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Global and Finite Convergence of a Generalized Newton Method for Absolute Value Equations

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Abstract We investigate an efficient method for solving the absolute value equation $Ax - |x| = b$ when the interval matrix $[A - I, A + I]$ is regular. A generalized Newton method which combines the semismooth and the smoothing Newton steps is proposed. We establish global and finite convergence of the method. Preliminary numerical results indicate that the generalized Newton method is promising.

Keywords Absolute value equation · Interval matrix · Generalized Newton method · Global and finite convergence

1 Introduction

We consider an efficient method for solving the absolute value equation (AVE)

$$Ax - |x| = b, \tag{1}$$

where $A \in R^{n\times n}$, $b \in R^n$ are given and $|\cdot|$ represents the absolute value componentwise. The study of the AVE is inspired from the interval linear equations [1] as well as from the well-known linear complementarity problem (LCP), which is a uniform framework for many mathematical programs [2, 3].

Theoretical analysis focuses on the theorem of alternatives, various equivalent reformulations, and the existence and nonexistence of solutions. Reference [1] provides a theorem of the alternatives for a more general form of AVE, $Ax + B|x| = b$, and

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enlightens the relation between the AVE and the interval matrix. In [4], the AVE is shown to be equivalent to the bilinear program, the generalized LCP, and the standard LCP if 1 is not an eigenvalue of A. Based on the LCP reformulation, sufficient conditions for the existence and nonexistence of solutions are given.

It is worth mentioning that any LCP can be reduced to the AVE [3], which owns a very special and simple structure. Hence how to solve the AVE directly attracts much attention. In [3], a finite succession of linear programs (SLP) is employed to solve a concave minimization formulation for the AVE. The method involves a linear program each step and terminates at a point which satisfies the necessary optimality condition. In the recent interesting paper [5] of Mangasarian, a semismooth Newton method is proposed for solving the AVE, which largely shortens the computation time than the SLP method. It shows that the semismooth Newton iterates are well defined and bounded when the singular values of A exceed 1. However, the global linear convergence of the method is only guaranteed under more stringent condition than the singular values of A exceed 1.

The concept of interval matrix arises from the linear interval equations [6], and we will show that it also relates closely to the AVE. Given two matrices $\underline{A} = (\underline{A}_{ij})$ and $\overline{A} = (\overline{A}_{ij})$, an interval matrix $A^I = [\underline{A}, \overline{A}] = \{A : \underline{A} \le A \le \overline{A}\}$, where $\underline{A} \le \overline{A}$ refers to $\underline{A}_{ij} \le \overline{A}_{ij}$ for each i, j. A^I is called regular if each $A \in A^I$ is nonsingular. Throughout the paper, we always assume that the interval matrix $[A - I, A + I]$ is regular, which includes the special case that the singular values of A exceed 1 considered in [5].

In Sect. 2, we establish global and finite convergence of a generalized Newton method proposed for the AVE. The method utilizes both the semismooth and the smoothing Newton steps, in which the semismooth Newton step guarantees the finite convergence and the smoothing Newton step contributes to the global convergence. Sect. 3 provides numerical experiments on a thousand AVEs of 1000 dimension, which indicate that our proposed generalized Newton method can efficiently solve large scale AVEs.

We sum up here briefly our notation. I refers to the identity matrix, and $\|\cdot\|$ represents the Euclidean norm. Given a matrix $A = (A_{ij})$, the matrix $|A|$ is defined by $|A|_{ij} = |A_{ij}|$ for each i, j. Let $\varrho(A)$ be the spectral radius of the matrix A. For $x \in R^n$, $\mathrm{sign}(x)$ denotes a vector with component 1, 0 or -1 corresponding to the component of x which is positive, zero or negative, respectively. We use $\mathrm{diag}(x)$ to denote a diagonal matrix corresponding to x.

2 Generalized Newton Method

Let us denote

$$F(x) = Ax - |x| - b. \tag{2}$$

The AVE is equivalent to the nonsmooth equations $F(x) = 0$.

