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Extended Well-Posedness of Quasiconvex Vector Optimization Problems

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Abstract The notion of extended-well-posedness has been introduced by Zolezzi for scalar minimization problems and has been further generalized to vector minimization problems by Huang. In this paper, we study the extended well-posedness properties of vector minimization problems in which the objective function is C-quasiconvex. To achieve this task, we first study some stability properties of such problems.

Keywords Vector optimization · Well-posedness · Stability

1 Introduction

The notion of well-posedness for scalar optimization problems has been deeply studied (see e.g. Refs. [1, 2] for a review of the topic). Basically, two different approaches are known. The former is due to Hadamard (Ref. [3]), and it concerns the stability of the optimal solutions with respect to perturbations of the optimization problem (i.e. of the objective function and the feasible region). For this reason, Hadamard well-posedness is often called also “stability” and in this paper we follow this convention. The latter has been introduced by Tykhonov (see Ref. [4]) and it is based on the convergence of minimizing sequences. The relations between the two approaches have

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been widely studied (see e.g. Refs. [1, 2]). In Ref. [5] the notion of extended well-posedness has been proposed. In some sense this notion unifies the ideas of Tykhonov and Hadamard well-posedness, allowing perturbations of the objective function (but not of the feasible region).

The development of the well-posedness notion for vector optimization problems is less systematic. Several definitions have been proposed with regard to vector minimization problems (see e.g. Ref. [6]). Among recent contributions to this topic we recall Refs. [7–10]. In particular, in Ref. [9] vector well-posedness notions have been separated into two types: pointwise notions and global notions. The first class includes those definitions which consider a fixed solution point and deal with well-posedness of the vector optimization problem at this point. The second class considers those definitions which involve the efficient frontier as a whole. Moreover, the notion of extended well-posedness has been generalized to vector optimization problems by Huang in Refs. [11, 12].

One of the main tasks in studying well-posedness is to find classes of problems that enjoy such property (see e.g. Refs. [1, 2]). It is worth to recall here that scalar optimization problems with quasiconvex objective function are well-posed in the extended sense (Ref. [2]). The search for classes of well-posed optimization problems is being performed also in the vector case. For instance, in Refs. [7–9] it has been shown that vector quasiconvex functions enjoy well-posedness properties of Tykhonov type and in Ref. [13] it has been proved that vector convex functions enjoy stability properties.

The previous considerations arise naturally the question whether vector optimization problems with convex or quasiconvex objective function are well-posed in the extended sense.

In this paper, we study an even more general problem. Indeed, we slightly generalize Huang's definition of extended well-posedness in order to consider also perturbations of the feasible region of the problem. This leads to consider a well-posedness notion for vector optimization problems that fully combines the features of stability and Tykhonov well-posedness. Our main result shows that, under some assumptions, vector quasiconvex functions enjoy such well-posedness property (and a fortiori enjoy Huang's extended well-posedness property). We show also that for convex vector functions these well-posedness properties can be proved under simpler assumptions. In order to prove such results, in Sect. 3 we first investigate the stability properties of quasiconvex vector minimization problems.

2 Preliminaries

Consider a function $f : \mathbb{R}^m \to \mathbb{R}^l$, let X be a closed convex subset of $\mathbb{R}^m$ and let $C \subseteq \mathbb{R}^l$ be a closed convex pointed cone with nonempty interior. We deal with the vector optimization problem

$$\mathrm{VP}(f, X) \quad C\text{-}\min f(x), \quad x \in X.$$

We recall that a point $x \in X$ is said to be an efficient solution of $\mathrm{VP}(f, X)$ when

$$(f(X) - f(x)) \cap (-C) = \{0\},$$

while $x \in X$ is said to be a weakly efficient solution of VP(f, X) when

$$(f(X) - f(x)) \cap (-\text{int}\, C) = \emptyset.$$

We denote by Eff(f, X) the set of all efficient solutions of the problem VP(f, X) and by Min(f, X) the set of all minimal points, i.e. the image of Eff(f, X) through the objective function f. Further WEff(f, X) is the set of weakly efficient solutions and WMin(f, X) the image of WEff(f, X) through the objective function f.

Convexity and its generalizations play a crucial role to define classes of functions which imply well-posedness of VP(f, X). We briefly recall some classic definitions.

Definition 2.1 (Ref. [14]) A function $f : \mathbb{R}^m \to \mathbb{R}^l$ is said to be:

(i) C-convex if

$$f\left(\lambda x^1 + (1-\lambda)x^2\right) - \lambda f\left(x^1\right) - (1-\lambda) f\left(x^2\right) \in -C,$$

for every $x^1, x^2 \in \mathbb{R}^m$ and $\lambda \in [0, 1]$.

(ii) C-quasiconvex if, for every $y \in \mathbb{R}^l$ the level sets

$$\text{Lev}(f, y) := \left\{x \in \mathbb{R}^m : f(x) \in y - C\right\}$$

are either empty or convex.

(iii) Strictly C-quasiconvex when, $\forall y \in \mathbb{R}^l$ and $x^1, x^2 \in X$, $x^1 \neq x^2$, $t \in (0, 1)$,

$$f\left(x^1\right), f\left(x^2\right) \in y - C$$

imply $f(tx^1 + (1-t)x^2) \in y - \text{int}\, C$.

