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Regularization Algorithms for Solving Monotone Ky Fan Inequalities with Application to a Nash-Cournot Equilibrium Model

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Abstract We make use of the Banach contraction mapping principle to prove the linear convergence of a regularization algorithm for strongly monotone Ky Fan inequalities that satisfy a Lipschitz-type condition recently introduced by Mastroeni. We then modify the proposed algorithm to obtain a line search-free algorithm which does not require the Lipschitz-type condition. We apply the proposed algorithms to implement inexact proximal methods for solving monotone (not necessarily strongly monotone) Ky Fan inequalities. Applications to variational inequality and complementarity problems are discussed. As a consequence, a linearly convergent derivative-free algorithm without line search for strongly monotone nonlinear complementarity problem is obtained. Application to a Nash-Cournot equilibrium model is discussed and some preliminary computational results are reported.

Keywords Ky Fan inequality · Variational inequality · Complementarity problem · Linear convergence · Lipschitz property · Proximal point algorithm · Equilibria · Nash-Cournot model

1 Introduction

Let C be a nonempty closed convex set in a real Hilbert space $\mathcal{H}$ and $f : C \times C \to \mathbb{R}$. We consider the following problem:

$$\text{(P)} \quad \text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \quad \text{for all } y \in C.$$

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We will refer to this problem as the Ky Fan inequality due to his results in this field [1]. Problem (P) is very general in the sense that it includes, as special cases, the optimization problem, the variational inequality, the saddle point problem, the Nash equilibrium problem in noncooperative games, the Kakutani fixed point problem and others (see for instance [2–9] and the references quoted therein). The interest of this problem is that it unifies all these particular problems in a convenient way. Moreover, many methods devoted to solving one of these problems can be extended, with suitable modifications, to solving Problem (P). It is worth mentioning that when f is convex and subdifferentiable on C with respect to the second variable, then (P) can be formulated as a generalized variational inequality of the form

$$\text{Find } x^* \in C,\ z^* \in \partial_2 f(x^*, x^*) \text{ such that } \langle z^*, y - x^*\rangle \geq 0, \quad \text{for all } y \in C,$$

where $\partial f_2(x^*, x^*)$ denotes the subdifferential of $f(x^*, .)$ at x^*.

In recent years, methods for solving Problem (P) have been studied extensively. One of the most popular methods is the proximal point method. This method was introduced first by Martinet [10] for variational inequalities and then was extended by Rockafellar [11] for finding the zero point of a maximal monotone operator. Moudafi [6] and Konnov [12] further extended the proximal point method to Problem (P) with monotone and weakly monotone bifunctions respectively.

Another solution-approach to Problem (P) is the auxiliary problem principle. This principle was introduced first to optimization problems by Cohen [13] and then extended to variational inequalities in [14]. Recently, Mastroeni [4] further extended the auxiliary problem principle to Problem (P) involving strongly monotone bifunctions satisfying a certain Lipschitz-type condition. Noor [8] used the auxiliary problem principle to develop iterative algorithms for solving (P) where the bifunctions f were supposed to be partially relaxed strongly monotone.

Other solution methods well developed in mathematical programming and variational inequalities such as the gap function, extragradient and bundle methods recently have been extended to Problem (P) [5, 9, 12, 15].

In this paper, first we make use of the Banach contraction mapping principle to prove linear convergence of a regularization algorithm for strongly monotone Ky Fan inequalities that satisfy a Lipschitz-type condition introduced in [4]. Then, we apply the algorithm to strongly monotone Lipschitzian variational inequalities. As a consequence, we obtain a new linearly convergent derivative-free algorithm for strongly monotone complementarity problems. The obtained linear convergence rate allows the algorithm to be coupled with inexact proximal point methods for solving monotone (not necessarily strong) problem (P) satisfying the Lipschitz-type condition introduced in [4]. Finally, we propose a line-search free algorithm for the strong monotone problem (P) which does not require the Lipschitz-type condition as the algorithm presented in Sect. 2.

The rest of the paper is organized as follows. In Sect. 2, we describe an algorithm for a strongly monotone problem (P) and prove its linear convergence-rate. This algorithm is then applied in Sect. 3 to strongly monotone variational inequalities and complementarity problems. A new derivative-free linearly convergent algorithm without line search for strongly monotone complementarity problems is described at the end of this section. Section 4 is devoted to present an algorithm which does not require the above mentioned Lipschitz-type condition. In Sect. 5, we apply the algorithms obtained in the Sects. 3 and 4 to implement inexact proximal point methods for solving monotone (not necessarily strong) Problem (P). We close the paper with some computational experiments and results for a Nash-Cournot equilibrium model.

2 Linearly Convergent Algorithm

First of all, we recall the following well-known definitions on monotonicity that we need in the sequel.

Definition 2.1 (See e.g. [2]) Let $f : C \times C \to \mathbb{R} \cup \{+\infty\}$. The bifunction f is said to be monotone on C if $f(x, y) + f(y, x) \leq 0$, for all $x, y \in C$. It is said to be strongly monotone on C with modulus $\tau > 0$ if $f(x, y) + f(y, x) \leq -\tau\|x - y\|^2$, for all $x, y \in C$.

Throughout the paper we suppose that the bifunction f satisfies the following blanket assumption.

Assumption A For each $x \in C$, the function $f(x, \cdot)$ is proper, closed convex and subdifferentiable on C with respect to the second variable.

For each $x \in C$, we define the mapping S by taking

$$S(x) := \underset{y \in C}{\operatorname{argmin}}\{\rho f(x, y) + (1/2)\|y - x\|^2\}, \tag{1}$$

where $\rho > 0$. As usual, we refer to ρ as a regularization parameter. Since the objective function is strongly convex, problem (1) admits a unique solution. Thus the mapping S is well defined and single valued.

The following lemma can be found, for example, in [4] (see also [15]).

Lemma 2.1 *Let S be defined by* (1). *Then, x^* is a solution to* (P) *if and only if* $x^* = S(x^*)$.

Lemma 2.1 suggests an iterative algorithm for solving (P) by taking $x^{k+1} = S(x^k)$. It has been proved in [4] that, with suitable values of the regularization parameter ρ, the sequence $\{x^k\}_{k \geq 0}$ converges strongly to the unique solution of (P) when f is strongly monotone and satisfies the following Lipschitz-type condition introduced by Mastroeni in [4].

There exists constants $L_1 > 0$ and $L_2 > 0$ such that

$$\begin{aligned} &f(x, y) + f(y, z) \geq f(x, z) - L_1\|x - y\|^2 - L_2\|y - z\|^2, \\ &\forall x, y, z \in C. \end{aligned} \tag{2}$$

Applying this inequality with $x = z$, we obtain

$$f(x, y) + f(y, x) \geq -(L_1 + L_2)\|x - y\|^2, \quad \forall x, y \in C.$$

Thus, if in addition f is strongly monotone on C with modulus τ, then $\tau \le L_1 + L_2$. For convenience of presentation, we refer to L_1 and L_2 as the Lipschitz constants for f.

