_\Omega u_0\chi_\omega\rho\,dx\,dt + \int_\Omega y^0\rho(0)\,dx$$

is continuous on V. Therefore, the Lax-Milgram theorem allows us to say that, for every $y^0 \in L^2(\Omega)$ and for any $z \in L^2(Q)$, there exists one and only one solution $\rho_\theta = \rho_\theta(z)$ in V of the variational equation,

$$\begin{aligned} a(\rho_\theta,\rho) &= \int_0^T\int_\Omega L^*\rho_\theta\, L^*\rho\,dx\,dt + \int_0^T\int_\omega (\rho - P\rho)(\rho_\theta - P\rho_\theta)\,dx\,dt \\ &= \int_0^T\int_\Omega u_0\chi_\omega\rho\,dx\,dt + \int_\Omega y^0\rho(0)\,dx, \quad \forall \rho \in V. \end{aligned} \tag{59}$$

Proposition 2.6 *For any $y^0 \in L^2(\Omega)$ and for any $z \in L^2(Q)$, let ρ_θ be the unique solution of (59), let*

$$u_\theta = -(\rho_\theta\chi_\omega - P\rho_\theta) \tag{60}$$

and

$$y_\theta = L^*\rho_\theta. \tag{61}$$

Then, the pair (u_θ, y_θ) is such that (11), (10) and (5) hold. Moreover, there exists

$$C = C\left(\Omega, \omega, K, T, \sum_{i=1}^{M}\|e_i\|_{L^2(Q)}\right) > 0$$

such that

$$\|\rho_\theta\|_V \le C\|y^0\|_{L^2(\Omega)}, \tag{62a}$$

$$\|u_\theta\|_{L^2(\omega_T)} \le C\|y^0\|_{L^2(\Omega)}, \tag{62b}$$

$$\|y_\theta\|_{L^2(0,T;H_0^1(\Omega))} \le C\|y^0\|_{L^2(\Omega)}. \tag{62c}$$

Proof One proceeds exactly as in [9, 13], using the variational equation (59) and inequality (28). □

Proposition 2.7 *For any $y^0 \in L^2(\Omega)$ and for any $z \in L^2(Q)$, there exists a unique control $u = u(z)$ such that*

$$\|u(z)\|_{L^2(\omega_T)} = \min_{\bar{u}(z)\in\mathcal{E}} |\bar{u}(z)|_{L^2(\omega_T)}, \tag{63}$$

where

$$\mathcal{E} = \left\{\bar{u}(z) \in L^2(\omega_T) | \, (\bar{u}(z), \bar{y}(z) = y(\bar{u}(z))) \text{ verifies } (11), (10), (5)\right\}.$$

Moreover, there exists

$$C = C\left(\Omega, \omega, K, T, \sum_{i=1}^{M} \|e_i\|_{L^2(Q)}\right) > 0$$

such that, for every $z \in L^2(Q)$,

$$\|u(z)\|_{L^2(\omega_T)} \leq C\|y^0\|_{L^2(\Omega)}. \tag{64}$$

Proof According to Proposition 2.6, the pair $(u_\theta(z), y_\theta(z))$ satisfies (11), (10) and (5). Consequently, the set $\mathcal{E}$ is non empty. Since $\mathcal{E}$ is also a closed convex subset of $L^2(\omega_T)$, we deduce that there exists a unique *control variable* $u(z)$ of minimal norm in $L^2(\omega_T)$ such that $(u(z), y(z) = y(u(z)))$ solves (11), (10) and (5). This means that

$$\|u(z)\|_{L^2(\omega_T)} \leq \|u_\theta(z)\|_{L^2(\omega_T)}.$$

Hence, using (62b), we obtain (64). □

Proposition 2.8 *Let $u(z)$ be the unique control verifying (63). Let also P be the orthogonal projection operator from $L^2(\omega \times (0, T))$ into $\mathcal{U}$. Then,*

$$u(z) = -(\rho(z)\chi_\omega - P\rho(z)\chi_\omega), \tag{65}$$

where $\rho(z) \in V$ is solution of

$$L^*\rho(z) = 0, \quad \text{in } Q, \tag{66a}$$

$$\rho(z) = 0, \quad \text{on } \Sigma. \tag{66b}$$

Moreover, there exists

$$C = C\left(\Omega, \omega, K, T, \sum_{i=1}^{M} \|e_i\|_{L^2(Q)}\right) > 0$$

such that, for any $z \in L^2(Q)$,

$$\|\rho(z)\|_V \le C\|y^0\|_{L^2(\Omega)}, \tag{67a}$$

$$\|\rho(z)\|_{L^2(\omega_T)} \le C\|y^0\|_{L^2(\Omega)}. \tag{67b}$$

Proof The proof of Proposition 2.8 uses a penalization method. We also refer to [9] for more details. □