In the following, we set Lev$(f, y, X) = $ Lev$(f, y) \cap X$. The proof of the next proposition is immediate and we omit it.

Proposition 2.1 *Let f be continuous and strictly C-quasiconvex. Then*:

(i) WEff$(f, X) = $ Eff(f, X);

(ii) $\forall y \in$ Min(f, X), $f^{-1}(y)$ *is a singleton.*

Since we deal with set convergence in Euclidean spaces, we shall consider the Kuratowski-Painlevé set-convergence (see e.g. Ref. [2]). Let A_n be a sequence of subsets of $\mathbb{R}^m$. Set

$$\text{Ls}A_n := \left\{x \in \mathbb{R}^m \;:\; x = \lim_{k\to+\infty} x^k,\; x^k \in A_{n_k},\; n_k \text{ a subsequence of the integers}\right\},$$

$$\text{Li}A_n := \left\{x \in \mathbb{R}^m \;:\; x = \lim_{k\to+\infty} x^k,\; x^k \in A_k,\; \text{eventually}\right\}.$$

The set LsA_n is called the upper limit of the sequence of sets A_n, while the set Li A_n is called the lower limit of A_n. We say that the sequence A_n converges in the sense of Kuratowski to the set A when

$$\operatorname{Ls} A_n \subseteq A \subseteq \operatorname{Li} A_n$$

and we denote this convergence by $A_n \xrightarrow{K} A$.

When needed, we may use a fixed vector $e \in \operatorname{int} C$. If no confusion occurs, we may omit to explicitely assume what e represents in the following sections.

3 Stability of Quasiconvex Vector Minimization Problems

Stability properties of vector optimization problems have been studied in Ref. [13] when the objective function is C-convex. Here we give some extensions of these properties to the case of a C-quasiconvex function which have a relevant role in the next section. Let $f_n : \mathbb{R}^m \to \mathbb{R}^l$ be a sequence of functions and let X_n be a sequence of subsets of $\mathbb{R}^m$. Together with problem $\mathrm{VP}(f, X)$, we consider problems $\mathrm{VP}(f_n, X_n)$ and we investigate the behaviour of the sets $\mathrm{WEff}(f_n, X_n)$, $\mathrm{Eff}(f_n, X_n)$, $\mathrm{WMin}(f_n, X_n)$, $\mathrm{Min}(f_n, X_n)$, when f_n and X_n “approach” to f and X respectively.

In the following, B denotes the closed unit ball both in $\mathbb{R}^m$ and in $\mathbb{R}^l$. From the context it will be clear to which space we refer.

Lemma 3.1 *Let $f_n : \mathbb{R}^m \to \mathbb{R}^l$ and $f : \mathbb{R}^m \to \mathbb{R}^l$ be continuous C-quasiconvex functions, $y \in \mathbb{R}^l$ and $y^n \to y$. Assume that:*

(i) *$f_n \to f$ in the continuous convergence.*
(ii) $X_n \xrightarrow{K} X$.
(iii) $\mathrm{Lev}(f, y, X)$ *is nonempty and bounded.*

Then, $\forall \varepsilon > 0$, it holds that

$$\mathrm{Lev}(f_n, y^n, X_n) \subseteq \mathrm{Lev}(f, y, X) + \varepsilon B,$$

eventually.

Proof Assume the contrary. Then, one can find a number $\bar{\varepsilon} > 0$ such that, $\forall n$ of some subsequence, there exists a point $x^n \in \mathrm{Lev}(f_n, y^n, X_n)$ with

$$x^n \notin \mathrm{Lev}(f, y, X) + \bar{\varepsilon} B.$$

Let $\hat{x} \in \mathrm{Lev}(f, y, X)$. Since $X_n \xrightarrow{K} X$, we can find a sequence $\hat{x}^n \in X_n$ such that $\hat{x}^n \to \hat{x}$. Since f is continuous we have $f(\hat{x}^n) \to f(\hat{x}) \in y - C$ and hence for $\alpha > 0$, we get $f_n(\hat{x}^n) \in y - C + \alpha e$, eventually, i.e.

$$\hat{x}^n \in \mathrm{Lev}(f_n, y + \alpha e, X_n). \tag{1}$$

Moreover, since $y^n \to y$, for $e \in \operatorname{int} C$ and $\alpha > 0$, we have $y^n \in y - C + \alpha e$, eventually and hence it follows $f_n(x^n) \in y - C + \alpha e$, eventually. Let $x^n(t) = tx^n + (1-t)\hat{x}^n$, $t \in [0, 1]$. From the C-quasiconvexity of f_n we obtain the existence of an integer $\bar{n} = \bar{n}(\alpha)$ such that $f_n(x^n(t)) \in y - C + \alpha e$ for every $t \in [0, 1]$

and $n > \bar{n}$. For every $n > \bar{n}$, we can find a number $t_n \in [0, 1]$, which satisfies

$$x^n(t_n) \in \partial[\mathrm{Lev}(f, y, X) + \bar{\varepsilon}B].$$

Indeed, it is enough to observe that, since $\hat{x}^n \to \hat{x}$, it holds $\hat{x}^n \in \mathrm{Lev}(f, y, X) + \bar{\varepsilon}B$, eventually, while $x^n \notin \mathrm{Lev}(f, y, X) + \bar{\varepsilon}B$.

