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Regularity Conditions in Differentiable Vector Optimization Revisited

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Abstract This work is concerned with differentiable constrained vector optimization problems. It focus on the intrinsic connection between positive linearly dependent gradient sets and the distinct notions of regularity that come to play in this context. The main aspect of this contribution is the development of regularity conditions, based on the positive linear dependence or independence of gradient sets, for problems with general nonlinear constraints, without any convexity hypothesis. Being easy to verify, these conditions might be useful to define termination criteria in the development of algorithms.

Keywords Nonlinear vector optimization · Weak Pareto points · Regularity · Constraint qualifications

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1 Introduction

The role of the constraint qualifications is well known in nonlinear programming. These conditions are properties of the feasible points that, whenever satisfied by a minimizer, ensure that the Karush-Kuhn-Tucker (KKT) condition holds at such a point. Among the most known and used constraint qualifications, we can mention the linear independence of the gradients of the active constraints, Mangasarian–Fromovitz constraint qualification, quasinormality constraint qualification [1] and the constant positive linear dependence (CPLD), recently introduced by Qi and Wei [2] and further analyzed by Andreani, Martínez and Schuverdt [3].

On the other hand, in nonlinear vector optimization, qualification conditions involving only the constraints are not enough to ensure, in general, the existence of the nonzero multipliers associated to the objective functions. Therefore, one needs conditions involving the problem functions, which in spite of the potential confusion, are frequently called in the literature "constraint qualifications" [4, 5]. In this work, we prefer to use the term *regularity conditions* whenever the problem functions involve the objective functions and use the term *constraint qualification* if they do not include the objective functions. The concept of total regularity has been introduced by Castellani, Mastroeni and Pappalardo [6] in the context of linear separation of sets, following the ideas of the alternative theorem of generalized systems [7]. Bigi and Pappalardo deepened the analysis to the more general theory of separation sets [8] and extended this scheme to the vector optimization case [9].

There are many contributions dealing with constraint qualifications and regularity conditions for the vector optimization problem. In most of them, as a consequence, necessary conditions for Pareto optimal points are obtained. Among the conditions most closely related to those of the current work, we can mention those of Maeda [5], Preda and Chitescu [10] and Giorgi, Jiménez and Novo [11].

For the differentiable vector optimization problem with inequality constraints, Maeda reviews several constraint qualifications, establishes relationships among them and derives KKT necessary conditions for Pareto points. Preda and Chitescu develop similar results to Maeda for the semidifferentiable case, but their conclusions are limited to problems under convexity assumptions. More recently, the previous results were extended by Giorgi, Jiménez and Novo to the equality and inequality cases, assuming Dini or Hadamard differentiable functions. These authors introduce new constraint qualifications, analyze the relationships among them, and give several KKT necessary conditions to a local Pareto minimal point.

In this work, regularity conditions established in the mentioned papers are revisited within the framework of the positive linear dependence (PLD) and positive linear independence (PLI) properties of the sets of gradient vectors. In mathematical programming, these concepts were employed in [12, 13]. More recently, these notions have been used by Qi and Wei [2] and by Andreani, Martínez and Schuverdt [3] in the context of sequential quadratic programming and augmented Lagrangian methods, respectively.

We assume that the feasible set is stated explicitly by means of equality and inequality constraints and apply these properties to characterize the classes of multipliers introduced by Bigi and Pappalardo [9].

The main contribution of this paper is to bring the positive linear dependence into the vector optimization scenario, not only restating already known regularity conditions within this perspective, but also presenting new results along such point of view, without any convexity assumption. Besides being easily computable, since they rest upon gradient evaluations, the new conditions might be useful in the development of algorithms, particularly in the definition of termination criteria.

Through this work, we use different tools to present and prove the results as well as to relate them to others in the literature. For instance, the classic theorems of the alternative can be stated in a compact way in terms of sets of vectors, which is a primal approach. Our proofs, based on the geometric and algebraic concepts of PLD and PLI, follow a primal-dual point of view. In order to enlarge the reader's perspective, we restate also the results in terms of sets of vectors.

This paper is organized as follows. We start by defining the general problem, some notation and basic properties in Sect. 2. In Sect. 3, the different classes of regularity for feasible points are characterized and the relationships with known results are established. Finally, comments and concluding remarks are drawn in Sect. 4.