Theorem 2.1 *Assume that the hypotheses of Theorem 1.1 are satisfied. For any* $z \in L^2(Q)$, *let* $u_0 = u_0(z) \in \mathcal{U}_\theta$ *be defined by* (17) *and let* $u = u(z)$ *be the solution of* (63). *Then, the control* $v = v(z)$ *defined by*

$$v = (u_0 + u)\chi_\omega \tag{68}$$

is such that the pair $(v, y(v))$ *verifies the null controllability problem with constraints on the state associated to the linearized system* (9), (4) *and* (5), *and there exists*

$$C = C\left(\Omega, \omega, K, T, \sum_{i=1}^{M} \|e_i\|_{L^2(Q)}\right) > 0$$

such that

$$\|v\|_{L^2(Q)} \le C\|y^0\|_{L^2(\Omega)}. \tag{69}$$

Proof We proved above that, for any $z \in L^2(Q)$, there exists a unique control $u = u(z) \in \mathcal{U}^\perp$, solution of (63) such that the pair $(u, y = y(u))$ verifies (11) and (5). Therefore, Proposition 2.1 allows to say that the control $v = (u_0 + u)\chi_\omega$ with $u_0 \in \mathcal{U}_\theta$ is such that $(v, y(v))$ satisfies the null controllability problem with constraints on the state associated to the linearized system (9), (4) and (5). Then, using (53b) and (64), we deduce that

$$\|v\|_{L^2(Q)} \le C\|y^0\|_{L^2(\Omega)}.$$

□

3 Proof of Theorem 1.1

In Sect. 2.3, we showed that, for every $z \in L^2(Q)$, there exists a control $v \in L^2(Q)$ verifying (68) such that the pair $(v, y(v))$ satisfies the null controllability problem with constraints on the state associated to the linearized system (9), (4) and (5). Thus, we have constructed a nonlinear map

$$S : L^2(Q) \to L^2(Q),$$

such that, for every $z \in L^2(Q)$, $S(z) = y(v(z)) = y(z)$ is the solution of (9) with $v(z) = (u_0 + u)\chi_\omega$, $u_0(z) \in \mathcal{U}_\theta$ and $u(z) \in \mathcal{U}^\perp$. Now, proving that S has a fixed point $y \in L^2(Q)$, such that $S(y) = y$, since $a(y)y = f(y)$, will be sufficient to finish the proof of Theorem 1.1.

Proposition 3.1 *Let f be a real function of class $\mathcal{C}^1$, globally Lipschitz verifying (2). Then*:

(i) *S is continuous.*
(ii) *S is compact.*
(iii) *The range of S is bounded*; *i.e.*,

$$\exists M > 0 : \|S(z)\|_{L^2(Q))} \le M, \quad \forall z \in L^2(Q).$$

3.1 Proof of the Continuity of S

We proceed in five steps.

Step 1. Let $(z_n) \in L^2(Q)$ be such that $z_n \to z$ strongly in $L^2(Q)$. Then, we can extract a subsequence of (z_n), still denoted (z_{nk}), such that $z_{nk} \to z$ almost everywhere in Q. Therefore, f being a function of class $\mathcal{C}^1$, the function a defined by (7) is continuous and we have

$$a(z_{nk}) \to a(z), \quad \text{almost everywhere in } Q.$$

And since Q is bounded and $|a(z_{nk})| \le K$ almost everywhere in Q the Lebesgue theorem allows us to write

$$a(z_{nk}) \to a(z), \quad \text{strongly in } L^2(Q). \tag{70}$$

Step 2. Since Theorem 2.1 holds for every $z \in L^2(Q)$, it also holds for $z_{nk} \in L^2(Q)$. Thus, the control $v(z_{nk})$ is such that the solution $y_{nk} = y(z_{nk})$ of