Since $\mathrm{Lev}(f, y, X) + \bar{\varepsilon}B$ is compact, without loss of generality we can assume $x^n(t_n) \to \tilde{x} \in \partial[\mathrm{Lev}(f, y, X)+\bar{\varepsilon}B]$ and from $f_n \to f$ in the continuous convergence, we get also $f_n(x^n(t_n)) \to f(\tilde{x}) \in y - C + \alpha e$. Since $X_n \overset{K}{\to} X$, we get $\tilde{x} \in X$ and since α is arbitrary we conclude $f(\tilde{x}) \in y - C$, i.e. $\tilde{x} \in \mathrm{Lev}(f, y, X)$, which is a contradiction. □

Theorem 3.1 *Let $f_n : \mathbb{R}^m \to \mathbb{R}^l$, $f : \mathbb{R}^m \to \mathbb{R}^l$ be continuous, C-quasiconvex functions with $f_n \to f$ in the continuous convergence and $X_n \overset{K}{\to} X$. Assume that the level sets of f, $\mathrm{Lev}(f, y, X)$, are bounded when nonempty.*

(i) *If $y \in \mathrm{Min}(f, X)$, there exists a sequence $y^n \in \mathrm{Min}(f_n, X_n)$ such that $y^n \to y$, i.e. $\mathrm{Li}\,\mathrm{Min}(f_n, X_n) \supseteq \mathrm{Min}(f, X)$.*
(ii) *If $y \in \mathrm{Min}(f, X)$, there exist $\bar{x} \in f^{-1}(y)$ and a sequence $x^n \in \mathrm{Eff}(f_n, X_n)$, which admits a subsequence x^{n_k} converging to $\bar{x}$.*
(iii) *If f is strictly C-quasiconvex, then we have:*
   (a) $\mathrm{Min}(f_n, X_n) \overset{K}{\to} \mathrm{Min}(f, X)$;
   (b) $\mathrm{Eff}(f_n, X_n) \overset{K}{\to} \mathrm{Eff}(f, X)$.

Proof (i) Let $y \in \mathrm{Min}(f, X)$ and consider the level set $\mathrm{Lev}(f, y, X) = f^{-1}(y)$. The assumptions ensure $f^{-1}(y)$ is compact. Let $\bar{x} \in f^{-1}(y)$.

From $\bar{x} \in X$, and $X_n \overset{K}{\to} X$, we get the existence of a sequence $z^n \in X_n$, $z^n \to \bar{x}$. Since $f_n \to f$ in the continuous convergence, we get $f_n(z^n) \to f(\bar{x})$ and hence, for $e \in \mathrm{int}\,C$, we can find a sequence $\alpha_n \to 0^+$ such that

$$f_n(z^n) \in y + \alpha_n e - C,$$

i.e. $z^n \in \mathrm{Lev}(f_n, w^n, X_n)$ with $w^n = y + \alpha_n e \to y$. Using Lemma 3.1, for every $\varepsilon > 0$ we get

$$\mathrm{Lev}(f_n, w^n, X_n) \subseteq \mathrm{Lev}(f, y, X) + \varepsilon B = f^{-1}(y) + \varepsilon B, \tag{2}$$

eventually. From the assumptions we get that both $\mathrm{Lev}(f_n, w^n, X_n)$ and $f_n(\mathrm{Lev}(f_n, w^n, X_n))$ are compact. Hence $\mathrm{Min}(f_n, \mathrm{Lev}(f_n, w^n, X_n))$ is nonempty (see Ref. [14]). From the assumptions and (2), we get

$$f\left(\mathrm{Lev}(f_n, w^n, X_n)\right) \subseteq f\left(\mathrm{Lev}(f, y, X) + \varepsilon B\right),$$

eventually and hence

$$\begin{aligned}\mathrm{Min}\left(f_n, \mathrm{Lev}(f_n, w^n, X_n)\right) &\subseteq f\left(\mathrm{Lev}(f, y, X) + \varepsilon B\right)\\ &= f\left(f^{-1}(y) + \varepsilon B\right),\end{aligned}$$

eventually. Since f is continuous, for every $\delta > 0$ there exists $\varepsilon > 0$ such that $f(f^{-1}(y) + \varepsilon B) \subseteq y + \delta B$ and hence

$$\mathrm{Min}(f_n, \mathrm{Lev}(f_n, w^n, X_n)) \subseteq y + \delta B,$$

eventually. Let $y^n \in \mathrm{Min}(f_n, \mathrm{Lev}(f_n, w^n, X_n))$. Then we can assume $y^n \to y$, and the proof is complete observing that $\mathrm{Min}(f_n, \mathrm{Lev}(f_n, w^n, X_n)) \subseteq \mathrm{Min}(f_n, X_n)$.