The following theorem shows that the sequence $\{x^k\}_{k\ge 0}$ defined by $x^{k+1} = S(x^k)$ linearly converges to the unique solution of (P) under the same condition as in [4].

Theorem 2.1 *Suppose that f is strongly monotone on C with modulus τ and satisfies the Lipschitz-type condition* (2)*. Then, for any starting point $x^0 \in C$, the sequence $\{x^k\}_{k\ge 0}$ defined by*

$$x^{k+1} := \operatorname*{argmin}_{y\in C}\{\rho f(x^k, y) + (1/2)\|y - x^k\|^2\} \tag{3}$$

satisfies

$$\|x^{k+1} - x^*\|^2 \le \alpha\|x^k - x^*\|^2, \quad \forall k \ge 0, \tag{4}$$

provided $0 < \rho \le 1/(2L_2)$, where x^* is the unique solution of (P) *and $\alpha := 1 - 2\rho(\tau - L_1)$.*

Proof For each $k \ge 0$, let

$$f_k(x) := \rho f(x^k, x) + (1/2)\|x - x^k\|^2.$$

Then, by the convexity of $f(x^k, \cdot)$, the function f_k is strongly convex on C with modulus 1, which implies

$$f_k(x^{k+1}) + (w^k)^T(x - x^{k+1}) + (1/2)\|x - x^{k+1}\|^2 \le f_k(x), \quad \forall x \in C, \tag{5}$$

where $w^k \in \partial f_k(x^{k+1})$. Since x^{k+1} is the solution of problem (3), $(w^k)^T(x - x^{k+1}) \ge 0$ for every $x \in C$. Thus, from (5), it follows that

$$f_k(x^{k+1}) + (1/2)\|x - x^{k+1}\|^2 \le f_k(x), \quad \forall x \in C. \tag{6}$$

Applying (6) with $x = x^*$ and using the definition of f_k, we obtain

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\le 2\rho[f(x^k, x^*) - f(x^k, x^{k+1})]\\ &\quad + \|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2.\end{aligned} \tag{7}$$

Since f is strongly monotone on C with modulus τ,

$$f(x^k, x^*) \le -f(x^*, x^k) - \tau\|x^k - x^*\|^2.$$

Substituting this inequality into (7), we have

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\le (1 - 2\rho\tau)\|x^k - x^*\|^2\\ &\quad + 2\rho[-f(x^*, x^k) - f(x^k, x^{k+1})] - \|x^{k+1} - x^k\|^2.\end{aligned} \tag{8}$$

Now, applying the Lipschitz-type condition (2) with $x = x^*$, $y = x^k$, and $z = x^{k+1}$, we obtain

$$-f(x^k, x^{k+1}) - f(x^*, x^k) \leq -f(x^*, x^{k+1}) + L_1\|x^* - x^k\|^2 + L_2\|x^k - x^{k+1}\|^2 \\ \leq L_1\|x^* - x^k\|^2 + L_2\|x^k - x^{k+1}\|^2. \quad (9)$$

The latter inequality in (9) follows from $f(x^*, x^{k+1}) \geq 0$, since x^* is the solution of (P). Substituting into (8), we obtain

$$\|x^{k+1} - x^*\|^2 \leq [1 - 2\rho(\tau - L_1)]\|x^k - x^*\|^2 - (1 - 2\rho L_2)\|x^{k+1} - x^k\|^2. \quad (10)$$

By the assumption $0 < \rho \leq 1/(2L_2)$, it follows from (10) that

$$\|x^{k+1} - x^*\|^2 \leq \big[1 - 2\rho(\tau - L_1)\big]\|x^k - x^*\|^2, \quad (11)$$

which proves the theorem. □

The following corollary is immediate from Theorem 2.1.

Corollary 2.1 *Let* $L_1 < \tau$ *and* $0 < \rho \leq 1/(2L_2)$. *Then*,

$$\|x^{k+1} - x^*\| \leq r\|x^k - x^*\|, \quad \forall k \geq 0,$$

where $0 < r := \sqrt{1 - 2\rho(\tau - L_1)} < 1$.

Remark 2.1 Since $\tau \leq L_1 + L_2$ and $0 < \rho \leq 1/(2L_2)$, it is easy to see that $2\rho(\tau - L_1) < 1$. Thus, r attains its minimal value at $\rho = 1/(2L_2)$.

Based upon Theorem 2.1 and Corollary 2.1, we can develop a linearly convergent algorithm for solving problem (P) where f is τ-strongly monotone on C and satisfies (2) with positive constants L_1, L_2 such that $L_1 < \tau$. As usual, we call a point $x \in C$ an ε-solution to (P) if $\|x - x^*\| \leq \varepsilon$, where x^* is an exact solution of (P).

Algorithm A1 (Strongly Monotone Problem)
Initialization. Choose a tolerance $\varepsilon \geq 0$ and $0 < \rho \leq 1/(2L_2)$. Take $x^0 \in C$.
Iteration k, $k = 0, 1, \ldots$. Execute Steps 1 and 2 below:

Step 1. Compute x^{k+1} by solving the strongly convex program

$$(\mathrm{P}_k) \quad x^{k+1} = \operatorname*{argmin}_{y \in C}\{\rho f(x^k, y) + (1/2)\|y - x^k\|^2\}.$$

Step 2. If $\|x^{k+1} - x^k\| \leq \varepsilon(1 - r)/r$, with $r := \sqrt{1 - 2\rho(\tau - L_1)}$, then terminate: x^{k+1} is an ε-solution to (P). Otherwise, increase k by 1 and go to iteration k.

Note that, by the contraction property

$$\|x^{k+1} - x^*\| \leq r\|x^k - x^*\|, \quad \text{with } r < 1,$$

it is easy to see that

$$\|x^{k+1} - x^*\| \le r/(1-r)\|x^{k+1} - x^k\|, \quad \forall k \ge 0.$$

Hence,

$$\|x^{k+1} - x^*\| \le r^{k+1}/(1-r)\|x^0 - x^1\|, \quad \forall k \ge 0.$$

Thus, if

$$\|x^{k+1} - x^k\| \le \varepsilon(1-r)/r \quad \text{or} \quad r^{k+1}/(1-r)\|x^0 - x^1\| \le \varepsilon,$$

then indeed

$$\|x^{k+1} - x^*\| \le \varepsilon.$$

In this case, we can terminate the algorithm to obtain an ε-solution. Clearly, Algorithm A1 terminates after a finite number of iterations when $\varepsilon > 0$.

Remark 2.2 This algorithm has been presented in [4], but its linear convergence was not proved there.