$$\frac{\partial y_{nk}}{\partial t} - \Delta y_{nk} + a(z_{nk}) y_{nk} = v(z_{nk})\chi_\omega, \quad \text{in } Q, \tag{71a}$$

$$y_{nk} = 0, \quad \text{on } \Sigma, \tag{71b}$$

$$y_{nk}(0) = y^0, \quad \text{in } \Omega \tag{71c}$$

satisfies

$$\int_0^T \int_\Omega y_{nk}\, e_i\, dx\, dt = 0, \quad 1 \le i \le M, \tag{72}$$

and

$$y_{nk}(T) = 0, \quad \text{in } \Omega. \tag{73}$$

More precisely,

$$v(z_{nk}) = (u_0(z_{nk}) + u(z_{nk}))\chi_\omega, \tag{74}$$

where, on the one hand, in view of (17),

$$u_0(z_{nk}) \in \operatorname{Span}\left(\frac{1}{\theta} p_1(z_{nk})\chi_\omega, \dots, \frac{1}{\theta} p_M(z_{nk})\chi_\omega\right)$$

verifies

$$-\int_\Omega y^0 p_i(z_{nk})(0)\,dx = \int_0^T\int_\omega u_0(z_{nk})p_i(z_{nk})\,dx\,dt, \quad 1\le i\le M, \tag{75}$$

with $p_i(z_{nk})$ solution of

$$-\frac{\partial p_i(z_{nk})}{\partial t} - \Delta p_i(z_{nk}) + a(z_{nk})p_i(z_{nk}) = e_i, \quad \text{in } Q, \tag{76a}$$

$$p_i(z_{nk}) = 0, \quad \text{on } \Sigma, \tag{76b}$$

$$p_i(z_{nk})(T) = 0, \quad \text{in } \Omega. \tag{76c}$$

On the other hand, if we denote by $P_{nk} = P(z_{nk})$ the orthogonal projection operator from $L^2(\omega_T)$ into $\mathcal{U}(z_{nk}) = \text{Span}(p_1(z_{nk}),\dots,p_M(z_{nk}))$, in view of (65),

$$u(z_{nk}) = -(\rho(z_{nk})\chi_\omega - P_{nk}\rho(z_{nk})), \tag{77}$$

$\rho(z_{nk})\in V$, solution of

$$-\frac{\partial \rho(z_{nk})}{\partial t} - \Delta\rho(z_{nk}) + a(z_{nk})\rho(z_{nk}) = 0, \quad \text{in } Q, \tag{78a}$$

$$\rho(z_{nk}) = 0, \quad \text{on } \Sigma. \tag{78b}$$

Furthermore, according to (67), (53b), (53a), (64) and (69), there exists a positive constant

$$C = C\left(\Omega,\omega,K,T,\sum_{i=1}^M \|e_i\|_{L^2(Q)}\right) > 0$$

independent of z_{nk} such that

$$\|\rho(z_{nk})\|_V \le \|y^0\|_{L^2(\Omega)}, \tag{79a}$$

$$\|\rho(z_{nk})\|_{L^2(\omega_T)} \le C\|y^0\|_{L^2(\Omega)}, \tag{79b}$$

$$\|u_0(z_{nk})\|_{L^2(\omega_T)} \le C\|y^0\|_{L^2(\Omega)}, \tag{79c}$$

$$\|\theta u_0(z_{nk})\|_{L^2(\omega_T)} \le C\|y^0\|_{L^2(\Omega)}, \tag{79d}$$

$$\|u(z_{nk})\|_{L^2(\omega_T)} \le C\|y^0\|_{L^2(\Omega)}, \tag{79e}$$

$$\|v(z_{nk})\|_{L^2(Q)} \le C\|y^0\|_{L^2(\Omega)}. \tag{79f}$$

Consequently, we can extract subsequences $(\rho(z_{nk}))$, $(u_0(z_{nk}))$, $(\theta u_0(z_{nk}))$ and $(u(z_{nk}))$ (still denoted $(\rho(z_{nk}))$, $(u_0(z_{nk}))$, $(\theta u_0(z_{nk}))$ and $(u(z_{nk}))$) such that

$$\rho(z_{nk}) \rightharpoonup \tilde{\rho}, \quad \text{weakly in } V, \tag{80a}$$

$$\rho(z_{nk}) \rightharpoonup \tilde{\rho}, \quad \text{weakly in } L^2(\omega_T), \tag{80b}$$

$$u_0(z_{nk}) \rightharpoonup \tilde{w}, \quad \text{weakly in } L^2(\omega_T), \tag{80c}$$

$$\theta u_0(z_{nk}) \rightharpoonup \tilde{w}_1, \quad \text{weakly in } L^2(\omega_T), \tag{80d}$$

$$u(z_{nk}) \rightharpoonup \tilde{u}, \quad \text{weakly in } L^2(\omega_T). \tag{80e}$$