2 Preliminaries

Let us consider the following vector optimization problem (VOP),

$$\begin{aligned} \min \quad & F(x) = (f_1(x), f_2(x), \ldots, f_r(x))^T, \\ \text{s.t.} \quad & h_i(x) = 0, \quad i \in I, \\ & g_j(x) \le 0, \quad j \in J, \end{aligned}$$

where $f_k : \mathbb{R}^n \to \mathbb{R}$, $k \in K \equiv \{1, \ldots, r\}$, $h_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I \equiv \{1, \ldots, m\}$, $g_j : \mathbb{R}^n \to \mathbb{R}$, $j \in J \equiv \{1, \ldots, p\}$ are continuously differentiable functions, and for which the order is with respect to the cone $\operatorname{int}(\mathbb{R}^r_+) \cup \{0\}$, that is, given $F : \mathbb{R}^n \to \mathbb{R}^r$, $F(\hat{x}) \le F(\tilde{x}) \Leftrightarrow F(\hat{x}) - F(\tilde{x}) \in \operatorname{int}(\mathbb{R}^r_+) \cup \{0\}$. We define the constraint set as

$$X = \{x \in \mathbb{R}^n : h_i(x) = 0,\ i \in I,\ g_j(x) \le 0,\ j \in J\}$$

and the active set index related to the inequality constraints as

$$A(x) = \{j \in J : g_j(x) = 0\}.$$

We recall the associated notion of weak Pareto optimality in the following definition.

Definition 2.1 A vector $x^\star \in X$ is said to be a weakly Pareto optimal point if there does not exist another vector $x \in X$ such that $f_k(x) < f_k(x^\star)$ for all $k \in K$.

When the ordering cone is $\mathbb{R}^r_+$, the vector optimization problem is known as the Pareto optimization problem and its optimality related notion gives rise to Pareto optimal solutions. Since this work focuses on the properties of the feasible points rather than the optimal solutions, we do not put emphasis on the different optimality notions.

The next result states the Fritz-John necessary condition for weak Pareto optimality.

Theorem 2.1 [14] *Let us consider the VOP problem and let the functions involved be continuously differentiable at $x^\star \in X$. A necessary condition for $x^\star$ to be a weak Pareto point is that there exist vectors $\theta \in \mathbb{R}^r_+$, $\lambda \in \mathbb{R}^p_+$ and $\mu \in \mathbb{R}^m$ such that*

$$\sum_{k\in K} \theta_k \nabla f_k(x^\star) + \sum_{j\in J} \lambda_j \nabla g_j(x^\star) + \sum_{i\in I} \mu_i \nabla h_i(x^\star) = 0, \tag{1}$$

$$\lambda_j g_j(x^\star) = 0, \quad j \in J, \tag{2}$$

$$(\theta, \lambda, \mu) \neq (0, 0, 0). \tag{3}$$

The nonzero vector (θ, λ, μ) satisfying (1)–(3) is generally known as the Fritz-John multiplier vector. Let $M(x^\star)$ denote the set of such vectors associated to $x^\star$.

Unfortunately, the Fritz-John condition does not guarantee by itself that the multipliers θ_k, $k \in K$, associated to the objective functions are all nonzero. In fact, from (1)–(3), one cannot even ensure the existence of at least a single $\theta_k > 0$. To strengthen this result, it is necessary to have some regularity on the problem. Following the classification made by Bigi and Pappalardo [9], we state the notions of regularity that we are going to use in this work.

Definition 2.2 Given $x \in X$ such that $M(x) \neq \emptyset$, we say that:

(a) x is weak-regular if there exists $(\theta, \lambda, \mu) \in M(x)$ with $\theta \neq 0$.
(b) x is totally weak-regular if, for all $(\theta, \lambda, \mu) \in M(x)$, there exists $k \in K$ such that $\theta_k \neq 0$.
(c) x is regular if there exists $(\theta, \lambda, \mu) \in M(x)$ with $\theta_k > 0$ for all $k \in K$.
(d) x is totally regular if, for all $(\theta, \lambda, \mu) \in M(x)$, one has $\theta_k > 0$ for all $k \in K$.

Naturally, notion (d) implies (b) and (c), and in turn notions (b) and (c) imply (a). Note that notions (b) and (c) are not related, as can be seen from the examples provided in [9].

These four notions of regularity were already characterized by Bigi [15] in terms of alternative theorems and by means of Mangasarian-Fromovitz related conditions. In this work, we analyze the relationships between such notions and the concepts of positive linear dependence and independence.

It is worth mentioning that, although the regularity analysis is essentially of interest for solution candidates, the regularity notion is a property of feasible points. For the VOP, the existence of Fritz-John multipliers at a feasible point allows these multipliers to take part in (1) in several ways, so that the objective functions effectively influence the relation (1) in different degrees. The distinct regularity notions of Definition 2.2 provide such possible combinations; and the existence of Fritz-John multipliers associated to feasible points is usually enough to obtain theoretical results, avoiding the assumption of a weak Pareto point.

2.1 Properties of Gradient Sets

We review the concepts of positive linear dependence and independence of vectors introduced by Davis [12] and used by many authors, like Robinson and Meyer [16], Robinson [17] and Qi and Wei [2]. These concepts are crucial in the development of the results of the next section.