3 Application to Variational Inequality and Complementarity Problems

Let $C \subseteq \mathcal{H}$ be a nonempty, closed, convex set as before, φ be a proper, closed, convex function on C, and let $F : \mathcal{H} \to \mathcal{H}$ be a multivalued mapping. Suppose that

$$C \subseteq \operatorname{dom} F := \{x \in \mathcal{H} : F(x) \neq \emptyset\}.$$

Consider the following generalized (or multivalued) variational inequality:

$$\text{(VIP)} \quad \text{Find } x^* \in C,\ w^* \in F(x^*) \text{ such that } (w^*)^T(y - x^*) \ge 0, \quad \text{for all } y \in C.$$

It is well known [3] that, when C is a closed convex cone, then (VIP) becomes the following complementarity problem:

$$\text{(CP)} \quad \text{Find } x^* \in C,\ w^* \in F(x^*) \text{ such that } w^* \in C^*,\ (w^*)^T x^* = 0,$$

where

$$C^* := \{w \mid w^T x \ge 0,\ \forall x \in C\}$$

is the polar cone of C.

We recall the following well known definitions (see e.g. [3]).

(i) The multivalued mapping F is said to be monotone on C if

$$(u - v)^T(x - y) \ge 0, \quad \forall x, y \in C,\ \forall u \in F(x),\ \forall v \in F(y).$$

(ii) F is said to be strongly monotone on C with modulus τ (shortly τ-strongly monotone) if

$$(u-v)^T(x-y) \geq \tau\|x-y\|^2, \quad \forall x, y \in C,\ \forall u \in F(x),\ \forall v \in F(y).$$

(iii) F is said to be Lipschitz on C with constant L (shortly L-Lipschitz) if

$$\sup_{u\in F(x)} \inf_{v\in F(y)} \|u-v\| \leq L\|x-y\|, \quad \forall x, y \in C.$$

Define the bifunction f by taking

$$f(x,y) := \sup_{u\in F(x)} u^T(y-x) + \varphi(y) - \varphi(x). \tag{12}$$

The lemma below follows immediately from Proposition 4.2 in [9].

Lemma 3.1 *Let f be given by* (12). *The following statements hold*:

(i) *If F is τ-strongly monotone* (*resp. monotone*) *on C, then f is τ-strongly monotone* (*resp. monotone*) *on C.*
(ii) *If F is Lipschitz on C with constant $L > 0$, then f satisfies the Lipschitz-type condition* (2); *namely, for any $\delta > 0$, we have*

$$f(x,y) + f(y,z) \geq f(x,z) - (L/(2\delta))\|x-y\|^2 - ((L\delta)/2)\|y-z\|^2. \tag{13}$$

Suppose that $F(x)$ is closed and bounded and that f is defined by (12). Then, Problem (VIP) is equivalent to Problem (P) in the sense that their solution sets coincide. Lemma 3.1 allows us to apply Algorithm A1 to strongly monotone mixed variational inequalities.

Remark 3.1 In order to apply Algorithm A1 for strongly monotone variational inequality problems, it must hold that $L_1 < \tau$. By Lemma 3.1, $L_1 = L/(2\delta)$. Hence, $L_1 < \tau$ whenever $\delta > L/(2\tau)$.

Now, we apply Algorithm A1 to the complementarity Problem (CP) when $C = \mathbb{R}^n_+$ and F is a single-valued and strongly monotone on C with modulus τ. In this case, Problem (CP) takes the form

$$\text{Find } x^* \geq 0 \text{ such that } F(x^*) \geq 0, \quad F(x^*)^T x^* = 0. \tag{14}$$

Note that, in this case, the subproblem

$$(\mathrm{P}_k) \quad x^{k+1} = \operatorname*{argmin}_{y\in C}\{\rho f(x^k, y) + (1/2)\|x - x^k\|^2\}$$

defined in Algorithm A1 takes the form

$$x^{k+1} = \operatorname*{argmin}_{y\in C}\{\rho F(x^k)^T(y-x^k) + (1/2)\|y-x^k\|^2\},$$

which in turns is

$$x^{k+1} = P_{\mathbb{R}^n_+}(x^k - \rho F(x^k)),$$

where $P_{\mathbb{R}^n_+}$ is the Euclidean projection of the point $x^k - \rho F(x^k)$ onto $\mathbb{R}^n_+$. It is easy to verify that, if $y = (y_1, \dots, y_n)^T$ is the Euclidean projection of $x = (x_1, \dots, x_n)^T$ onto $\mathbb{R}^n_+$, then for every $i = 1, \dots, n$ one has

$$\begin{aligned} y_i &= x_i, \quad \text{if } x_i \geq 0, \\ x_i &= 0, \quad \text{otherwise.} \end{aligned}$$

Suppose that F is single valued, τ-strongly monotone, and L-Lipschitz continuous on $\mathbb{R}^n_+$. Then, Algorithm A1 applied to the complementarity problem (CP) collapses into the following algorithm.

Algorithm A2 (Strongly Monotone Complementarity Problem)
Initialization. Fixed a tolerance $\varepsilon \geq 0$. Choose δ and ρ such that $\delta > L/(2\tau)$, $0 < \rho \leq 1/(L\delta)$. Take $x^0 \geq 0$.
Iteration k, $k = 0, 1, \dots$. Execute Steps 1 and 2 below:

Step 1. Compute $x^{k+1} = (x_1^{k+1}, \dots, x_n^{k+1})^T$ by taking

$$\begin{aligned} x_i^{k+1} &:= x_i^k, \quad \text{if } \rho F_i(x^k) \leq x_i^k, \\ x^{k+1} &:= 0, \quad \text{otherwise,} \end{aligned}$$

where the subindex i stands for the ith coordinate of a vector.

Step 2. If $\|x^{k+1} - x^k\| \leq \varepsilon(1 - r)/r$, with $r := \sqrt{1 - 2\rho(L/(2\delta) - \tau)}$, then terminate: x^{k+1} is an ε-solution to (14). Otherwise, increase k by 1 and go to iteration k.

The validity and linear convergence of Algorithm A2 are immediate from those of Algorithm A1. Algorithm A2 is quite different from the derivative-free algorithm of Mangasarian and Solodov [16]. In fact, our algorithm is based upon the contraction mapping approach and does not use a line search, whereas the algorithm in [16] is based upon a gap function using a line search technique defined by the derivative of the cost mapping F.

4 Avoiding the Lipschitz-Type Condition

In the previous section, we suppose that f satisfies the Lipschitz-type condition (2). This assumption sometimes is not fulfilled; if it does, the constants L_1 and L_2 are not always easy to estimate. In this section, we consider the case where the bifunction f does not necessarily satisfy the Lipschitz-type condition (2).

In the following algorithm, we do not require the Lipschitz-type condition (2).

Algorithm A3
Initialization. Choose two sequences $\{\sigma_k\}_{k\geq 0} \subset (0,1)$ and $\{\rho_k\}_{k\geq 0} \in (0,+\infty)$ such that

$$\sum_{k=0}^{\infty} \rho_k \sigma_k = \infty, \qquad \sum_{k=0}^{\infty} \sigma_k^2 < \infty,$$

and $\rho_k \sigma_k \in (0, 1/(2\tau))$ for all $k \geq 0$. Take $x^0 \in C$.
Iteration k, $k = 0, 1, \ldots$. Execute Steps 1 and 2 below:

Step 1. Find $w^k \in \mathcal{H}$ such that

$$\rho_k f(x^k, y) + (w^k)^T (y - x^k) \geq 0, \quad \forall y \in C, \tag{15}$$

where $\rho_k > 0$ is a regularization parameter.