Hence, from (74) and (79f), we obtain

$$v(z_{nk}) \rightharpoonup \tilde{v} = (\tilde{w} + \tilde{u})\chi_\omega \quad \text{weakly in } L^2(Q). \tag{81}$$

Step 3. Since y_{nk} is solution of (71), using (79f) we have

$$\|y_{nk}\|_{W(0,T)} \le C\left(\Omega, \omega, K, T, \sum_{i=1}^{M} \|e_i\|_{L^2(Q)}\right) \|y^0\|_{L^2(\Omega)}, \tag{82}$$

where

$$W(0,T) = \left\{\rho \in L^2(0,T; H_0^1(\Omega)),\ \frac{\partial \rho}{\partial t} \in L^2(0,T; H^{-1}(\Omega))\right\}.$$

Hence, there exists $\tilde{y} \in W(0,T)$ such that

$$y_{nk} \rightharpoonup \tilde{y}, \quad \text{weakly in } W(0,T). \tag{83}$$

Moreover, the embedding of $W(0,T)$ into $L^2(0,T,L^2(\Omega))$ being compact, we have

$$y_{nk} \to \tilde{y}, \quad \text{strongly in } L^2(Q). \tag{84}$$

Therefore, passing to the limit in (71), (72) and (73), while using (70), (81), (83) and (84), we deduce that $(\tilde{v}, \tilde{y} = y(\tilde{v}))$ verifies

$$\frac{\partial \tilde{y}}{\partial t} - \Delta \tilde{y} + a(z)\tilde{y} = \tilde{v}\chi_\omega, \quad \text{in } Q, \tag{85a}$$

$$\tilde{y} = 0, \quad \text{on } \Sigma, \tag{85b}$$

$$\tilde{y}(0) = y^0, \quad \text{in } \Omega, \tag{85c}$$

$$\int_0^T \int_\Omega \tilde{y}\, e_i \, dx\, dt = 0, \quad 1 \le i \le M. \tag{86}$$

and

$$\tilde{y}(T) = 0 \quad \text{in } \Omega. \tag{87}$$

Step 4. Since $p_i(z_{nk})$ is solution of (76), we deduce on the one hand that

$$\|p_i(z_{nk})\|_{L^2(0,T;H_0^1(\Omega))} \le C(\Omega, T, K)\|e_i\|_{L^2(Q)}, \tag{88}$$

and on the other hand that $p_i(z_{nk})$ verifies

$$-\frac{\partial p_i(z_{nk})}{\partial t} - \Delta p_i(z_{nk}) = c_{nk}, \quad \text{in } Q,$$
$$p_i(z_{nk}) = 0, \quad \text{on } \Sigma,$$
$$p_i(z_{nk})(T) = 0, \quad \text{in } \Omega,$$

where $c_{nk} = e_i - a(z_{nk})p_i(z_{nk})$ is uniformly bounded in $L^2(Q)$ according to (88) and (8). Then, from the regularizing effect of the heat equation, $p_i(z_{nk})$ is bounded in $\Xi^{1,2}(Q)$.

Therefore, we can extract a subsequence of $(p_i(z_{nk}))$ (still called $p_i(z_{nk})$) such that

$$p_i(z_{nk}) \rightharpoonup q_i, \quad \text{weakly in } \Xi^{1,2}(Q). \tag{89}$$

Hence, using the compactness embedding of $\Xi^{1,2}(Q)$ into $L^2(0,T,H_0^1(\Omega))$, we have

$$p_i(z_{nk}) \to q_i \quad \text{strongly in } L^2(0,T,H_0^1(\Omega)),\ 1 \le i \le M. \tag{90}$$