Definition 2.3 Let $V=\{v_1,\dots,v_q\}$ and $W=\{w_1,\dots,w_\ell\}$ be two finite sets of vectors in $\mathbb{R}^n$. The pair of sets (V,W) is positive linearly dependent (PLD) if there exist scalars $\alpha\in\mathbb{R}^q$ and $\beta\in\mathbb{R}^\ell$ such that $\alpha\ge 0$, $(\alpha,\beta)\neq(0,0)$ and

$$\sum_{i=1}^{q}\alpha_i v_i+\sum_{j=1}^{\ell}\beta_j w_j=0.$$

Otherwise, the pair of sets (V,W) is positive linearly independent (PLI).

Remark 2.1 A single set $V=\{v_1,\dots,v_q\}\subset\mathbb{R}^n$ is PLD if there exists $\alpha\in\mathbb{R}^q$ such that $\alpha\ge 0$ and $\sum_{i=1}^{q}\alpha_i v_i=0$. Otherwise, the set V is PLI.

Next, let us introduce some notation associated to a given set of vectors $V=\{v_1,\dots,v_q\}\subset\mathbb{R}^n$:

$$V^-=\{u\in\mathbb{R}^n:\langle v_i,u\rangle<0,\ i=1,\dots,q\},$$
$$V^\perp=\{u\in\mathbb{R}^n:\langle v_i,u\rangle=0,\ i=1,\dots,q\},$$
$$V^\star=\{u\in\mathbb{R}^n:\langle v_i,u\rangle\le 0,\ i=1,\dots,q\}.$$

Proposition 2.1 *The following properties hold*:

(a) *The set V is PLI if and only if $V^-\neq\emptyset$ (i.e., the set of inequalities $\langle v_i,u\rangle<0$, $i=1,\dots,q$, has at least a nonzero solution $u\in\mathbb{R}^n$).*
(b) *The pair (V,W) is PLI if and only if W is linearly independent and $V^-\cap W^\perp\neq\emptyset$.*
(c) *Given the set $\widehat{V}=\{\widehat{v}_1,\dots,\widehat{v}_p\}\subset\mathbb{R}^n$, the pair $(V\cup\widehat{V},W)$ is PLI if and only if W is linearly independent and $V^-\cap\widehat{V}^-\cap W^\perp\neq\emptyset$.*

Proof Using the Gordan and Motzkin Theorems of the Alternative [18], the proofs are straightforward. □

Given $x\in\mathbb{R}^n$, let us introduce the following gradient sets associated to the VOP functions:

$$\mathcal{F}(x)=\{\nabla f_k(x):k\in K\},\qquad \mathcal{H}(x)=\{\nabla h_i(x):i\in I\},$$
$$\mathcal{G}(x)=\{\nabla g_j(x):j\in J\},\qquad \mathcal{G}_A(x)=\{\nabla g_j(x):j\in A(x)\}.$$

The next lemma states an equivalence property between the set of Fritz-John multipliers and the set of gradients of the VOP functions.

Lemma 2.1 *The set $M(x^\star) \neq \emptyset$ if and only if the pair $(\mathcal{F}(x^\star) \cup \mathcal{G}_A(x^\star), \mathcal{H}(x^\star))$ is PLD.*

Proof Whenever $M(x^\star) \neq \emptyset$, the Fritz-John conditions (1)–(3) are equivalent to

$$\sum_{k\in K} \theta_k \nabla f_k(x^\star) + \sum_{j\in A(x^\star)} \lambda_j \nabla g_j(x^\star) + \sum_{i\in I} \mu_i \nabla h_i(x^\star) = 0,$$
$$\lambda_j \geq 0, \quad \text{for all } j \in A(x^\star),$$
$$\lambda_j = 0, \quad \text{for all } j \in J - A(x^\star),$$
$$\theta_k \geq 0, \quad \text{for all } k \in K,$$

with $(\theta, \lambda, \mu) \neq (0, 0, 0)$. Then, by using Definition 2.3, such relationships are equivalent to saying that $(\mathcal{F}(x^\star) \cup \mathcal{G}_A(x^\star), \mathcal{H}(x^\star))$ is PLD. □

3 Regularity Conditions Based on PLD and PLI

Some preliminary properties of weak-regular and totally weak-regular points, already established in [15] under a different perspective, are presented to further enlighten the connections between the gradient vector sets of the constraints and the objective functions of the VOP.

The next theorem provides necessary and sufficient conditions for a point to be weak-regular.