Proposition 2.1 *The AVE* (1) *is uniquely solvable for any $b \in R^n$ if the interval matrix $[A - I, A + I]$ is regular.*

Proof Since $[A-I, A+I]$ is regular, we know that $(A-I)^{-1}$ exists and hence 1 is not an eigenvalue of A. By Proposition 2 of [4], the AVE (1) can be reduced to the following LCP:

$$0 \le z \perp (A+I)(A-I)^{-1}z + q \ge 0,$$

where

$$q = ((A+I)(A-I)^{-1} - I)b,$$
$$z = (A-I)x - b.$$

By Theorem 1.2 of [6], $(A+I)(A-I)^{-1}$ is a P-matrix, which implies that the LCP has a unique solution for any $b \in R^n$ [2]. From the relation between the AVE (1) and the LCP, we can easily deduce that the AVE (1) is uniquely solvable for any $b \in R^n$. □

According to Corollary 3.2 and Theorem 4.2 of [7], the interval matrix $[A-I, A+I]$ is regular if one of the following conditions holds:

(i) $\varrho(|A^{-1}|) < 1$.
(ii) The singular values of A exceed 1.

The interval matrix $[A-I, A+I]$ is regular is weaker than that the singular values of A exceed 1, since (ii) cannot include (i), which can be seen from the following example.

Example 2.1 Let

$$A = \begin{pmatrix} 1.5 & -3 \\ 0 & 1.5 \end{pmatrix}.$$

By simple computation, the spectral radius $\varrho(|A^{-1}|) = 0.6667 < 1$ and hence (i) holds. However, (ii) does not hold, since the minimal singular value of A is 0.3860, which is smaller than 1.

In fact, (ii) does not imply (i); see Example 2.2 below.

Example 2.2 Let

$$A = \begin{pmatrix} 1 & -0.01 \\ 0.01 & 1 \end{pmatrix}.$$

The minimal singular value of A equals 1.0001 which exceeds 1, while the spectral radius $\varrho(|A^{-1}|) = 1.0099 > 1$.

We also provide an interval matrix $[A-I, A+I]$ satisfying the regularity condition, but neither condition (i) or (ii) holds in the following example.

Example 2.3 Let

$$A = \begin{pmatrix} 1 & 3 \\ -0.01 & 1 \end{pmatrix}.$$

Both conditions (i) and (ii) fail, since the spectral radius $\varrho(|A^{-1}|) = 1.1390 > 1$ and the minimal singular value of A is $0.0973 < 1$. However, $[A - I, A + I]$ is regular, since $C_1 = (A - I)^{-1}(A + I)$, $C_2 = C_1^{-1}$, $C_3 = (A - \mathrm{diag}(1, -1))^{-1}(A + \mathrm{diag}(1, -1))$, $C_4 = C_3^{-1}$ are P-matrices, and by Theorem 5.1 (R) and (B1) of [6], the interval matrix $[A - I, A + I]$ is regular if and only if the four matrices C_1, C_2, C_3, C_4 are P-matrices.

It is obvious that F in (2) is a globally Lipschitzian function in R^n. Hence, F is differentiable almost everywhere. Let $\mathcal{D}_F$ be the set of points at which F admits differentiability. The Clarke generalized Jacobian of F at x, denoted by $\partial F(x)$, is defined in [8] by

$$\partial F(x) = \mathrm{co}\Big\{ \lim_{x^k \to x,\ x^k \in \mathcal{D}_F} F'(x^k) \Big\},$$

where co denotes the convex hull and $F'(x)$ is the usual derivative whenever F is differentiable at x. It is clear that

$$\partial F(x) = \{A - D_x\}. \tag{3}$$

Here,

$$D_x = \mathrm{diag}(d_x), \tag{4}$$

for all $d_x \in R^n$ such that, for each $i = 1, 2, \ldots, n$,

$$(d_x)_i = \begin{cases} 1, & \text{if } x_i > 0, \\ \delta, & \text{if } x_i = 0, \\ -1, & \text{if } x_i < 0, \end{cases}$$

where δ is any number in the interval $[-1, 1]$.