From the energy inequality for $p_i(z_{nk})$ and (88), it follows that

$$\|p_i(z_{nk})(0)\|_{L^2(\Omega)} \le C(\Omega,T,K)\|e_i\|_{L^2(Q)}. \tag{91}$$

Therefore, passing to the limit in (76) while using (70), (89), (90), we get

$$-\frac{\partial q_i}{\partial t} - \Delta q_i + a(z)q_i = e_i, \quad \text{in } Q,$$
$$q_i = 0, \quad \text{on } \Sigma,$$
$$q_i(T) = 0, \quad \text{in } \Omega,$$

and in view of (91),

$$p_i(z_{nk})(0) \rightharpoonup q_i(0), \quad 1 \le i \le M \text{ weakly in } L^2(\Omega). \tag{92}$$

Thus, for each e_i $1 \le i \le M$, q_i is solution of (13). Hence, thanks to uniqueness of the solution of (13),

$$q_i(z) = p_i(z), \quad 1 \le i \le M. \tag{93}$$

Step 5. Since $\theta u_0(z_{nk}) \in \text{Span}(p_1(z_{nk})\chi_\omega, \dots, p_M(z_{nk})\chi_\omega)$ and satisfies (79d), using Lemma 2.3 with $H = L^2(\omega_T)$, $h^n = \theta u_0(z_{nk})$, $p_i^n = p_1(z_{nk})$ while taking into account (80d), (90) and (93), we deduce that there exists $\tilde{\alpha}_j \in \mathbb{R}$, $1 \le j \le M$ such that

$$\theta u_0(z_{nk}) \to \tilde{w}_1 = \sum_{j=1}^{M} \tilde{\alpha}_j p_j(z)\chi_\omega, \quad \text{strongly in } L^2(\omega_T).$$

Hence, $\frac{1}{\theta}$ being bounded in $L^\infty(Q)$ and $u_0(z_{nk})$ verifying (80c), it follows that

$$u_0(z_{nk}) \to \tilde{w} = \sum_{j=1}^{M} \tilde{\alpha}_j \frac{1}{\theta} p_j(z)\chi_\omega, \quad \text{strongly in } L^2(\omega_T).$$

Passing to the limit in (75), while using (80c), (90), (92) and (93), we have

$$-\int_{\Omega} y^0\, p_i(0)\, dx = \int_0^T\int_{\omega} \tilde{w}\, p_i\, dx\, dt, \quad 1 \le i \le M.$$

Therefore, the uniqueness of $u_0 \in \mathcal{U}_\theta$ which verifies (17), allows us to conclude that $u_0(z) = \tilde{w}$.

Next, since $u(z_{nk}) \in \mathcal{U}^\perp(z_{nk}) = \mathrm{Span}(p_1(z_{nk})\chi_\omega, \dots, p_M(z_{nk})\chi_\omega)^\perp$, we have

$$\int_0^T\int_{\omega} u(z_{nk})\, p_i(z_{nk})\, dx\, dt = 0, \quad 1 \le i \le M.$$

Consequently, passing to the limit in this identity while using (80e), (90) and (93), we deduce that

$$\int_0^T\int_{\omega} \tilde{u}\, p_i\, dx\, dt = 0, \quad 1 \le i \le M.$$

This means that $\tilde{u} \in \mathcal{U}^\perp = \mathrm{Span}(p_1\chi_\omega, \dots, p_M\chi_\omega)^\perp$.

Now, as $\rho(z_{nk}) \in V$ verifies (78) and (79b), if we apply inequality (24) to $\rho(z_{nk})$ we obtain that $\rho(z_{nk})$ is bounded in $(]\beta, T-\beta[; H^2(\Omega))$, $\forall \beta > 0$. Then, we have in particular, for every $\beta > 0$,

$$\rho(z_{nk}) \rightharpoonup \tilde{\rho}, \quad \text{weakly in } L^2(]\beta, T-\beta[\times\Omega),$$
$$\rho(z_{nk}) \rightharpoonup \tilde{\rho}, \quad \text{weakly in } L^2(]\beta, T-\beta[\times\Gamma).$$

This implies that

$$\rho(z_{nk}) \rightharpoonup \tilde{\rho}, \quad \text{weakly in } D'(Q),$$
$$\rho(z_{nk}) \rightharpoonup \tilde{\rho}, \quad \text{weakly in } D'(\Sigma).$$