(ii) Let $y^n \in \mathrm{Min}(f_n, \mathrm{Lev}(f_n, w^n, X_n))$ be the sequence previously found at point (i) and let $x^n \in f^{-1}(y^n)$. We have

$$x^n \in \mathrm{Eff}(f_n, \mathrm{Lev}(f_n, w^n, X_n)) \subseteq \mathrm{Eff}(f_n, X_n).$$

Since $\mathrm{Eff}(f_n, \mathrm{Lev}(f_n, w^n, X_n)) \subseteq \mathrm{Lev}(f_n, w^n, X_n) \subseteq f^{-1}(y) + \varepsilon B$, eventually, ε is arbitrary and $f^{-1}(y)$ is compact, we obtain the existence of a subsequence of x^n converging to some point $\bar{x} \in f^{-1}(y)$.

(iii) At point (i) we have proved $\mathrm{Li}\,\mathrm{Min}(f_n, X_n) \supseteq \mathrm{Min}(f, X)$. It remains to prove $\mathrm{Ls}\,\mathrm{Min}(f_n, X_n) \subseteq \mathrm{Min}(f, X)$. Let $y^n \in \mathrm{Min}(f_n, X_n)$ and assume y^n admits a convergent subsequence y^{n_k}. Since f is strictly C-quasiconvex we have $\mathrm{Min}(f, X) = \mathrm{WMin}(f, X)$ (see Proposition 2.1). Assume by contradiction $y^{n_k} \to y \notin \mathrm{Min}(f, X)$. Hence there exists $\bar{x} \in X$ such that $f(\bar{x}) - y \in -\mathrm{int}\, C$. Since $\bar{x} \in X$ and $X_n \overset{K}{\to} X$, there exists a sequence $x^n \in X_n$, with $x^n \to \bar{x}$. From $f(\bar{x}) - y \in -\mathrm{int}\, C$, recalling $f_{n_k}(x^{n_k}) \to f(\bar{x})$, it follows easily $f_{n_k}(x^{n_k}) - y^{n_k} \in -\mathrm{int}\, C - \alpha e \subseteq -\mathrm{int}\, C$, eventually, which contradicts $y^{n_k} \in \mathrm{Min}(f_{n_k}, X_{n_k})$ and a) is proved. To prove b) it is enough to recall $f^{-1}(y)$ is a singleton and the proof easily follows from ii). □

Remark 3.1 One can easily check that in Theorem 3.1 the boundedness assumption on the level sets can be replaced with the weaker requirement that $f^{-1}(y)$ is bounded for every $y \in \mathrm{Min}(f, X)$. This condition is certainly satisfied when f is strictly C-quasiconvex (see Proposition 2.1).

Remark 3.2 It is known that every C-convex functions is continuous (see Ref. [15]). Hence, when f and f_n are C-convex functions, the continuity assumption in Theorem 3.1 is superfluous. Further, in this case, in Theorem 3.1, it is enough to require the existence of $y \in \mathbb{R}^l$ such that $\mathrm{Lev}(f, y, X)$ is nonempty and bounded. Indeed for a C-convex function, the boundedness of one of the nonempty level sets $\mathrm{Lev}(f, y, X)$ is equivalent to the boundedness of all the level sets (see Ref. [13]).

4 Extended Well-Posedness of Quasiconvex Vector Minimization Problems

The concept of extended well-posedness for vector (and also set-valued) optimization problems is due to Huang (see Refs. [11, 12]), who has generalized a concept introduced in the scalar case by Zolezzi (see Ref. [5]). In this section we introduce a slight generalization of Huang's definition and we show that vector optimization problems with quasiconvex or convex objective functions enjoy, under certain assumptions, this well-posedness property. Throughout this section we assume $\mathrm{WEff}(f, X) \neq \emptyset$. Further $d(y, A) = \inf\{\|y - a\|, a \in A\}$ denotes the distance of a point $y \in \mathbb{R}^l$ from a set $A \subseteq \mathbb{R}^l$. We introduce first the following definition.

Definition 4.1 Let $f_n : \mathbb{R}^m \to \mathbb{R}^l$ be a sequence of functions, let $f : \mathbb{R}^m \to \mathbb{R}^l$ and let X_n be a sequence of subsets of $\mathbb{R}^m$. Problem VP(f, X) satisfies property (P) (with respect to the perturbations defined by the sequences f_n and X_n) when, for every sequence $x^n \in X_n$ such that

$$(f_n(X_n) - f_n(x^n)) \cap (-\operatorname{int} C - \varepsilon_n e) = \emptyset, \tag{3}$$

for some sequence $\varepsilon_n \to 0^+$, there exists a subsequence x^{n_k} of x^n such that $d(x^{n_k}, \mathrm{WEff}(f, X)) \to 0$, as $k \to +\infty$.

It can be shown that the previous definition does not depend on the choice of the vector $e \in \operatorname{int} C$. The proof of this statement can be given along the lines of Proposition 3.3 in Ref. [7].

Observe that when $\mathrm{WEff}(f, X)$ is compact, the requirement $d(x^{n_k}, \mathrm{WEff}(f, X)) \to 0$, amounts to the existence of a point $\bar{x} \in \mathrm{WEff}(f, X)$ such that x^{n_k} converges to $\bar{x}$.