(a) If $w^k = 0$, then terminate: x^k is the solution of (P).
(b) If $w^k \neq 0$, go to Step 2.

Step 2. Set $z^{k+1} = x^k + \sigma_k w^k$ and $x^{k+1} = P_C(z^{k+1})$, where P_C stands for the Euclidean projection on C.

Remark 4.1 Note that the main subproblem in Algorithm A3 is problem (15). This problem can be solved, for example, as follows:

(i) Suppose that the convex program $\min_{y\in C} f(x^k, y)$ admits a solution. Let

$$m_k := -\min_{y\in C} f(x^k, y) < +\infty.$$

Take $w^k \in \mathcal{H}$ such that $(w^k)^T(y - x^k) \geq \rho_k m_k$, for all $y \in C$. Then, it is easy to seethat w^k is a solution to (15).
(ii) Since $f(x, \cdot)$ is convex and subdifferentiable on C, we have

$$f(x^k, y) - f(x^k, x^k) \geq (g^k)^T(y - x^k), \quad \forall y \in C,\ g^k \in \partial_2 f(x^k, x^k).$$

Since $f(x^k, x^k) = 0$, it follows that $w^k = -\rho_k^{-1} g^k$ satisfies the inequality

$$\rho_k f(x^k, y) + (w^k)(y - x^k) \geq 0, \quad \forall y \in C.$$

Hence, w^k solves the subproblem (15).

Now, we are in position to prove convergence of Algorithm A3.

Theorem 4.1 *Suppose that f is strongly monotone with modulus τ on C. Let $\{x^k\}_{k\geq 0}$ be the sequence generated by Algorithm A3. Then, one has*

$$\|x^{k+1} - x^*\|^2 \leq (1 - 2\tau\rho_k\sigma_k)\|x^k - x^*\|^2 + \sigma_k^2 \|w^k\|^2, \quad \forall k \geq 0, \tag{16}$$

where x^* is the unique solution of (P). *Moreover, if the sequence $\{w^k\}_{k\geq 0}$ is bounded, then $\{x^k\}$ converges to the solution x^* of* (P).

Proof Let x^* be the unique solution of (P). Since $x^{k+1} = P_C(z^{k+1})$, we have

$$\|x^{k+1} - x^*\|^2 \leq \|z^{k+1} - x^*\|^2 - \|z^{k+1} - x^{k+1}\|^2. \tag{17}$$

Substituting

$$z^{k+1} = x^k + \sigma_k x^k$$

in (17), we obtain

$$\begin{aligned}\|z^{k+1} - x^*\|^2 &= \|x^k + \sigma_k w^k - x^*\|^2 \\ &= \|x^k - x^*\|^2 + 2\sigma_k (w^k)^T (x^k - x^*) + \sigma_k^2 \|w^k\|^2. \end{aligned} \tag{18}$$

Applying (15) with $y = x^*$, we obtain

$$\rho_k f(x^k, x^*) \geq (w^k)^T (x^k - x^*). \tag{19}$$

Since f is strongly monotone on C with modulus τ and since x^* is a solution to (P), we have

$$\rho_k f(x^k, x^*) \leq -\rho_k \tau \|x^k - x^*\|^2 - \rho_k f(x^*, x^k) \leq -\tau \rho_k \|x^k - x^*\|^2. \tag{20}$$

From (18)–(20) it follows that

$$\|z^{k+1} - x^*\|^2 \leq (1 - 2\tau\rho_k\sigma_k)\|x^k - x^*\|^2 + \sigma_k^2 \|w^k\|^2. \tag{21}$$

Substituting (21) into (17), we obtain

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\leq (1 - 2\tau\rho_k\sigma_k)\|x^k - x^*\|^2 + \sigma_k^2 \|w^k\|^2 - \|z^{k+1} - x^{k+1}\|^2 \\ &\leq (1 - 2\tau\rho_k\sigma_k)\|x^k - x^*\|^2 + \sigma_k^2 \|w^k\|^2,\end{aligned}$$

which proves inequality (16).

To prove $\lim_{k\to\infty} x^k = x^*$, using the assumption of boundedness of the sequence $\{w^k\}$, from (16) we have

$$\|x^{k+1} - x^*\|^2 \leq (1 - 2\tau\rho_k\sigma_k)\|x^k - x^*\|^2 + \sigma_k^2 M, \quad \forall k, \tag{22}$$

where $M > 0$ is a constant. Let $\lambda_k = 2\tau\rho_k\sigma_k$; by the assumption on the sequences $\{\rho_k\}$ and $\{\sigma_k\}$, we have that $\lambda_k \in (0, 1)$, for all $k \geq 0$, and $\sum_{k=0}^{\infty} \lambda_k = \infty$. On the other hand, since $\sum_{k=1}^{\infty} \sigma_k^2 < +\infty$, it is easy to see from (22) that $\|x^{k+1} - x^*\| \to 0$, as $k \to +\infty$. The theorem thus is proved. □

Note that, since Algorithms A3 is not linearly convergent, we cannot use $\|x^{k+1} - x^k\|$ to check whether or not the iterate x^{k+1} is an ε-solution as in Algorithm A1. Instead, we may use the value of a gap function at the iterate to check its ε-solution. The following two gap functions have been defined for Problem (P) (see e.g. [5]):

$$g(x) := \sup_{y\in C}\{f(x, y)\} \tag{23}$$

and

$$h(x) := \max_{y\in C}\{-f(x,y) - (1/(2\lambda))\|y-x\|^2\}, \tag{24}$$

where $\lambda > 0$ is a regularization parameter.

The function g is the Auslender gap function and h is the Fukushima gap function extended to Problem (P). Since $f(x,\cdot)$ is convex on C, evaluating these functions amounts to solving convex programs. Note that the convex program defining $g(x)$ may not have a solution; if it has a solution, it may not be unique. The Fukushima gap function can avoid this inconvenience because the objective function of the maximization program defining $h(x)$ is strongly concave. It has been shown in [4] that these are gap functions, which means that, for the g-function, $g(x) \geq 0$, for every $x \in C$, and $g(x) = 0$, $x \in C$ if and only if x solves (P). The same properties are also true for the h-function.

For checking the ε-solution of an iterate, we use the following lemma that is an immediate consequence of Propositions 4.1 and 4.2 in [5].

Lemma 4.1 *Let f be strongly monotone on C with modulus $\tau > 0$. Then, for any $\lambda > 0$, we have*:

(i) $g(x) \geq \tau\|x-x^*\|^2$, *for all $x \in C$.*
(ii) $h(x) \geq (\tau - 1/(2\lambda))\|x-x^*\|^2$, *for all $x \in C$, where x^* is an arbitrary solution of* (P).

By Lemma 4.1, if one of the following inequalities hold true:

(i) $g(x^k) \leq \tau\varepsilon$,
(ii) $h(x^k) \leq (\tau - 1/(2\lambda))\varepsilon$,

then x^k is an ε-solution to (P).