Theorem 3.1 *Let us consider the VOP and $x^\star \in X$.*

(a) *If there exist $\mathcal{G}_0(x^\star) \subset \mathcal{G}_A(x^\star)$ and $\mathcal{H}_0(x^\star) \subset \mathcal{H}(x^\star)$ such that the pair $(\mathcal{G}_0(x^\star), \mathcal{H}_0(x^\star))$ is PLI and $(\mathcal{F}(x^\star) \cup \mathcal{G}_0(x^\star), \mathcal{H}_0(x^\star))$ is PLD, then $M(x^\star) \neq \emptyset$ and $x^\star$ is weak-regular.*
(b) *If $x^\star$ is weak-regular, then $\mathcal{F}(x^\star)$ is PLD, or there exist $\mathcal{G}_0(x^\star) \subset \mathcal{G}_A(x^\star)$ and $\mathcal{H}_0(x^\star) \subset \mathcal{H}(x^\star)$ such that the pair $(\mathcal{G}_0(x^\star), \mathcal{H}_0(x^\star))$ is PLI and $(\mathcal{F}(x^\star) \cup \mathcal{G}_0(x^\star), \mathcal{H}_0(x^\star))$ is PLD.*

Proof (a) The proof is immediate.

(b) Let us suppose that $x^\star$ weak-regular. Then, there exists $(\theta, \lambda, \mu) \in M(x^\star)$ such that $\theta \neq 0$ and

$$\sum_{k\in \widetilde{K}} \theta_k \nabla f_k(x^\star) + \sum_{j\in A(x^\star)} \lambda_j \nabla g_j(x^\star) + \sum_{i\in I} \mu_i \nabla h_i(x^\star) = 0, \tag{4}$$

where $\widetilde{K} = \{k \in K \mid \theta_k > 0\} \subseteq K$. Thus,

$$\sum_{k\in \widetilde{K}} \theta_k \nabla f_k(x^\star) = -\left(\sum_{i\in I} \mu_i \nabla h_i(x^\star) + \sum_{j\in A(x^\star)} \lambda_j \nabla g_j(x^\star)\right) = u \in \mathbb{R}^n.$$

(i) If $u=0$, then $\{\nabla f_k(x^\star) : k \in \widetilde{K}\}$ is PLD and so $\mathcal{F}(x^\star)$ is PLD.
(ii) If $u \neq 0$ then, by using Caratheodory's lemma [1, p. 689], there exist index sets $J_0 \subset A(x^\star)$ and $I_0 \subset I$ such that, defining $\mathcal{G}_0(x^\star) \equiv \{\nabla g_j(x^\star) : j \in J_0\}$ and $\mathcal{H}_0(x^\star) \equiv \{\nabla h_i(x^\star) : i \in I_0\}$, the pair $(\mathcal{G}_0(x^\star), \mathcal{H}_0(x^\star))$ is PLI and from (4) it follows that $(\{\nabla f_k(x^\star) : k \in \widetilde{K}\} \cup \mathcal{G}_0(x^\star), \mathcal{H}_0(x^\star))$ is PLD. Thus $(\mathcal{F}(x^\star) \cup \mathcal{G}_0(x^\star), \mathcal{H}_0(x^\star))$ is PLD. □

In the next theorem, a necessary and sufficient condition for $x^\star$ to be a totally weak-regular point is stated.

Theorem 3.2 *A point $x^\star \in X$ for which $M(x^\star) \neq \emptyset$ is totally weak-regular if and only if $(\mathcal{F}(x^\star) \cup \mathcal{G}_A(x^\star), \mathcal{H}(x^\star))$ is PLD and $(\mathcal{G}_A(x^\star), \mathcal{H}(x^\star))$ is PLI.*

Proof ($\Rightarrow$) If $(\mathcal{G}_A(x^\star), \mathcal{H}(x^\star))$ is PLD, then there exists a pair $(\bar{\lambda}, \bar{\mu}) \neq (0,0)$ with $\bar{\lambda} \geq 0$ such that

$$\sum_{j \in A(x^\star)} \bar{\lambda}_j \nabla g_j(x^\star) + \sum_{i \in I} \bar{\mu}_i \nabla h_i(x^\star) = 0.$$

Thus $(\theta, \lambda, \mu) \equiv (0, \bar{\lambda}, \bar{\mu}) \in M(x^\star)$ and $x^\star$ would not be totally weak-regular.

($\Leftarrow$) Let us assume that $(\mathcal{F}(x^\star) \cup \mathcal{G}_A(x^\star), \mathcal{H}(x^\star))$ PLD; then, by Lemma 2.1, there exists $(\theta, \lambda, \mu) \in M(x^\star) \neq \emptyset$, $(\theta, \lambda, \mu) \neq (0,0,0)$, $\theta \geq 0$, $\lambda \geq 0$. This means that

$$\sum_{k \in K} \theta_k \nabla f_k(x^\star) + \sum_{j \in A(x^\star)} \lambda_j \nabla g_j(x^\star) + \sum_{i \in I} \mu_i \nabla h_i(x^\star) = 0. \tag{5}$$

Now, since $(\mathcal{G}_A(x^\star), \mathcal{H}(x^\star))$ is PLI, there does not exist a pair $(\alpha, \beta) \neq (0,0)$ with $\alpha \geq 0$ such that

$$\sum_{j \in A(x^\star)} \alpha_j \nabla g_j(x^\star) + \sum_{i \in I} \beta_i \nabla h_i(x^\star) = 0.$$

Therefore, since $\lambda \geq 0$, it follows from (5) that

$$\sum_{k \in K} \theta_k \nabla f_k(x^\star) \neq 0, \quad \text{with } \theta_k \geq 0,$$

and $x^\star$ is totally weak-regular. □

The next definition recalls the Mangasarian–Fromovitz constraint qualification (MFCQ, [19]).