Theorem 4.1 *Let $f : \mathbb{R}^m \to \mathbb{R}^l$ and $f_n : \mathbb{R}^m \to \mathbb{R}^l$ be continuous and C-quasiconvex, with $f_n \to f$ in the continuous convergence. Let X_n be a sequence of closed convex subsets of $\mathbb{R}^m$ such that $X_n \xrightarrow{K} X$. Assume that, for every $y \in \mathbb{R}^l$, $\mathrm{Lev}(f, y, X)$ is bounded and let $\mathrm{WEff}(f, X)$ be bounded. Assume further that there exists $\bar{n} \in \mathbb{N}$ such that $\mathrm{Lev}(f_n, y, X_n)$ is bounded for every $y \in \mathbb{R}^l$ and for every $n > \bar{n}$. Then, problem VP(f, X) satisfies property (P) with respect to the perturbations defined by the sequences f_n and X_n.*

Proof Let

$$\mathrm{WEff}_{\varepsilon_n e}(f_n, X_n) = \{x \in X_n : (f_n(X_n) - f_n(x)) \cap (-\operatorname{int} C - \varepsilon_n e) = \emptyset\}.$$

Assume that VP(f, X) does not satisfy property (P). Then we can find sequences $\varepsilon_n \to 0^+$, $x^n \in \mathrm{WEff}_{\varepsilon_n e}(f_n, X_n)$, such that, for some $\delta > 0$, it holds that $x^n \notin \mathrm{WEff}(f, X) + \delta B$, eventually.

We claim that for every sufficiently large n there exists a point $z^n \in \partial[\mathrm{WEff}(f, X) + \delta B]$ such that $z^n \in \mathrm{WEff}_{\varepsilon_n e}(f_n, X_n)$. Indeed, if such a z^n does not exist, we would have for some n

$$\mathrm{WEff}_{\varepsilon_n e}(f_n, X_n) \subseteq \operatorname{int}\left[\mathrm{WEff}(f, X) + \delta B\right] \cup \left[\mathrm{WEff}(f, X) + \delta B\right]^c. \tag{4}$$

Clearly $\mathrm{WEff}_{\varepsilon_n e}(f_n, X_n) \cap [\mathrm{WEff}(f, X) + \delta B]^c \neq \emptyset$. We now prove that

$$\mathrm{WEff}_{\varepsilon_n e}(f_n, X_n) \cap \operatorname{int}\left[\mathrm{WEff}(f, X) + \delta B\right] \neq \emptyset, \tag{5}$$

eventually. Since

$$\mathrm{WEff}(f_n, X_n) \subseteq \mathrm{WEff}_{\varepsilon_n e}(f_n, X_n),$$

it is enough to prove that

$$\mathrm{WEff}(f_n, X_n) \cap \operatorname{int}[\mathrm{WEff}(f, X) + \delta B] \neq \emptyset, \tag{6}$$

eventually. Let $y \in f(X)$ be fixed. The level set $\mathrm{Lev}(f, y, X)$ is nonempty since $f^{-1}(y) \subseteq \mathrm{Lev}(f, y, X)$ and from the assumptions we obtain that both $\mathrm{Lev}(f, y, X)$ and $f(\mathrm{Lev}(f, y, X))$ are compact.

It follows (see Ref. [14]) that $\mathrm{Min}(f, \mathrm{Lev}(f, y, X))$ is nonempty and since $\mathrm{Min}(f, \mathrm{Lev}(f, y, X)) \subseteq \mathrm{Min}(f, X)$ also $\mathrm{Min}(f, X)$ is nonempty.

Let $y \in \mathrm{Min}(f, X)$. From Theorem 3.1(ii), we get the existence of a point $\bar{x} \in f^{-1}(y) \subseteq \mathrm{Eff}(f, X)$ and a sequence $v^n \in \mathrm{Eff}(f_n, X_n)$, which admits a subsequence converging to $\bar{x}$. Avoiding relabeling, we can assume, without loss of generality, $v^n \to \bar{x}$.

Recalling $\mathrm{Eff}(f, X) \subseteq \mathrm{WEff}(f, X)$, it follows easily that (6) holds and hence (5) holds.

Since there exists $\bar{n} \in \mathbb{N}$ such that $\mathrm{Lev}(f_n, y, X_n)$ is bounded $\forall y \in \mathbb{R}^l$ and for all $n > \bar{n}$, the sets $\mathrm{WEff}_{\varepsilon_n e}(f_n, X_n)$ are connected, nonempty and closed for $n > \bar{n}$ (see Theorem 4 in Ref. [7]) and hence (4) cannot hold. It follows the existence of a sequence $z^n \in \partial[\mathrm{WEff}(f, X) + \delta B] \cap \mathrm{WEff}_{\varepsilon_n e}(f_n, X_n)$. Since $\mathrm{WEff}(f, X)$ is compact, we can assume z^n converges to a point $\bar{z}$ and since $X_n \xrightarrow{K} X$, it follows $\bar{z} \in X$. Since $z^n \in \mathrm{WEff}_{\varepsilon_n e}(f_n, X_n)$ it follows $\bar{z} \in \mathrm{WEff}(f, X)$. Indeed, if $\bar{z} \notin \mathrm{WEff}(f, X)$, there exists $x \in X$ such that $f(x) - f(\bar{z}) \in -\mathrm{int}\, C$ and hence we can find a positive number $\bar{\delta}$, such that

$$f(x) - f(\bar{z}) \in -\mathrm{int}\, C - \bar{\delta} e. \tag{7}$$

Since $x \in X$, there exists a sequence $w^n \to x$, $w^n \in X_n$ and from (7), we obtain $f_n(w^n) - f_n(z^n) \in -\mathrm{int}\, C - \bar{\delta} e$, eventually, which contradicts to $z^n \in \mathrm{WEff}_{\varepsilon_n e}(f_n, X_n)$. To complete the proof it is enough to observe that from $z^n \in \partial[\mathrm{WEff}(f, X) + \delta B]$ we get the contradiction $\bar{z} \notin \mathrm{WEff}(f, X)$. □