5 Application to the Proximal Point Method

In the preceding section, in order to ensure the convergence, we require that f is strongly monotone on C. This requirement may not be fulfilled in some applications.

In [6], Moudafi has extended the proximal point method [11] to Problem (P), where f is monotone. However, in [6] he does not discuss how to solve the subproblems raised in the proximal point method. In this section, we make used of the linear convergence rate obtained in the preceding section to implement inexact proximal point algorithms. Each iteration $k = 1, 2, \ldots$ of the proximal point method for solving (P) requires that the following subproblem to be solved:

(P_k) Find $x^{k+1} \in C$ such that

$$c_k f(x^{k+1}, y) + (x^{k+1} - x^k)^T (y - x^{k+1}) \geq 0, \quad \text{for all } y \in C,$$

where $c_k > 0$ is a regularization parameter. Since the computation of the exact solution of this subproblem can be quite difficult or even impossible in practice, the use of approximate solutions is essential for devising implementable algorithms. Rockafellar [11] suggests approximation criteria that enable one to replace the exact problem by an approximation problem. Using the ideas of Rockafellar, for Problem (P), Moudafi [6] has proposed the following approximation problem:

$$(\mathrm{P}_{\varepsilon_k})\quad \text{Find } x^{k+1}\in C \text{ such that}$$
$$c_k f(x^{k+1}, y) + (x^{k+1}-x^k)^T(y-x^{k+1}) \geq -\varepsilon_k, \quad \text{for all } y \in C.$$

It has been proved in [6] that, if f is upper hemicontinuous, monotone on C and if $f(x,\cdot)$ is proper, closed convex for each fixed $x \in C$, then the sequence $\{x^k\}_{k\geq 0}$ generated by the proximal point algorithm using the approximation subproblems $(\mathrm{P}_{\varepsilon_k})$ weakly converges to a solution of (P) provided $0 < c < c_k < +\infty$ for all $k \geq 0$ large, and $\varepsilon_k \geq 0$ is such that $\sum_{k=0}^{\infty} \varepsilon_k < +\infty$.

In the sequel, instead of approximate solution defined by $(\mathrm{P}_{\varepsilon_k})$, we use the usual definition of ε_k-solution. Recall that $x \in C$ is an ε_k-solution to (P) if $\|x - x^*\| \leq \varepsilon_k$, where x^* is an exact solution of (P). We show that, if x^k is an ε_k solution to the subproblem (P_k), then the sequence $\{x^k\}$ weakly converges to a solution of (P) provided $\varepsilon_k \searrow 0$ (not necessarily $\sum_{k=1}^{\infty} \varepsilon_k < +\infty$ as in the approximate rules that have been used in [6, 11]).

To this end, for each $k \geq 0$, we define the bifunction f_k on C by taking

$$f_k(x,y) := c_k f(x,y) + (x-x^k)^T(y-x). \tag{25}$$

The following lemma says that the bifunction in subproblem (P_k) is strongly monotone and satisfies the Lipschitz-type condition (2).

Lemma 5.1 *Suppose that f is monotone on C and satisfies Lipschitz-type condition* (2) *with positive constants L_1, L_2. Then, for any $c_k > 0$, it holds true that*:

(i) *The bifunction f_k is strongly monotone with modulus* 1.
(ii) *The bifunction f_k satisfies the Lipschitz-type condition* (2); *namely,*

$$f_k(x,y)+f_k(y,x) \geq f_k(x,z) - (c_k L_1 + 1/(4t))\|x-y\|^2 - (c_k L_2 + t)\|y-z\|^2, \tag{26}$$

for all $x, y, z \in C$ and $t > 0$.

Proof Since f is monotone on C, we have

$$\begin{aligned} &f_k(x,y) + f_k(y,x) \\ &\quad = c_k f(x,y) + (x-x^k)^T(y-x) + c_k f(y,x) + (y-x^k)^T(x-y) \\ &\quad \leq -\|x-y\|^2, \end{aligned}$$

which proves (i). Let

$$g_k(x,y) := (x-x^k)^T(y-x).$$

We first show that g_k satisfies the condition (2). Indeed,

$$\begin{aligned} & g_k(x,y)+g_k(y,z)-g_k(x,z) \\ & \quad = (x-x^k)^T(y-x)-(y-x^k)^T(z-y)-(x-x^k)^T(z-x) \\ & \quad = (y-x)^T(z-y) \le \|y-x\|\|z-y\|. \end{aligned} \tag{27}$$

Using the well-known elementary inequality

$$2\|y-x\|\|z-y\| \le (1/(2t))\|y-x\|^2 + 2t\|z-y\|^2, \quad \forall t>0,$$

we obtain from (27) that

$$g_k(x,y)+g_k(y,z) \ge g_k(x,z) - 1/(4t)\|y-x\|^2 - t\|z-y\|^2, \quad \forall t>0.$$

Since f satisfies (2) with constants L_1, L_2 and since

$$f_k(x,y) = c_k f(x,y) + g_k(x,y),$$

it follows that f_k also satisfies (2) with constants

$$L_{k1} = \rho_k L_1 + 1/(4t) \quad \text{and} \quad L_{k2} = c_k L_2 + t.$$

Namely,

$$f_k(x,y)+f_k(y,x) \ge f_k(x,z) - (c_k L_1 + 1/(4t))\|x-y\|^2 - (c_k L_2 + t)\|y-z\|^2,$$

for all $x,y,z \in C$ and $t>0$. The statement (ii) is proved. □

Lemma 5.1 allows us to apply Algorithm A1 to solve the subproblem (P_k). Coupling Algorithm A1 with the inexact proximal point algorithms, we obtain implementable algorithms for solving (P). For simplicity of notation, we take $c_k \equiv c > 0$ for all k.

Let

$$L_{k1} := cL_1 + 1/(4t), \qquad L_{k2} := cL_2 + t \tag{28}$$

be the Lipschitz constants for f_k and let

$$r_k := \sqrt{1 - 2c\rho_k(1 - L_{k1})}, \tag{29}$$

with $L_{k1} < \tau \equiv 1$ and $0 < \rho_k \le 1/(2L_{k2})$, where ρ_k denotes the regularization parameter for subproblem (P_k).

Algorithm A4 (BFP Algorithm for Monotone Problems)
Initialization. Choose $t>0$, $c>0$ and a positive sequence $\{\varepsilon_k\}_{k\ge 0}$ such that:

$$\varepsilon_k \searrow 0 \quad \text{and} \quad L_{k1} \equiv cL_1 + 1/(4t) \in (0,1).$$

Take $x^0 \in C$.
Outer Loop Main Iteration $k = 0,1,\ldots$. Choose ρ_k such that $0 < \rho_k \le 1/(2(cL_2+t))$. Take $x^{k,0} := x^k$.
Inner Loop Iteration $j = 0,\ldots,J_k$.