Definition 3.1 The MFCQ holds at $x \in X$ if:

(i) the set $\{\nabla h_i(x) : i \in I\}$ is linearly independent;
(ii) there exists $d \in \mathbb{R}^n$ such that

$$\nabla g_j(x)^T d < 0, \quad j \in A(x),$$
$$\nabla h_i(x)^T d = 0, \quad i \in I.$$

Remark 3.1 The condition established in Definition 3.1 was used by Da Cunha and Polak [14, Corollary 77, p. 114] to conclude that the multiplier vectors associated to the set $\mathcal{F}(x)$ are not zero. Subsequently, Bigi [15, Theorem 2.2.2(i), p. 42] proved that this property is a necessary and sufficient condition for a point to be totally weak-regular.

Remark 3.2 Using the gradient sets introduced in the previous section and Proposition 2.1(b), the MFCQ can be written as: the pair $(\mathcal{G}_A(x), \mathcal{H}(x))$ is PLI. Hence, Theorem 3.2 is equivalent to Theorem 2.2.2(i) of [15].

Now, we introduce the strict positive linear dependence (SPLD) regularity condition and the positive linear independence regularity condition (PLIRC). These conditions are based on the positive linear dependence or independence of sets of vectors, stated in Definition 2.3.

The main feature of the SPLD condition is that it is weaker than the Mangasarian-Fromovitz regularity condition (MFRC) introduced by Bigi [15] and recalled here in Definition 3.4. From the theoretical viewpoint, in the convergence theorems, the weaker is the regularity condition satisfied at a stationary point, the stronger are the related convergence results.

Concerning the second regularity condition introduced (PLIRC), we prove that it is necessary and sufficient for total regularity at a feasible point at which there exists a Fritz-John multiplier.

Definition 3.2 Let x be a feasible point of the VOP. The strict positive linear dependence (SPLD) regularity condition holds at x if, for each $s \in K$, there exist sets $\mathcal{F}_s(x) \subset \mathcal{F}(x)$, $\nabla f_s(x) \in \mathcal{F}_s(x)$ such that $(\mathcal{F}_s(x) \cup \mathcal{G}_A(x), \mathcal{H}(x))$ is PLD with corresponding scalar $\alpha_s > 0$.

In the next theorem, the SPLD regularity condition is shown to be a necessary and sufficient condition for the regularity of a feasible point for which there exists a Fritz-John multiplier.

Theorem 3.3 *Let us consider the VOP. The SPLD condition holds at $x^\star \in X$ if and only if, $M(x^\star) \neq \emptyset$ and $x^\star$ is a regular point.*

Proof ($\Rightarrow$) Let us assume first that $x^\star$ fulfills the SPLD condition. Then, for each $s \in K$,

$$\alpha_s^{(s)} \nabla f_s(x^\star) + \sum_{k \in K, k \neq s} \alpha_k^{(s)} \nabla f_k(x^\star) + \sum_{j \in A(x^\star)} \beta_j^{(s)} \nabla g_j(x^\star) + \sum_{i \in I} \gamma_i^{(s)} \nabla h_i(x^\star) = 0, \tag{6}$$

with $\alpha_s^{(s)} > 0$, $\alpha_k^{(s)} \geq 0$, $\beta_j^{(s)} \geq 0$, $\gamma^{(s)} \in \mathbb{R}^m$. Summing (6) over all s and rearranging conveniently the coefficients, we obtain

$$\sum_{k=1}^{r} \theta_k \nabla f_k(x^\star) + \sum_{j \in A(x^\star)} \lambda_j \nabla g_j(x^\star) + \sum_{i \in I} \mu_i \nabla h_i(x^\star) = 0,$$

where

$$\theta_k = \alpha_k^{(k)} + \sum_{s=1, s\neq k}^{r} \alpha_k^{(s)} > 0,$$

for all $k \in K$, with $\alpha_k^{(k)} > 0$ and where $\alpha_k^{(s)} \geq 0$ is the coefficient of $\nabla f_k(x^\star)$ in (6). In case $\nabla f_k(x^\star) = 0$, then the corresponding coefficient is taken $\theta_k > 0$. In a similar way, the coefficients λ_j and μ_i are determined,

$$\lambda_j = \sum_{s=1}^{r} \beta_j^{(s)}, \qquad \mu_i = \sum_{s=1}^{r} \gamma_i^{(s)}.$$

Hence, we are able to obtain (θ, λ, μ), where $\theta \in \mathbb{R}^r$ has all components greater than zero, $\mu \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}_+^{|A(x^\star)|}$, possibly including some $\lambda_j = 0$ to complete the vector $\lambda \in \mathbb{R}_+^p$, and $x^\star$ is regular.