We know from [9] that F is semismooth in R^n, since F is locally Lipschitzian and

$$\lim_{\substack{V \in \partial F(x + th') \\ h' \to h,\ t \downarrow 0}} \{Vh'\}$$

exists for any $h \in R^n$. Furthermore, a semismooth Newton method is proposed there to overcome the nonsmoothness of F, by utilizing the iteration

$$x^{k+1} = x^k - V_k^{-1} F(x^k), \quad \text{for any invertible } V_k \in \partial F(x^k).$$

The above iteration can be simplified as

$$x^{k+1} = (A - D_{x^k})^{-1} b, \tag{5}$$

by setting $V_k = A - D_{x^k}$ from (3) and substituting $F(x^k)$ according to (2).

One special element in $\{D_x\}$ is

$$D_x^0 = \mathrm{diag}(\mathrm{sign}(x)). \tag{6}$$

If we choose $D_{x^k} = D^0_{x^k}$, then the iteration (5) can be expressed as

$$x^{k+1} = (A - D^0_{x^k})^{-1} b,$$

which coincides to the semismooth iteration employed by Mangasarian [5].

Lemma 2.1 *Suppose that the interval matrix* $[A - I, A + I]$ *is regular and* x^* *is the unique solution of the AVE* (1). *Then, for any* x *such that*

$$x_i^* > 0 \implies x_i > 0 \quad \textit{and} \quad x_i^* < 0 \implies x_i < 0, \quad \forall i \in \{1, 2, \ldots, n\}, \tag{7}$$

the semismooth iteration (5) *reaches* x^* *in one iteration, i.e.,*

$$x^* = (A - D_x)^{-1} b.$$

Proof It is easy to see that, for any x satisfying the 'sign match' property (7),

$$|x^*| = D_{x^*} x^* = D_x x^*,$$

and hence,

$$\begin{aligned} 0 = F(x^*) &= Ax^* - |x^*| - b \\ &= Ax^* - D_x x^* - b \\ &= (A - D_x)x^* - b. \end{aligned}$$

Since $[A - I, A + I]$ is regular, any element $A - D_x \in \partial F(x)$ is invertible. We get

$$x^* = (A - D_x)^{-1} b,$$

as desired. □

Remark 2.1 Note that the 'sign match' property (7) holds if x is sufficiently near x^*. Hence, by Lemma 2.1, global and finite convergence is then obvious for any algorithm which combines the semismooth iterate with any other globally convergent algorithm for the AVE. That is, at each iteration the algorithm checks whether the semismooth formula solves the AVE. If not, it provides some sufficient decrease in a suitable objective function.

The smoothing method for solving $F(x) = 0$, with the framework mentioned below, is easy to implement and owns global convergence under mild assumptions.

- Construct a smoothing function $f(\cdot, \epsilon)$ to approximate F.

- Use f_x to construct the Newton iterate

$$x^{k+1} = x^k - t_k f_x(x^k, \epsilon_k)^{-1} F(x^k),$$

such that the function value of $\|f(\cdot, \epsilon_k)\|^2$ has a sufficient descent.
- Force the smoothing parameter ϵ_k tending to 0 by some delicate strategy, which leads to a solution of $F(x) = 0$.

Let $\rho : R \to [0, \infty)$ be any density function satisfying

$$\kappa := \int_R |s|\rho(s)ds < \infty.$$

Reference [10] provides the smoothing function $Q(t, \epsilon)$ of $|t|$ by

$$Q(t, \epsilon) := \int_R |t - \epsilon s|\rho(s)ds, \quad t \in R,\ \epsilon > 0. \tag{8}$$

It is worth mentioning that $Q(t, \epsilon)$ is easy to compute, if some concrete density functions are chosen [10]. Since $|x|$ is separable, the smoothing function $f(x, \epsilon)$ of $F(x)$ can be constructed by

$$f(x, \epsilon) = Ax - (Q(x_1, \epsilon), Q(x_2, \epsilon), \ldots, Q(x_n, \epsilon))^T - b. \tag{9}$$

Lemma 2.2

(i) *For any $x \in R^n$ and $\epsilon > 0$,*

$$\|f(x, \epsilon) - F(x)\| \le \sqrt{n}\kappa\epsilon.$$

(ii) *For any $\epsilon > 0$, $f(\cdot, \epsilon)$ is continuously differentiable on R^n and*

$$f_x(x, \epsilon) = A - D_{x,\epsilon},$$

where $D_{x,\epsilon} = \mathrm{diag}(d_{x,\epsilon})$ for some $d_{x,\epsilon} \in [-1, 1]^n$.