Remark 4.1 Actually, we cannot apply Lemma 3.1, to achieve boundedness of $\mathrm{Lev}(f_n, y, X_n)$ from the same property for $\mathrm{Lev}(f, y, X)$. Indeed here we require something stronger, namely that it can be fixed the same $\bar{n}$ for every y, while Lemma 3.1 implies only that such $\bar{n}$ exists for every y, possibly depending on it.

When f and f_n are C-convex functions, the assumptions of Theorem 4.1 can be simplified. Indeed, we get the following:

Corollary 4.1 *Let $f : \mathbb{R}^m \to \mathbb{R}^l$ and $f_n : \mathbb{R}^m \to \mathbb{R}^l$ be C-convex functions, with $f_n \to f$ in the continuous convergence and assume that* $\mathrm{WEff}(f, X)$ *is bounded. Then, problem* $\mathrm{VP}(f, X)$ *satisfies property* (P) *with respect to the perturbations defined by f_n and X_n.*

Proof It is known (see Ref. [15]) that C-convex functions are continuous. If $\bar{y} = f(\bar{x})$, with $\bar{x} \in \mathrm{WEff}(f, X)$, the level set $\mathrm{Lev}(f, \bar{y}, X)$ is clearly nonempty and further we have $\mathrm{Lev}(f, \bar{y}, X) \subseteq \mathrm{WEff}(f, X)$. Indeed, assume there exists a point $x' \in \mathrm{Lev}(f, \bar{y}, X) \backslash \mathrm{WEff}(f, X)$. Hence $f(x') \in f(\bar{x}) - C$ and we can find a point $x'' \in X$ such that $f(x'') \in f(x') - \mathrm{int}\, C$. This entails $f(x'') \in f(x') - \mathrm{int}\, C \subseteq f(\bar{x}) - \mathrm{int}\, C$, which contradicts to $\bar{x} \in \mathrm{WEff}(f, X)$.

The inclusion $\mathrm{Lev}(f, \bar{y}, X) \subseteq \mathrm{WEff}(f, X)$ proves $\mathrm{Lev}(f, \bar{y}, X)$ is bounded. From Lemma 3.1, we get

$$\mathrm{Lev}(f_n, \bar{y}, X_n) \subseteq \mathrm{Lev}(f, \bar{y}, X) + \varepsilon B, \tag{8}$$

eventually. Hence there exists $\bar{n} \in \mathbb{N}$ such that $\mathrm{Lev}(f_n, \bar{y}, X_n)$ is bounded for $n > \bar{n}$. Since f_n are C-convex, this implies that for $n > \bar{n}$ all the level sets of f_n are bounded (see Ref. [13]). Hence, the assumptions of Theorem 4.1 hold and the proof is complete. □

The boundedness assumption on $\mathrm{WEff}(f, X)$ cannot be avoided, as the following example shows.

Example 4.1 Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, z) = (z^2, e^x)$, $C = \mathbb{R}^2_+$ and $X = \mathbb{R}^2$, $f_n = f$, and $X_n = X$, for every n.

The objective function is C-convex, the set $\mathrm{WMin}(f, X) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = 0\}$, while $\mathrm{WEff}(f, X) = \{(x, z) \in \mathbb{R}^2 : z = 0\}$.

The sequence $(x^n, z^n) = (-n, -n)$ satisfies (3), but does not admit any subsequence (x^{n_k}, z^{n_k}) such that $d(f(x^{n_k}, z^{n_k}), \mathrm{WEff}(f, X)) \to 0$.

Now, let (P, ρ) be a metric space and let $p^* \in P$ be a fixed point. Let L be a closed ball in P with center p^* and positive radius. Let $I : \mathbb{R}^m \times L \to \mathbb{R}^l$ be vector-valued functions such that

$$I(x, p^*) = f(x), \quad \forall x \in X,$$

and let $Z : L \rightsquigarrow \mathbb{R}^m$ be a set-valued function.

The perturbed problem corresponding to the parameter p is denoted by

$$\mathrm{VP}(I(x, p), Z(p)) \quad C\text{-}\min I(x, p), \quad x \in Z(p).$$

In this framework we formulate a notion of extended well-posedness which is a generalization of well-posedness in the strongly extended sense formulated in Refs. [11, 12].