($\Leftarrow$) Now, if $x^\star$ is regular, then there exists $(\bar{\theta}, \bar{\lambda}, \bar{\mu}) \in M(x^\star)$ such that $\bar{\theta}_k > 0$, for all $k \in K$, $\bar{\lambda} \in \mathbb{R}_+^p$, $\bar{\mu} \in \mathbb{R}^m$. That is,

$$\sum_{k=1}^{r} \bar{\theta}_k \nabla f_k(x^\star) + \sum_{j \in A(x^\star)} \bar{\lambda}_j \nabla g_j(x^\star) + \sum_{i \in I} \bar{\mu}_i \nabla h_i(x^\star) = 0,$$

and the SPLD condition holds at $x^\star$ for all $s \in K$. □

Remark 3.3 It is worth noticing that, due to the alternative result of Motzkin, the fulfillment of the SPLD condition at $x^\star$ can be rewritten as follows: for each $s \in K$, the system

$$\nabla f_s(x^\star)^T d < 0, \tag{7}$$

$$\nabla f_k(x^\star)^T d \leq 0, \quad k \in K, k \neq s, \tag{8}$$

$$\nabla g_j(x^\star)^T d \leq 0, \quad j \in A(x^\star), \tag{9}$$

$$\nabla h_i(x^\star)^T d = 0, \quad i \in I, \tag{10}$$

has no solution $d \in \mathbb{R}^n$. Therefore, the result expressed by our Theorem 3.3 is a restatement of Theorem 2.2.1(ii) of Bigi's thesis [15, p. 40].

Remark 3.4 For the sake of completeness, the system (7)-(10) is expressed in terms of the gradient sets as follows: for all $s \in K$,

$$\{\nabla f_s\}^- \cap \mathcal{F}_s^\star \cap \mathcal{G}_A^\star \cap \mathcal{H}^\perp = \emptyset,$$

where $\mathcal{F}_s = \mathcal{F} - \{\nabla f_s\}$ and all the gradients are evaluated at $x^\star$.

The next example shows that the SPLD regularity condition does not imply total regularity.

Example 3.1 Let us consider the VOP with $n=2, r=2, m=1, p=2$, $f_1(x)=-x_1$, $f_2(x)=x_2$, $h_1(x)=x_1-x_2^2$, and $g_1(x)=x_1^2-x_2$, $g_2(x)=-x_1$. We see that $x^\star=(0,0)^T$ is a locally weak Pareto optimal point. The system obtained from the Fritz-John necessary condition is given by

$$-\theta_1-\lambda_2+\mu_1=0,$$
$$\theta_2-\lambda_1=0.$$

The SPLD holds at $x^\star$ because $(\theta,\lambda,\mu)=(1,1;1,0;1)\in M(x^\star)$, but $x^\star$ is not totally weak-regular because $(\theta_1,\theta_2;\lambda_1,\lambda_2;\mu_1)=(0,0;0,1;1)\in M(x^\star)$ as well.

In the sequel, a regularity condition for VOP is introduced, which results to be necessary and sufficient for total regularity.

Definition 3.3 Let x be a feasible point of the VOP. The positive linear independence regularity condition (PLIRC) holds at x if:

(i) the pair $(\mathcal{G}_A(x),\mathcal{H}(x))$ is PLI;
(ii) for each $s\in K$, there does not exist $\alpha\in\mathbb{R}^{r-1}$, $\alpha_k\geq 0$, $\alpha\neq 0$, $\beta_j\geq 0$, $j\in A(x)$, $\gamma\in\mathbb{R}^m$ such that

$$\sum_{k\in K-\{s\}}\alpha_k\nabla f_k(x)+\sum_{j\in A(x)}\beta_j\nabla g_j(x)+\sum_{i\in I}\gamma_i\nabla h_i(x)=0. \tag{11}$$

Remark 3.5 Due to Motzkin's theorem, the second condition of PLIRC is equivalent to: for each $s\in K$, there exists $d\in\mathbb{R}^n$ such that

$$\nabla f_k(x)^T d<0,\quad k\in K, k\neq s, \tag{12}$$
$$\nabla g_j(x)^T d\leq 0,\quad j\in A(x), \tag{13}$$
$$\nabla h_i(x)^T d=0,\quad i\in I. \tag{14}$$