Proof (i) For any $t \in R$ and $\epsilon > 0$, we derive from (8) that $|Q(t, \epsilon) - |t|| \le \kappa\epsilon$, and hence,

$$\begin{aligned}\|f(x, \epsilon) - F(x)\| &= \|(|x_1| - Q(x_1, \epsilon), |x_2| - Q(x_2, \epsilon), \ldots, |x_n| - Q(x_n, \epsilon))\| \\ &\le \sqrt{n}\kappa\epsilon.\end{aligned}$$

(ii) For any $\epsilon > 0$, $Q(\cdot, \epsilon)$ is continuously differentiable on R and

$$Q_t(t, \epsilon) = \int_{-\infty}^{t/\epsilon} \rho(s)ds - \int_{t/\epsilon}^{\infty} \rho(s)ds \in [-1, 1]. \tag{10}$$

We have from (9) that

$$f_x(x, \epsilon) = A - D_{x,\epsilon}$$

where $D_{x,\epsilon} = \operatorname{diag}(d_{x,\epsilon})$ for $d_{x,\epsilon} = (Q_{x_1}(x_1,\epsilon), Q_{x_2}(x_2,\epsilon), \ldots, Q_{x_n}(x_n,\epsilon))^T \in [-1,1]^n$. □

Let $f(x,\epsilon)$ be any smoothing function defined by (9). We then denote

$$\Theta(x) = \frac{1}{2}\|F(x)\|^2,$$

$$\theta_k(x) = \frac{1}{2}\|f(x,\epsilon_k)\|^2.$$

We propose a generalized Newton method for solving the AVE (1), which is a hybrid method of the semismooth Newton method [9] and the smoothing Newton method [11].

Algorithm 2.1 Generalized Newton Method.

Given $\rho, \alpha, \eta \in (0,1)$ and an initial point $x^0 \in R^n$, choose a scalar $\sigma \in (0, \frac{1}{2}(1-\alpha))$. Let $\beta_0 = \|F(x^0)\|$ and $\epsilon_0 = \frac{\alpha}{2\sqrt{n}\kappa}\beta_0$. For $k \geq 0$, do the steps below:

Step 1. Choose any D_{x^k} satisfying (4) and find a solution $\hat{x}^{k+1}$ by solving

$$(A - D_{x^k})x = b. \tag{11}$$

If $\|F(\hat{x}^{k+1})\| \leq \eta\beta_k$, let $x^{k+1} = \hat{x}^{k+1}$. Otherwise, go to Step 2.

Step 2. Find a solution d^k by solving the system of linear equations

$$F(x^k) + f_x(x^k,\epsilon_k)d = 0. \tag{12}$$

Let m_k be the smallest nonnegative integer m such that

$$\theta_k(x^k + \rho^m d^k) - \theta_k(x^k) \leq -2\sigma\rho^m\Theta(x^k). \tag{13}$$

Set $t_k = \rho^{m_k}$ and $x^{k+1} = x^k + t_k d^k$.

Step 3. 3.1 If $\|F(x^{k+1})\| = 0$, terminate.

3.2 If

$$0 < \|F(x^{k+1})\| \leq \max\{\eta\beta_k, \alpha^{-1}\|F(x^{k+1}) - f(x^{k+1},\epsilon_k)\|\}, \tag{14}$$

let

$$\beta_{k+1} = \|F(x^{k+1})\| \quad \text{and} \quad \epsilon_{k+1} = \min\left\{\frac{\alpha}{2\sqrt{n}\kappa}\beta_{k+1}, \frac{\epsilon_k}{2}\right\}. \tag{15}$$

3.3 Otherwise, let $\beta_{k+1} = \beta_k$ and $\epsilon_{k+1} = \epsilon_k$.

Lemma 2.3 *The level set*

$$L = \{x \in R^n \mid \Theta(x) \leq \Gamma\}$$

is bounded for any $\Gamma > 0$ *if the interval matrix* $[A - I, A + I]$ *is regular.*