Step 1. Compute $x^{k,j+1}$ by solving the strongly convex program:

$$x^{k,j+1} = \operatorname*{argmin}_{y \in C}\{\rho_k f_k(x^{k,j}, y) + (1/2)\|y - x^{k,j}\|^2\}. \tag{30}$$

Step 2. If

$$\|x^{k,j+1} - x^{k,j}\| \le (1 - r_k)\varepsilon_k / r_k, \quad r_k = \sqrt{1 - 2\rho_k(1 - L_{k1})},$$

terminate the inner loop. Set $x^{k+1} := x^{k,j+1}$ and go to the outer iteration k with $k := k + 1$. Otherwise, go to Step 1 of the inner iteration j with $j := j + 1$.

Note that, since f_k is strongly monotone and satisfies the condition (2), by Algorithm A1 the inner loop in Algorithm A4 must terminate after a finite number of iterations yielding an ε_k-solution of subproblem (P_k).

Theorem 5.1 *Suppose that, in addition to Assumption A, f is hemicontinuous on $C \times C$, monotone on C and satisfies the Lipschitz-type condition* (2). *Then, the sequence $\{x^k\}_{k \ge 0}$ generated by Algorithm A4 weakly converges to a solution of* (P). *Moreover, if $\sum_{k=1}^{\infty} \varepsilon_k < +\infty$, then the following estimate holds true*:

$$\|x^{k+1} - x^*\|^2 \le \|x^k - x^*\|^2 - \|x^{k+1} - x^k\|^2 + \delta_k, \quad \forall k \ge 0, \tag{31}$$

where

$$\delta_k := 6M(\varepsilon_{k-1} + \varepsilon_k) + \varepsilon_{k-1}^2 + 2\varepsilon_{k-1}\varepsilon_k,$$

with $M > 0$ being a constant.

Proof For each k, let $\overline{x}^k$ be the exact solution of Problem (P_k). By Theorem 1 in [6], the sequence $\{\overline{x}^k\}$ weakly converges to a solution, say, x^* of (P). Since x^{k+1} is an ε_k-solution of (P_k), we have

$$\|x^{k+1} - \overline{x}^{k+1}\| \le \varepsilon_k.$$

Thus, the sequence $\{x^k\}$ converges weakly to x^* too. Indeed, since $\overline{x}^k$ weakly converges to x^* and

$$\|x^k - \overline{x}^k\| \le \varepsilon_{k-1}, \quad \text{with } \varepsilon_k \to 0,$$

for every $w \in \mathcal{H}$, we have

$$w^T x^k = w^T(x^k - \bar{x}^k + \bar{x}^k) = w^T(x^k - \bar{x}^k) + w^T \bar{x}^k \to w^T x^*, \quad k \to +\infty,$$

which means that x^k weakly converges to x^*. Thus, both sequences $\{x^k - x^*\}$ and $\{\bar{x}^k - x^*\}$ are bounded. So, there is a positive constant M such that

$$\|x^k - x^*\| \le M, \qquad \|\bar{x}^k - x^*\| \le M. \tag{32}$$

From the Theorem 1 in [6], we have

$$\|\bar{x}^{k+1} - x^*\|^2 \le \|\bar{x}^k - x^*\|^2 - \|\bar{x}^{k+1} - \bar{x}^k\|^2. \tag{33}$$

Now, by using the elementary inequality

$$|\|a\| - \|b\|| \leq \|a + b\|,$$

we have

$$|\|x^{k+1} - x^*\| - \|\bar{x}^{k+1} - x^{k+1}\||^2 \leq \|\bar{x}^{k+1} - x^*\|^2,$$

which implies

$$\begin{aligned}
&\|x^{k+1} - x^*\|^2 - 2\|x^{k+1} - x^*\|\|x^{k+1} - \bar{x}^{k+1}\| \\
&\quad \leq \|x^{k+1} - x^*\|^2 - 2\|x^{k+1} - x^*\|\|x^{k+1} - \bar{x}^{k+1}\| + \|x^{k+1} - \bar{x}^{k+1}\|^2 \\
&\quad \leq \|\bar{x}^{k+1} - x^*\|^2.
\end{aligned}$$

Combining this inequality with

$$\|x^{k+1} - \bar{x}^{k+1}\| \leq \varepsilon_k,$$

we can write

$$\|x^{k+1} - x^*\|^2 - 2\varepsilon_k\|x^{k+1} - x^*\| \leq \|\bar{x}^{k+1} - x^*\|^2. \tag{34}$$

On the other hand,

$$\begin{aligned}
\|\bar{x}^k - x^*\|^2 &\leq (\|\bar{x}^k - x^k\| + \|x^k - x^*\|)^2 \\
&\leq \|x^k - x^*\|^2 + 2\|x^k - x^*\|\|\bar{x}^k - x^k\| + \|\bar{x}^k - x^k\|^2 \\
&\leq \|x^k - x^*\|^2 + 2\varepsilon_{k-1}\|x^k - x^*\| + \varepsilon_{k-1}^2.
\end{aligned} \tag{35}$$

From (33), (34), and (35), it follows that

$$\begin{aligned}
&\|x^{k+1} - x^*\|^2 - 2\varepsilon_k\|x^{k+1} - x^*\| \\
&\quad \leq \|x^k - x^*\|^2 + 2\varepsilon_{k-1}\|x^k - x^*\| + \varepsilon_{k-1}^2 - \|\bar{x}^{k+1} - \bar{x}^k\|^2.
\end{aligned} \tag{36}$$

Now, we estimate $\|\bar{x}^{k+1} - \bar{x}^k\|^2$ as follows:

$$\begin{aligned}
\|\bar{x}^{k+1} - \bar{x}^k\|^2 &\geq |\|\bar{x}^{k+1} - x^{k+1}\| - \|x^{k+1} - \bar{x}^k\||^2 \\
&= \|x^{k+1} - \bar{x}^{k+1}\|^2 - 2\|x^{k+1} - \bar{x}^{k+1}\|\|x^{k+1} - \bar{x}^k\| + \|x^{k+1} - \bar{x}^k\|^2 \\
&\geq \|x^{k+1} - \bar{x}^k\|^2 - 2\|x^{k+1} - \bar{x}^{k+1}\|\|x^{k+1} - \bar{x}^k\| \\
&\geq |\|x^{k+1} - x^k\| - \|x^k - \bar{x}^k\||^2 - 2\|x^{k+1} - \bar{x}^{k+1}\|\|x^{k+1} - \bar{x}^k\| \\
&\geq \|x^{k+1} - x^k\|^2 + \|x^k - x^*\|^2 - 2\|x^{k+1} - x^k\|\|x^k - \bar{x}^k\| \\
&\quad - 2\|x^{k+1} - \bar{x}^{k+1}\|\|x^{k+1} - \bar{x}^k\| \\
&\geq \|x^{k+1} - x^k\|^2 - 2\|x^{k+1} - x^k\|\|x^k - \bar{x}^k\| \\
&\quad - 2\|x^{k+1} - \bar{x}^{k+1}\|\|x^{k+1} - \bar{x}^k\|
\end{aligned}$$

$$
\begin{aligned}
&\geq \|x^{k+1}-x^k\|^2 - 2\varepsilon_{k-1}\|x^{k+1}-x^k\| - 2\varepsilon_k\|x^{k+1}-\bar{x}^k\| \\
&\geq \|x^{k+1}-x^k\|^2 - 2\varepsilon_{k-1}\|x^{k+1}-x^k\| \\
&\quad - 2\varepsilon_k(\|x^{k+1}-x^k\| + \|x^k-\bar{x}^k\|) \\
&\geq \|x^{k+1}-x^k\|^2 - 2\varepsilon_{k-1}\|x^{k+1}-x^k\| - 2\varepsilon_k\|x^{k+1}-x^k\| \\
&\quad - 2\varepsilon_k\varepsilon_{k-1}, \qquad (37)
\end{aligned}
$$

which follows from the inequalities

$$\|x^k-\bar{x}^k\| \leq \varepsilon_{k-1} \quad \text{and} \quad \|x^{k+1}-\bar{x}^{k+1}\| \leq \varepsilon_k.$$