Remark 3.6 By using the gradient sets and the notation introduced in Remark 3.4, the system (12)–(14) can be rewritten as: for all $s\in K$,

$$\mathcal{F}_s(x)^-\cap\mathcal{G}_A^\star(x)\cap\mathcal{H}^\perp(x)\neq\emptyset;$$

combining the first and the second conditions of the PLIRC, the relation above can be equivalently stated as: for all $s\in K$, $\mathcal{H}(x)$ is LI and $\mathcal{F}_s^-(x)\cap\mathcal{G}_A^-(x)\cap\mathcal{H}^\perp(x)\neq\emptyset$. Thus, applying Proposition 2.1(c), the PLIRC holds at x if and only if the pair of sets $(\mathcal{F}_s(x)\cup\mathcal{G}_A(x),\mathcal{H}(x))$ is PLI.

Theorem 3.4 *Let $x^\star\in X$ be such that $M(x^\star)\neq\emptyset$. The VOP is totally regular at $x^\star$ if and only if the PLIRC holds at $x^\star$.*

Proof ($\Rightarrow$) To prove that a totally regular point $x^\star$ does satisfy conditions (i) and (ii) of Definition 3.3, we assume that the thesis does not hold and prove that $x^\star$ cannot be totally regular.

If $x^\star \in X$ is such that the pair $(\mathcal{G}_A(x^\star), \mathcal{H}(x^\star))$ is PLD, then the pair $(\mathcal{G}(x^\star), \mathcal{H}(x^\star))$ is also PLD and there exists scalar sets $\{\eta_i\}_{i\in I}$, $\{\xi_j\}_{j\in J}$ with $\xi_j \geq 0$, for all $j \in J$, $\sum_{i\in I} |\eta_i| + \sum_{j\in J} \xi_j > 0$ and

$$\sum_{i\in I} \eta_i \nabla h_i(x^\star) + \sum_{j\in J} \xi_j \nabla g_j(x^\star) = 0.$$

Thus, $(0, \eta, \xi) \in M(x^\star)$ and $x^\star$ is not totally regular.

Now, if $(\mathcal{G}_A(x^\star), \mathcal{H}(x^\star))$ is PLI, but the second condition is not verified, then for each $s \in K$ there exists $(\alpha, \beta, \gamma) \in \mathbb{R}^{r-1}_+ \times \mathbb{R}^p_+ \times \mathbb{R}^m$, $(\alpha, \beta, \gamma) \neq (0, 0, 0)$, $\beta_j = 0$, $j \in J - A(x^\star)$ such that (11) holds. Therefore, for each $s \in K$, we have $(\theta, \beta, \gamma) \in M(x^\star)$, $\theta \in \mathbb{R}^r$, with a null sth component and $x^\star$ is not totally regular.

($\Leftarrow$) Let us assume that $x^\star$ is not totally regular. Since $M(x^\star) \neq \emptyset$, two cases should be considered.

Case 1. If $\theta_k = 0$ for all $k \in K$, the Fritz-John relationship (1) becomes

$$\sum_{j\in A(x^\star)} \lambda_j \nabla g_j(x^\star) + \sum_{i\in I} \mu_i \nabla h_i(x^\star) = 0,$$

so that the pair $(\mathcal{G}_A(x^\star), \mathcal{H}(x^\star))$ is PLD, which contradicts the positive linearly independence assumption.

Case 2. If there exists $\widehat{K} \subset K$ such that $\theta_k = 0$ for all $k \in \widehat{K}$, then the Fritz-John condition (1) becomes:

$$\sum_{k\in K-\widehat{K}} \theta_k \nabla f_k(x^\star) + \sum_{j\in A(x^\star)} \lambda_j \nabla g_j(x^\star) + \sum_{i\in I} \mu_i \nabla h_i(x^\star) = 0. \tag{15}$$

By the Motzkin alternative theorem, condition (ii) of Definition 3.3 is equivalent to: for each $s \in K$ there exists $d \in \mathbb{R}^n$, such that:

$$\nabla f_k(x^\star)^T d < 0, \quad k \in K, k \neq s, \tag{16}$$

$$\nabla g_j(x^\star)^T d \leq 0, \quad j \in A(x^\star), \tag{17}$$

$$\nabla h_i(x^\star)^T d = 0, \quad i \in I. \tag{18}$$

Then, taking $s \in \widehat{K}$ and multiplying (15) by d satisfying (16)–(18), we obtain

$$0 = \sum_{k\in K-\widehat{K}} \theta_k \nabla f_k(x^\star)^T d + \sum_{j\in A(x^\star)} \lambda_j \nabla g_j(x^\star)^T d < 0,$$

which is a contradiction. □

Another regularity condition from the literature (cf. [15]) that enlarges the perspective of the PLIRC condition is recalled below.