Proof We follow the analogous arguments used in Proposition 3 [5]. Suppose on the contrary that the level set L is unbounded for a certain $\Gamma > 0$. Then, there is an infinite sequence $\{x^k\}$ such that $\|x^k\| \to \infty$ and $\Theta(x^k) \le \Gamma$ for any k. Note that

$$\Theta(x) = \frac{1}{2}\|Ax - |x| - b\|^2 = \frac{1}{2}\|Ax - D_x^0 x - b\|^2,$$

where D_x^0 can only be chosen from a finite number of possible configurations. Thus, there exists an infinite subsequence $\{x^{k_j}\} \subset \{x^k\}$ such that

$$\left\{\frac{x^{k_j}}{\|x^{k_j}\|}\right\} \to \tilde{x}, \quad \|\tilde{x}\| = 1, \qquad D^0_{x^{k_j}} \equiv \bar{D} \in [-I, I].$$

Consequently,

$$\frac{\Gamma}{\|x^{k_j}\|^2} \ge \frac{\Theta(x^{k_j})}{\|x^{k_j}\|^2} = \frac{1}{2}\left\| A\frac{x^{k_j}}{\|x^{k_j}\|} - \bar{D}\frac{x^{k_j}}{\|x^{k_j}\|} - \frac{b}{\|x^{k_j}\|}\right\|^2 \ge 0.$$

Letting $k_j \to \infty$ yields

$$(A - \bar{D})\tilde{x} = 0.$$

Since $A - \bar{D} \in [A - I, A + I]$ and the interval matrix $[A - I, A + I]$ is regular, we know that $(A - \bar{D})^{-1}$ exists and hence $\tilde{x} = 0$, contradicting to the fact that $\|\tilde{x}\| = 1$. This completes the proof. □

Theorem 2.1 *Suppose that the interval matrix $[A - I, A + I]$ is regular. Then, for any starting point $x^0 \in R^n$, Algorithm* 2.1 *is well defined and the generated sequence $\{x^k\}$ globally and finitely converges to the unique solution of the AVE* (1).

Proof By Proposition 2.1, the AVE (1) has a unique solution x^* since the interval matrix $[A - I, A + I]$ is regular.

For any $x \in R^n$ and $\epsilon > 0$, both $A - D_x$ and $f_x(x, \epsilon)$ given in (4) and (ii) of Lemma 2.2 are nonsingular, since they are in the regular interval matrix $[A - I, A + I]$. Then, (11) and (12) of Algorithm 2.1 are uniquely solvable. By Lemma 3.1 [11], there exists a finite nonnegative integer m_k such that (13) holds. Hence, Algorithm 2.1 is well defined.

Now, for the finite and global convergence, by the construction of the algorithm, it is sufficient to show that Algorithm 2.1 terminates within finite steps. Suppose on the contrary that Algorithm 2.1 generates an infinite sequence $\{x^k\}$. By noting Lemma 2.2 and Lemma 2.3, we get from Theorem 3.1 of [11] that

$$\Theta(x^k) \le (1 + \alpha)^2 \Theta(x^0), \quad \text{for any } k,$$

and

$$\lim_{k\to\infty} F(x^k) = 0.$$

By Lemma 2.3, $\{x^k\}$ is bounded and hence an accumulation point exists. Let $\bar{x}$ be any accumulation point of $\{x^k\}$ and we get $F(\bar{x}) = 0$. Thus, the whole sequence $\{x^k\}$ converges to $x^* = \bar{x}$. Choose $\bar{k}$ to be the smallest integer such that $x^{\bar{k}}$ satisfies (7). Note that the regularity of the interval matrix guarantees that $A - D_{x^{\bar{k}}}$ is invertible. We have $\hat{x}^{\bar{k}+1} = x^*$ by Lemma 2.1. It is then easy to see that

$$0 = \|F(x^*)\| = \|F(\hat{x}^{\bar{k}+1})\| \leq \eta\beta_{\bar{k}}.$$

By the construction of Step 1 and the fact that the AVE (1) has a unique solution x^*,

$$x^{\bar{k}+1} = \hat{x}^{\bar{k}+1} = x^*.$$

Hence Algorithm 2.1 terminates at the iterate $\bar{k}+1$, which contradicts to that $\{x^k\}$ is an infinite sequence. This completes the proof. □

Remark 2.2 Omitting Step 1 in Algorithm 2.1, we get the smoothing Newton method [11]. According to Theorem 3.2 [11], under the assumption that $[A - I, A + I]$ is regular, it is easy to show that the smoothing Newton method is well defined and that the generated sequence $\{x^k\}$ globally and superlinearly converges to the unique solution of the AVE (1).