Definition 4.2 Problem $\mathrm{VP}(f, X)$ is well-posed, with respect to the perturbations defined by the sequences $I(\cdot, p)$ and $Z(p)$, when:

(i) $\mathrm{WEff}(f, X) \neq \emptyset$.
(ii) For any sequences $p^n \to p^*$ and $x^n \in Z(p^n)$ such that $\exists \varepsilon_n > 0$, $\varepsilon_n \to 0^+$, with

$$(I(Z(p^n), p^n) - I(x^n, p^n) + \varepsilon_n e) \cap (-\mathrm{int}\, C) = \emptyset, \tag{9}$$

there exists a subsequence x^{n_k} of x^n such that $d(x^{n_k}, \mathrm{WEff}(f, X)) \to 0$, as $k \to +\infty$.

Sequences x^n satisfying (9) are called asymptotically minimizing sequences.

Remark 4.2 Sequences x^n and x^{n_k} in Definition 4.2 may fail to be feasible for the original problem $\mathrm{VP}(f, K)$.

We regard this feature as an extension of Levitin-Polyak approach to well-posedness (see Ref. [18]).

Huang's notion is generalized in Definition 4.2 mainly by the following two facts:

(a) it allows for perturbations of the feasible region and not only of the objective function;
(b) requirement (ii) in Definition 4.2 weakens the convergence requirement of Huang's definition.

Theorem 4.2 *Let $I(\cdot, p)$ be continuous C-quasiconvex functions, and let $Z(p)$ be a closed convex subset of $\mathbb{R}^m$, for every $p \in L$.*

Assume the following:

(i) $\forall p^n \to p^*$ *and* $x^n \to x^*$, $x^n \in Z(p^n)$, *it holds that* $I(x^n, p^n) \to I(x^*, p^*) := f(x^*)$ *and* $X_n := Z(p^n) \xrightarrow{K} X$.
(ii) $\forall y \in \mathbb{R}^l$, $\mathrm{Lev}(f, y, X)$ *is bounded.*
(iii) $\forall p^n \to p^*$, *there exists* $\bar{n} \in \mathbb{N}$ *such that* $\mathrm{Lev}(I(\cdot, p^n), y, Z(p^n))$ *is bounded for every* $y \in \mathbb{R}^l$ *and for every* $n > \bar{n}$.
(iv) $\mathrm{WEff}(f, X)$ *is nonempty and bounded.*

Then, problem $\mathrm{VP}(f, X)$ *is well-posed* (*with respect to the perturbations defined by the sequences* $I(\cdot, p)$ *and* $Z(p)$).

Proof Let $p^n \to p^*$ and set $f_n(\cdot) = I(\cdot, p^n)$ and $X_n := Z(p^n)$, $\forall n$. The proof follows easily from Theorem 4.1. □

Corollary 4.2 *Assume $I(\cdot, p)$ are C-convex functions and let $Z(p)$ be a convex subset of $\mathbb{R}^m$, for every $p \in L$. Let assumptions* (i), (ii) *and* (iv) *of Theorem 4.2 hold. Then, problem* $\mathrm{VP}(f, X)$ *is well-posed* (*with respect to the perturbations defined by the sequences* $I(\cdot, p)$ *and* $Z(p)$).

Proof It is an immediate consequence of Corollary 4.1. □

It remains an open question whether, in the case of C-quasiconvex functions, the assumptions of Theorem 4.1 can be simplified. Proposition 4.3 below, shows however that, when $Z(p) = X$ for every $p \in L$, this is the case if we strenghten the convergence requirement on the sequences $I(\cdot, p^n)$. We need first to recall the notion of oriented distance from a point to a set.

Definition 4.3 For a set $A \subseteq \mathbb{R}^l$, the oriented distance function $D_A : \mathbb{R}^l \to \mathbb{R} \cup \{\pm\infty\}$ is defined as

$$D_A(y) = d(y, A) - d(y, A^c).$$

The function D_A has been introduced in Ref. [16] in the framework of nonsmooth scalar optimization.

The main properties of the function D_A are gathered in the following proposition (see e.g. Ref. [17]).

Proposition 4.1

(i) *If* $A \neq \emptyset$ *and* $A \neq \mathbb{R}^l$, *then* D_A *is real valued.*
(ii) $D_A(y) < 0$ *for every* $y \in \operatorname{int} A$, $D_A(y) = 0$ *for every* $y \in \partial A$ *and* $D_A(y) > 0$ *for every* $y \in \operatorname{int} A^c$.
(iii) *If* A *is closed, then it holds that* $A = \{y : D_A(y) \leq 0\}$.

Lemma 4.1 *Let* $f_n : \mathbb{R}^m \to \mathbb{R}^l$ *be a sequence of functions converging to* f *in the uniform convergence. Assume that, for every* $y \in \mathbb{R}^l$, $\operatorname{Lev}(f, y, X)$ *is bounded. Then, there exists* $\bar{n} \in \mathbb{N}$ *such that, for every* $n > \bar{n}$ *and for every* $y \in \mathbb{R}^l$, $\operatorname{Lev}(f_n, y, X)$ *is bounded.*

Proof We begin observing that under the assumptions, for every $y \in \mathbb{R}^l$ we have $D_{-C}(f(x) - y) \to +\infty$, as $\|x\| \to +\infty$, $x \in X$. Indeed, assume, on the contrary one can find a sequence $x^n \in X$, with $\|x^n\| \to +\infty$ and $D_{-C}(f(x^n) - y) \not\to +\infty$. We distinguish two subcases.