Definition 3.4 The Mangasarian-Fromovitz regularity condition (MFRC) is satisfied at $x \in X$ if:

(i) the set $\{\nabla h_i(x) : i \in I\}$ is linearly independent;
(ii) for all $s \in K$, there exists $d \in \mathbb{R}^n$ so that

$$\begin{aligned} \nabla f_k(x)^T d < 0, &\quad k \in K, k \neq s, \\ \nabla g_j(x)^T d < 0, &\quad j \in A(x), \\ \nabla h_i(x)^T d = 0, &\quad i \in I. \end{aligned}$$

Remark 3.7 Once again, we observe that the MFRC at x can be expressed in terms of the gradient sets as follows:

(i) the set $\mathcal{H}(x)$ is LI;
(ii) for all $s \in K$ the set $\mathcal{F}_s^-(x) \cap \mathcal{G}_A^-(x) \cap \mathcal{H}^\perp(x) \neq \emptyset$,

that is, from Proposition 2.1(c), for all $s \in K$, $(\mathcal{F}_s(x) \cup \mathcal{G}_A(x), \mathcal{H}(x))$ is PLI.

The following result establishes a relationship between the PLIRC and MFRC.

Theorem 3.5 *Let $x^\star \in X$ be such that $M(x^\star) \neq \emptyset$. The PLIRC holds at $x^\star$ if and only if the MFRC holds at $x^\star$.*

Proof The proof is immediate from Remarks 3.6 and 3.7. □

Remark 3.8 For vector optimization problems in which the index set $I = \emptyset$, the MFRC becomes the so-called Cottle-type constraint qualification (CCQ, [5]).

Our PLIRC is an alternative to the MFRC, with the same order of complexity as far as verification is concerned, but that exploits the positive linear independence notion. Moreover, although distinct than MFRC, our condition turns out to be equivalent to it. Therefore, a result similar to Bigi's on the boundedness of a normalized set of multipliers in case the MFRC holds [15, Theorem 2.3.2, p. 44] is valid for the PLIRC.

4 Conclusions

The main contribution of this work is the use of the concept of positive linear dependence within the vector optimization context, (i) restating already known regularity conditions within this perspective and (ii) presenting new results along such point of view, without any convexity assumption.

The positive linear dependence and independence are tools that allow a direct analysis of the interrelations among the gradients of the objective functions and the constraints. As a consequence, the distinct notions of regularity that might occur in the vector optimization field emerge naturally in connection with the Fritz-John necessary condition for optimality. Approaches involving separation sets, contingent cones, linearized approximations to the feasible set, etc., are related to the Fritz-John condition indirectly by means of alternative results like Motzkin's theorem.

For the general vector optimization problem, we have proved a sufficient condition for weak regularity, based on the notions of PLI and PLD (Theorem 3.1). Moreover,

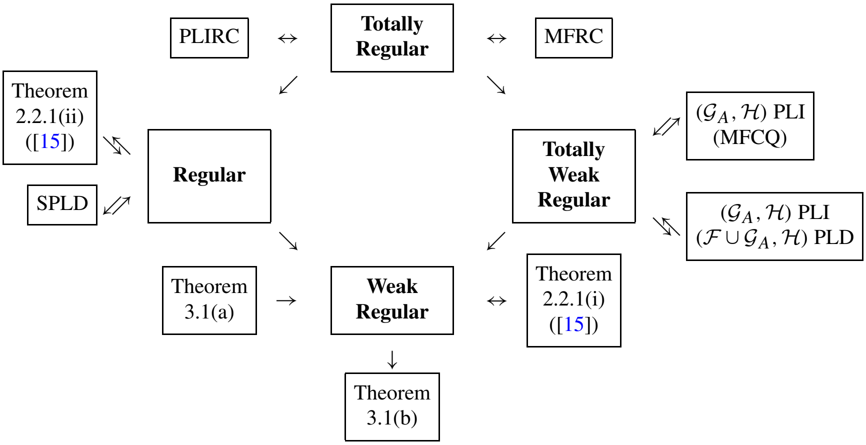

Fig. 1 Interrelations of regularity conditions for the VOP

we have shown that the positive linear dependence condition is necessary for totally weak regularity (Theorem 3.2). The SPLD regularity condition has been introduced for the VOP. It was proved to be necessary and sufficient for regularity in the general case (Theorem 3.3). Another regularity condition proposed for the VOP, namely the PLIRC, proved to be necessary and sufficient for total regularity (Theorem 3.4). Despite being distinct from the Mangasarian-Fromovitz regularity condition employed by Bigi, our PLIRC turned out to be equivalent to MFRC (Theorem 3.5). Due to this equivalence, a result on the boundedness of a normalized set of multipliers associated to a feasible point for which the PLIRC holds is valid. Finally, summarizing our discussion, we have put all the results and remarks together in the diagram of Fig. 1.

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