Therefore setting $L^*_{nk}\rho(z_{nk}) = -\frac{\partial\rho(z_{nk})}{\partial t} - \Delta\rho(z_{nk}) + a(z_{nk})\rho(z_{nk})$ and using (70), we have $L^*_{nk}\rho(z_{nk}) \rightharpoonup L^*\rho$ weakly in $D'(Q)$. Hence, in view of (78), we deduce that

$$L^*\tilde{\rho} = 0, \quad \text{in } Q,$$
$$\tilde{\rho} = 0, \quad \text{on } \Sigma.$$

According to (79a) and the definition of the norm on V, we have

$$\|P_{nk}\rho(z_{nk}) - \rho(z_{nk})\|_{L^2(\omega_T)} \le C\left(\Omega, \omega, K, T, \sum_{i=1}^M \|e_i\|_{L^2(Q)}\right)\|y^0\|_{L^2(\Omega)}; \qquad (94)$$

Applying inequality (27) to $\rho(z_{nk})$ while taking into account (78) and (79b), we obtain

$$\left\|\frac{1}{\theta}\rho(z_{nk})\right\|_{L^2(Q)} \le C\left(\Omega, \omega, K, T, \sum_{i=1}^M \|e_i\|_{L^2(Q)}\right)\|y^0\|_{L^2(\Omega)}. \qquad (95)$$

Then, proceeding as in the Step 1 of the proof of Proposition 2.4, while using (94) and (95), we deduce that

$$\|P_{nk}\rho(z_{nk})\|_{L^2(\omega_T)} \leq C\left(\Omega, \omega, K, T, \sum_{i=1}^{M}\|e_i\|_{L^2(Q)}\right)\|y^0\|_{L^2(\Omega)}. \tag{96}$$

Therefore, $P_{nk}\,\rho(z_{nk})$ being in $\mathcal{U}(z_{nk})$, using Lemma 2.3 with $H = L^2(\omega_T), h^n = P_{nk}\,\rho(z_{nk}),\ p_i^n = p_i(z_{nk})$, while taking into account (90) and (93), we obtain

$$P_{nk}\,\rho(z_{nk}) \to \delta \in \mathrm{Span}(p_1(z)\chi_\omega, \ldots, p_M(z)\chi_\omega).$$

This means that $\delta \in \mathcal{U}$. Now, using (77), (80e) and (80b), we get

$$u(z_{nk}) = -\rho(z_{nk})\chi_\omega + P_{nk}\rho(z_{nk}) \rightharpoonup -\rho\chi_\omega + \delta = \tilde{u}, \quad \text{weakly in } L^2(\omega_T).$$

Observing that $P(\tilde{u}) = 0$ and $P(\delta) = \delta$ because $\tilde{u} \in \mathcal{U}^\perp$ and $\delta \in \mathcal{U}$, from $-\rho\chi_\omega + \delta = \tilde{u}$, we derive $-P(\rho) + \delta = 0$. This means that $\delta = P(\rho)$ and $\tilde{u} = -\rho\chi_\omega + P\rho = u$.

Therefore, relation (81) allows us to say that $\tilde{v} = u_0(z) + u = v$ and it results that the pair (v, y) verifies (9), (4) and (5).

3.2 Proof of the Compactness of S

The argument above show that, when z lies in bounded subset B of $L^2(Q)$, $S(z) = y(z)$ lies in bounded set of $W(0,T)$. Since $W(0,T)$ is compact in $L^2(Q)$, we deduce that $S(B)$ is relatively compact in $L^2(Q)$. Consequently, S is a compact operator.

3.3 Proof of the Boundedness of the Range of S

Let $z \in L^2(Q)$. Since $S(z) = y(z)$ is solution of (9) with $v(z)$ satisfying (69), we have

$$\|y(z)\|_{L^2(0,T;H_0^1(\Omega))} \leq C\left(\Omega, \omega, K, T, \sum_{i=1}^{M}\|e_i\|_{L^2(Q)}\right)\|y^0\|_{L^2(\Omega)}.$$

Hence, the embedding of $L^2(0,T;H_0^1(\Omega))$ into $L^2(Q)$ being continuous, it follows that

$$\|y(z)\|_{L^2(Q)} \leq C\left(\Omega, \omega, K, T, \sum_{i=1}^{M}\|e_i\|_{L^2(Q)}\right)\|y^0\|_{L^2(\Omega)}.$$

Finally, in view of Proposition 3.1, the hypotheses of Schauder fixed-point Theorem are satisfied. Consequently, the operator S has a fixed point y. The proof of Theorem 1.1 is then complete. □

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