Substituting (37) into (36), we have

$$
\begin{aligned}
&\|x^{k+1}-x^*\|^2 - 2\varepsilon_k\|x^{k+1}-x^*\| \\
&\quad \leq \|x^k-x^*\|^2 - \|x^{k+1}-x^k\|^2 + 2\varepsilon_{k-1}\|x^k-x^*\| + \varepsilon_{k-1}^2 \\
&\qquad + 2\varepsilon_{k-1}\|x^{k+1}-x^k\| + 2\varepsilon_k\|x^{k+1}-x^k\| + 2\varepsilon_{k-1}\varepsilon_k,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\|x^{k+1}-x^*\|^2 &\leq \|x^k-x^*\|^2 - \|x^{k+1}-x^k\|^2 + 2\varepsilon_k\|x^{k+1}-x^*\| + 2\varepsilon_{k-1}\|x^k-x^*\| \\
&\quad + \varepsilon_{k-1}^2 + 2\varepsilon_{k-1}\|x^{k+1}-x^k\| + 2\varepsilon_k\|x^{k+1}-x^k\| + 2\varepsilon_{k-1}\varepsilon_k \\
&\leq \|x^k-x^*\|^2 - \|x^{k+1}-x^k\|^2 \\
&\quad + (4\varepsilon_k + 2\varepsilon_{k-1})\|x^{k+1}-x^*\| + (2\varepsilon_k + 4\varepsilon_{k-1})\|x^k-x^*\| \\
&\quad + \varepsilon_{k-1}^2 + 2\varepsilon_{k-1}\varepsilon_k.
\end{aligned}
$$

Combining the above relation with (32), it follows that

$$
\begin{aligned}
\|x^{k+1}-x^*\|^2 &\leq \|x^k-x^*\|^2 - \|x^{k+1}-x^k\|^2 \\
&\leq \|x^k-x^*\|^2 - \|x^{k+1}-x^k\|^2 + 6M(\varepsilon_{k-1}+\varepsilon_k) + \varepsilon_{k-1}^2 + 2\varepsilon_{k-1}\varepsilon_k.
\end{aligned}
$$

Setting

$$\delta_k := 6M(\varepsilon_{k-1}+\varepsilon_k) + \varepsilon_{k-1}^2 + 2\varepsilon_{k-1}\varepsilon_k,$$

we obtain

$$\|x^{k+1}-x^*\|^2 \leq \|x^k-x^*\|^2 - \|x^{k+1}-x^k\|^2 + \delta_k, \quad \forall k \geq 0.$$

From the assumption $\sum_{k=0}^{\infty}\varepsilon_k < +\infty$, it is easy to see that $\sum_{k=0}^{\infty}\delta_k < +\infty$. The inequality (31) thus is proved. □

Remark 5.1

(i) The main subproblem in each iteration k of Algorithm A4 is the problem (30). By the definition of f_k, this subproblem is a strongly convex mathematical program of the form

$$\min_{y\in C}\{c_k\rho_k f(x^{k,j}, y) + \rho_k(x^{k,j} - x^k)^T(y - x^{k,j}) + (1/2)\|y - x^{k,j}\|^2\}. \quad (38)$$

(ii) Applying (31) iteratively, we obtain

$$\|x^{k+1} - x^*\|^2 \le \|x^0 - x^*\|^2 + \sum_{j=0}^{k}\delta_j, \quad \forall k \ge 0.$$

which shows that convergence of the algorithm depends crucially on the starting point x^0.

In the case where the bifunction f does not satisfy the Lipschitz-type condition (2), we can use Algorithm A3 to find an ε_k-solution of subproblem ($\mathrm{P}_{\varepsilon_k}$). In this case, to check whether or not the iterate point x^{k+1} is an ε_k-solution of (P_k), we may use the Auslender or Fukushima gap function of the subequilibrium (P_k). Let g_k (resp. h_k) denote the Auslender (resp. Fukushima) gap function for Problem (P_k). Then, since the bifunction of (P_k) is strongly monotone with modulus 1, by Lemma 4.1, if either $g_k(x^{k+1}) \le \varepsilon_k$ or $h_k(x^{k+1}) \le (1 - 1/(2\lambda))\varepsilon_k$ hold true, then x^{k+1} is an ε_k-solution to (P_k).

6 Application to a Nash-Cournot Market Equilibrium Model

In this section, we use Algorithms A1 and A3 to solve the following well-known Nash-Cournot oligopolistic market equilibrium model that has been introduced in some books and research papers (see e.g. [3, 19] and the references therein).

Suppose that there are n-firms producing a common homogeneous commodity and that the price p_i of the goods produced by firm i depends on the commodities of all firms j, $j = 1, 2, \dots, n$. Let $h_i(x_i)$ denote the cost of the firm i, that is assumed to be dependent on only its production level x_i. Then, the profit of firm i can be given as

$$f_i(x_1, x_2, \dots, x_n) = x_i p_i(x_1, x_2, \dots, x_n) - h_i(x_i), \quad i = 1, 2, \dots, n. \quad (39)$$

Let $C_i \subset \mathbb{R}$, $i = 1, 2, \dots, n$, denote the strategy set of the firm i. Each firm i seeks to maximize its own profit by choosing the corresponding production level x_i. Let $C \subset \mathbb{R}^n$ denote the strategy set of the model. For convenience we write $x = (x_1, x_2, \dots, x_n)^T \in C$ and recall that $x^* = (x_1^*, x_2^*, \dots, x_n^*)^T \in C$ is an equilibrium point to this oligopolistic market equilibrium model if

$$f_i(x_1^*, \dots, x_{i-1}^*, y_i, x_{i+1}^*, \dots, x_n^*) \le f_i(x_1^*, \dots, x_n^*),$$
$$\forall y_i \in C_i,\ i = 1, 2, \dots, n. \quad (40)$$

Denote

$$\psi(x, y) := -\sum_{i=1}^{n} f_i(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n),$$
$$\phi(x, y) := \psi(x, y) - \psi(x, x).$$

The problem of finding an equilibrium point of this model can be formulated as follows:

$$\text{(P1)} \quad \text{Find } x^* \in C \text{ such that } \phi(x^*, y) \geq 0, \quad \text{for all } y \in C.$$