3 Numerical Experiments

In this section, we compare our proposed generalized Newton method (Algorithm 2.1) with the smoothing Newton method mentioned in Remark 2.2 and the semismooth Newton method [5], i.e., the next iterate point x^{k+1} is obtained by solving

$$(A - D^0_{x^k})x = b. \tag{16}$$

In our numerical experiment, we always choose $D^0_{x^k}$ in Step 1 of Algorithm 2.1. We randomly generate 1000 solvable AVEs following the procedure in [5]. Note that the singular values of each A exceed 1 and hence $[A - I, A + I]$ is regular.

In order that our numerical results can be repeated, we use the Matlab command

$$\text{rand}(\text{state}, i), \quad i = 1, 2, \ldots, 1000,$$

when generating the random matrix A. Numerical results were obtained by using Matlab 7 on a 1 G RAM, 2.60 Ghz Pentium(R) 4 processor. We stop the iteration of each algorithm if

$$\|Ax - |x| - b\| \leq 10^{-6}$$

or if the number of iterations exceeds 50. We say that the AVE is solved by a certain algorithm if it stops before 50 iterations.

Table 1 The semismooth Newton method

Number of problems solved	980
Average of $\|Ax - \|x\| - b\|$	0.0585×10^{-6}
Average of iterations	2.9429
Average CPU time (s)	8.6006

For each test AVE problem, we start the three algorithms from the same initial point

$$x^0 = \text{rand}(n, 1) - \text{rand}(n, 1).$$

We set the parameters in Algorithm 2.1 and the related smoothing Newton method described in Remark 2.1 as

$$\rho = 0.5, \qquad \alpha = 10^{-3}, \qquad \eta = 0.5, \qquad \sigma = 0.25(1 - \alpha).$$

Three well-known density functions are used to construct the smoothing functions $f(x, \epsilon)$ by (9) as follows:

(i) The neural networks smoothing function, by using

$$\rho(s) = \frac{e^{-s}}{(1 + e^{-s})^2}.$$

(ii) The Chen-Harker-Kanzow-Smale (CHKS) smoothing function, by using

$$\rho(s) = \frac{2}{(s^2 + 4)^{\frac{3}{2}}}.$$

(iii) The uniform smoothing function, by using

$$\rho(s) = \begin{cases} 1, & \text{if } -\frac{1}{2} \le s \le \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

We report the numerical results in Tables 1–3 for the semismooth Newton, the smoothing Newton and the generalized Newton methods, respectively. We record the number of problems solved, i.e., the test AVEs for which the algorithm meets the prescribed accuracy before 50 iterations. For the problems solved, we provide the average of $\|Ax - |x| - b\|$, the number of iterations as well as the CPU time. Here the average CPU time refers to the average time for generating and solving each of the problems solved. Average final smoothing parameters for the smoothing and the generalized Newton methods are given, as well as the total smoothing iterations for the generalized Newton method.

The generalized Newton method and the smoothing Newton method solve more than 998 of the total 1000 AVEs, while the semismooth Newton method solves 980 AVEs. The generalized Newton method and the semismooth Newton method converge very fast, within 3 iterations in average. For the generalized Newton method, the three smoothing functions perform well and similarly in our numerical test. In comparison, for the smoothing Newton method, the CHKS smoothing function and

Table 2 The smoothing Newton method

Smoothing function	(i)	(ii)	(iii)
Number of problems solved	999	998	999
Average of $\|Ax - \|x\| - b\|$	0.2531×10^{-6}	0.0127×10^{-6}	0.0066×10^{-6}
Average of iterations	6.6553	3.0180	2.9329
Average CPU time (s)	11.4656	8.6968	8.6315
Average final smoothing parameter	0.0289×10^{-10}	0.0010×10^{-10}	0.0043×10^{-10}

Table 3 The generalized Newton method (Algorithm 2.1)

Smoothing function	(i)	(ii)	(iii)
Number of problems solved	999	999	999
Average of $\|Ax - \|x\| - b\|$	0.0615×10^{-6}	0.0614×10^{-6}	0.0614×10^{-6}
Average of iterations	2.9749	2.9530	2.9489
Total smoothing iterations	43	24	20
Average CPU time (s)	8.7887	8.6140	8.6392
Average final smoothing parameter	0.0070×10^{-10}	0.0049×10^{-10}	0.0389×10^{-10}

the uniform smoothing function perform better than the neural networks smoothing function, in terms of the average of iterations and the CPU time.