1. The set $\{D_{-C}(f(x^n) - y), n \in \mathbb{N}\}$ is bounded. Then, without loss of generality, we can assume $D_{-C}(f(x^n) - y) \to \beta \in \mathbb{R}$ and the following two cases are possible.
   (i) $\beta < 0$. Then it holds $f(x^n) \in y - C$, eventually, which contradicts the boundedness of the level sets.
   (ii) $\beta \geq 0$. In this case, it is easily seen that we can choose $\alpha > 0$ such that $f(x^n) \in y + \alpha e - C$, eventually, contradicting again the boundedness of the level sets.
2. The set $\{D_{-C}(f(x^n) - y), n \in \mathbb{N}\}$ is unbounded. Since $D_{-C}(f(x^n) - y) \not\to +\infty$, it is possible to find a subsequence x^{n_k} of x^n such that $D_{-C}(f(x^{n_k}) - y) \to -\infty$. In this case it holds again $f(x^{n_k}) \in y - C$, eventually, which contradicts the boundedness of the level sets.

Assume now, ab absurdo, that for every n there exists $y^n \in \mathbb{R}^l$, such that $\operatorname{Lev}(f_n, y^n, X)$ is unbounded. Hence, for a fixed $\bar{n} \in \mathbb{N}$, we can find a sequence z^k, $k \in \mathbb{N}$, with $z^k \in X$, $\forall k$, $\|z^k\| \to +\infty$, as $k \to +\infty$ and $z^k \in \operatorname{Lev}(f_{\bar{n}}, y^{\bar{n}}, X)$, for every k, i.e.

$$f_{\bar{n}}(z^k) - y^{\bar{n}} \in -C, \quad \forall k.$$

We distinguish the following two cases:

(i) $f_{\bar{n}}(z^k) - y^{\bar{n}} \in -C$, for every k except a finite number. In this case we contradict the boundedness of the level set $\operatorname{Lev}(f, y^{\bar{n}}, X)$.
(ii) $f_{\bar{n}}(z^k) - y^{\bar{n}} \notin -C$ for infinitely many k. Without loss of generality we can assume $f_{\bar{n}}(z^k) - y^{\bar{n}} \notin -C$ for every k and we have

$$\begin{aligned} \left\| f(z^k) - f_{\bar{n}}(z^k) \right\| &= \left\| f(z^k) - y^{\bar{n}} - (f_{\bar{n}}(z^k) - y^{\bar{n}}) \right\| \\ &\geq D_{-C}(f(z^k) - y^{\bar{n}}). \end{aligned}$$

From $D_{-C}(f(z^k) - y^{\bar{n}}) \to +\infty$ as $k \to +\infty$, we obtain

$$\sup_{x \in \mathbb{R}^m} \|f(x) - f_{\bar{n}}(x)\| = +\infty.$$

Since $\bar{n}$ is arbitrary, we contradict the uniform convergence of f_n to f. □

The previous lemma does not hold (even in the quasiconvex case) if we assume $f_n \to f$ in the continuous convergence, as the following example shows.

Example 4.2 Let $f : \mathbb{R} \to \mathbb{R}$ and $f_n : \mathbb{R} \to \mathbb{R}$, be defined as

$$f(x) = |x|,$$
$$f_n(x) = \begin{cases} |x|, & x \in [-n, n], \\ |n|, & \text{otherwise}, \end{cases}$$

and let $X = \mathbb{R}$ and $C = \mathbb{R}_+$. We have $f_n \to f$ in the continuous convergence, but not in the uniform convergence and it can be easily seen that the level sets of f are bounded, but each function f_n admits unbounded level sets.

Proposition 4.2 *Let $f_n : \mathbb{R}^m \to \mathbb{R}^l$ and $f : \mathbb{R}^m \to \mathbb{R}^l$ be continuous C-quasiconvex functions and let* $\mathrm{WEff}(f, X)$ *be nonempty and bounded. Assume that $f_n \to f$ in the uniform convergence and that, for every $y \in \mathbb{R}^l$,* $\mathrm{Lev}(f, y, X)$ *is bounded. Then, problem* $\mathrm{VP}(f, X)$ *satisfies property* (P) (*with respect to the perturbations defined by the sequences f_n and X_n*).

Proof Recalling Theorem 4.1, it is enough to prove that there exists $\bar{n} > 0$ such that, for every $n > \bar{n}$ and for every $y \in \mathbb{R}^l$, $\mathrm{Lev}(f_n, y, X)$ is bounded. But this follows immediately from Lemma 4.1. □

The proof of the next result is an immediate consequence of Proposition 4.1.

Proposition 4.3 *Let $I(\cdot, p)$ be continuous C-quasiconvex functions $\forall p \in L$, let $Z(p) = X$, $\forall p \in L$ and assume that $\forall p^n \to p^*$ it holds* $\sup \|I(x, p^n) - f(x)\| \to 0$, *as $n \to +\infty$. If for every $y \in \mathbb{R}^l$,* $\mathrm{Lev}(f, y, X)$ *is bounded and* $\mathrm{WEff}(f, X)$ *is nonempty and bounded, then problem* $\mathrm{VP}(f, X)$ *is well-posed* (*with respect to the perturbations defined by the sequences $I(\cdot, p)$ and $Z(p)$*).

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