Suppose that the cost function h_i has the following form:

$$h_i(x_i) = \begin{cases} \underline{c}_i x_i + \frac{\underline{\beta}_i}{\underline{\beta}_i+1} \tau_i^{-1/\underline{\beta}_i} x_i^{(\underline{\beta}_i+1)/\underline{\beta}_i}, & \text{if } l_i \leq x_i < m_i, \\ \bar{c}_i x_i + \frac{\bar{\beta}_i}{\bar{\beta}_i+1} \tau_i^{-1/\bar{\beta}_i} x_i^{(\bar{\beta}_i+1)/\bar{\beta}_i}, & \text{if } m_i \leq x_i \leq u_i, \end{cases} \tag{41}$$

where $\underline{c}_i, \bar{c}_i, \underline{\beta}_i, \bar{\beta}_i, \tau_i, i = 1, \ldots, n$, are given positive parameters, l_i, u_i are the lower and upper bounds for the production level of firm i, and m_i is the change level of the cost function h_i, which depends on the market demand. To ensure the convexity and continuity of h_i, we choose the parameter $\bar{c}_i$ such that

$$\bar{c}_i = \underline{c}_i + \frac{\underline{\beta}_i}{\underline{\beta}_i + 1} \left(\frac{m_i}{\tau_i}\right)^{1/\underline{\beta}_i} - \frac{\bar{\beta}_i}{\bar{\beta}_i + 1} \left(\frac{m_i}{\tau_i}\right)^{1/\bar{\beta}_i}. \tag{42}$$

As in [18], we take the price function $p(\sigma)$ as

$$p(\sigma) = \left(\frac{5000}{\sigma}\right)^{1/\eta}, \quad \text{with } \eta = 1.1. \tag{43}$$

Suppose that the strategy set C of the model is the n-dimensional box given by

$$C := C_1 \times \cdots \times C_n, \tag{44}$$

where the interval $C_i := [l_i, u_i]$ is the strategy set of firm i, $i = 1, \ldots, n$. It is easy to see that the price function given by (43) with $\sigma := \sum_{i=1}^{n} x_i$ is convex on C and that h_i is convex on C_i. These properties imply that $\phi(x, \cdot)$ is convex with respect to the second variable y on C. Let $\partial_2\phi(x, x)$ denote the subgradient of the bifunction ϕ with respect to the second variable at x. It has been indicated in [17] that the function $G(x) := \partial_2\phi(x, x)$ is strongly monotone on C. We have used Algorithms A1 and A3 to find an equilibrium point of the Nash-Cournot market model where the cost and price functions are given by (41) and (43) respectively. For Algorithm A3, the two sequences $\{\sigma_k\}$ and $\{\rho_k\}$ have been chosen such that the conditions

$$\rho_k \sigma_k \in (0, 1/(2\tau)), \quad \sum_{k=0}^{\infty} \sigma_k = +\infty, \quad \sum_{k=0}^{\infty} \sigma_k^2 < +\infty$$

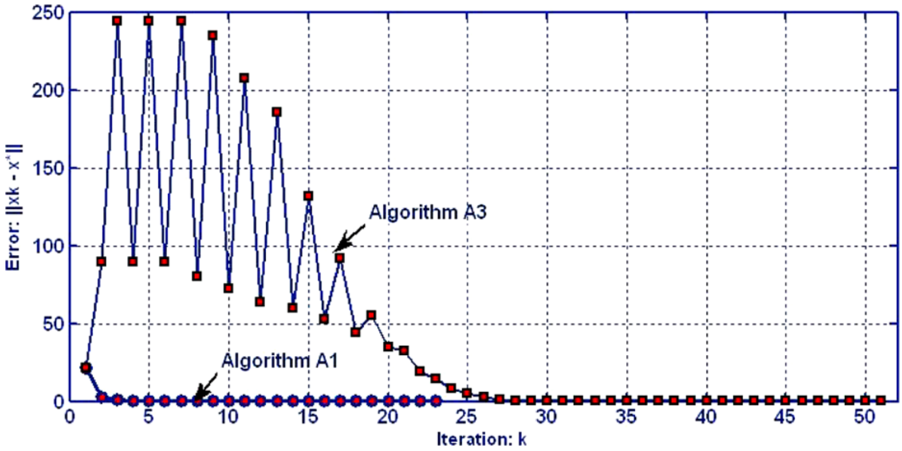

Fig. 1 Convergence behavior of Algorithm A1 and Algorithm A3 ($n = 6$, $\varepsilon = 10^{-8}$)

Table 1 Results computed with random data

Size	Algorithm A1			Algorithm A3		
	Iter	CPU_time(s)	Error	Iter	CPU_time(s)	Error
5	23	1.75	$O(10^{-8})$	51	0.49	$O(10^{-8})$
10	31	6.11	$O(10^{-8})$	76	1.31	$O(10^{-8})$
20	48	34.11	$O(10^{-8})$	129	9.99	$O(10^{-8})$
30	57	75.02	$O(10^{-8})$	121	7.33	$O(10^{-8})$
40	46	121.35	$O(10^{-8})$	116	8.53	$O(10^{-8})$
50	67	251.00	$O(10^{-8})$	128	9.13	$O(10^{-8})$
100	79	1058.39	$O(10^{-8})$	144	14.53	$O(10^{-8})$
150	47	1590.51	$O(10^{-8})$	123	18.00	$O(10^{-8})$
200	61	3415.56	$O(10^{-8})$	150	24.34	$O(10^{-8})$

are satisfied. Namely, we choose

$$\rho_k = 0.499/(\tau\sigma_k), \qquad \sigma_k = 1/(k+1)^{-0.55}.$$

Both algorithms were implemented on a PC with 1.7 GHz, 512 Mb-RAM and 100 Gb memory by the MATLAB software Version 7.0. The main subproblems were solved with the MATLAB Optimization Toolbox by using FMINCON and QUADPROG functions, respectively. The convergence behavior of Algorithms A1 and A3 is shown in Fig. 1. The horizontal and vertical axes show the iteration k and error $\mathtt{err} := \|x^k - x^*\|$, respectively.

To test Algorithms A1 and A3, we have implemented them with random data and with

$$C := \{x \in \mathbb{R}^n \mid 1 \le x_i \le 150,\ \forall i = 1, \dots, n\}.$$

The parameters $c_i, \underline{\beta}_i, \bar{\beta}_i, \tau_i$, for all $i = 1, \dots, n$, have been generated randomly in the intervals [2, 10], [0.5, 1.5], [0.4, 1.4], [5, 6]. In this case, the convexity of $\phi(x, \cdot)$ and the monotonicity of $G(x) := \partial_2 \phi(x, x)$ are still guaranteed. The computational results are reported in Table 1 below.

The results in Table 1 show that Algorithm A1 spends more CPU time than Algorithm A3. The reason is as follows: for Algorithm A3, by using (ii) of Remark 4.1, at each iteration, we need only to compute $g^k \in \partial_2 \phi(x^k, x^k)$; for Algorithm A1, at each iteration, we have to solve convex subprograms which, for this model, are not quadratic.

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