Note that the test problems can be fixed by i in the command rand(state, i). We provide the test problems that cannot reach the accuracy 10^{-6} before 50 iterations by the three methods, respectively.

- $i = 142$ for three methods,
- $i = 807$ for the smoothing method employing the smoothing function (ii),
- $i = 21, 30, 212, 236, 307, 342, 360, 425, 443, 456, 478, 519, 538, 685, 807, 945, 981, 982, 999$ for the semismooth Newton method.

For the above test problems that failed, we allow a larger number of iterations to see the performance of the three methods. It is interesting to find that a larger number of iterations for the semismooth Newton method does not affect the accuracy for all the 20 problems above. Their average accuracy is 6.0210×10^{-6} after 100 iterations. In fact, after 2 or 3 iterations, the accuracy $\|Ax - |x| - b\|$ keeps the same and has no improvement for each of the 20 problems.

The smoothing method employing the smoothing function (ii) can solve the problem $i = 807$ if allowed 61 iterations, with the accuracy $\|Ax - |x| - b\|$, CPU time (s), and final smoothing parameter 0.9856×10^{-6}, 59.6563, 7.9673×10^{-12} respectively.

The three methods fail to reach the accuracy 10^{-6} for the problem $i = 142$, even if we enlarge the number of iterations to 100, 200 and 500. The semismooth Newton method has no improvement in accuracy after 2 iterations, while the smoothing and the generalized Newton methods have tiny up-and-down fluctuations in accuracy after a few iterations. Tables 4–6 record the numerical results for this problem employing the three methods when the number of iterations reaches 100. We mention that no

Table 4 Problem $i = 142$ by the semismooth Newton method (100 iterations)

$\|Ax - \|x\| - b\|$	3.4173×10^{-5}
CPU time (s)	80.2969

Table 5 Problem $i = 142$ by the smoothing Newton method (100 iterations)

Smoothing function	(i)	(ii)	(iii)
$\|Ax - \|x\| - b\|$	1.6542×10^{-6}	1.6595×10^{-6}	1.5896×10^{-6}
CPU time (s)	84.7344	87.3750	87.1406
Final smoothig parameter	1.8293×10^{-11}	1.2824×10^{-11}	1.0521×10^{-10}

Table 6 Problem $i = 142$ by the generalized Newton method (100 iterations)

Smoothing function	(i)	(ii)	(iii)
$\|Ax - \|x\| - b\|$	1.6277×10^{-6}	1.5222×10^{-6}	1.5222×10^{-6}
CPU time (s)	156.3906	159.8906	158
Smoothing iterations	99	99	99
Final smoothing parameter	2.0168×10^{-11}	1.3357×10^{-11}	1.0685×10^{-10}

essential changes in the performance of each algorithm occur if a larger number of iterations, 200 or 500, is adopted.

The semismooth Newton method is very easy to implement and converges fast locally. In order to further understand the semismooth Newton method, we also did an additional computational experiment on a damped semismooth Newton method, which adds a line search for deciding the stepsize along the semismooth Newton direction. We find that the damped semismooth Newton method has good computational behavior. However, we cannot prove its global convergence, whose difficulty roots from the non-Fréchet differentiability of the function Θ. Therefore, one interesting question is whether we can design a globally convergent algorithm based on the semismooth Newton iteration with a sufficient decrease stepsize condition. We will consider this question in our future research.

4 Conclusions

We propose a generalized Newton method, which globally and finitely converges to the unique solution of the AVE (1) under the assumption that $[A - I, A + I]$ is regular. This assumption is weaker than the singular values of A exceed 1 in Proposition 3 [4]. Numerical experiments on 1000 randomly generated 1000-dimensional AVEs indicate that the generalized Newton method is stable and converges fast.

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