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General Maximum Principles for Partially Observed Risk-Sensitive Optimal Control Problems and Applications to Finance

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Abstract This paper is concerned with partially observed risk-sensitive optimal control problems. Combining Girsanov's theorem with a standard spike variational technique, we obtain some general maximum principles for the aforementioned problems. One of the distinctive differences between our results and the standard risk-neutral case is that the adjoint equations and variational inequalities strongly depend on a risk-sensitive parameter γ. Two examples are given to illustrate the applications of the theoretical results obtained in this paper. As a natural deduction, a general maximum principle is also obtained for a fully observed risk-sensitive case. At last, this result is applied to study a risk-sensitive optimal portfolio problem. An explicit optimal investment strategy and a cost functional are obtained. A numerical simulation result shows the influence of a risk-sensitive parameter on an optimal investment proportion; this coincides with its economic meaning and theoretical results.

Keywords Risk-sensitive optimal control · General maximum principle · Partial information · Nonzero sum differential game · Portfolio choices

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1 Introduction and Problem Formulation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a complete filtered probability space equipped with a natural filtration

$$\mathcal{F}_t = \sigma\{W(s), Y(s); 0 \le s \le t\},$$

where $(W(\cdot), Y(\cdot))$ is an $\Re^{d+r}$-valued standard Brownian motion defined on this probability space. Let $\mathcal{F} = \mathcal{F}_T$ and let $T > 0$ be a fixed-time horizon.

Consider the following stochastic control system:

$$dx^v(t) = b(t, x^v(t), v(t))dt + \sigma(t, x^v(t), v(t))dW(t), \qquad x^v(0) = x_0, \tag{1}$$

where $x^v(t) \in \Re^n$, $v(t) \in U \subseteq \Re^k$, $0 \le t \le T$,

$$b : [0, T] \times \Re^n \times U \to \Re^n, \qquad \sigma : [0, T] \times \Re^n \times U \to \Re^{n\times d},$$

and x_0 is an $\mathcal{F}_0$-measurable random variable with the law P_0 and independent of $(W(\cdot), Y(\cdot))$.

We assume that the state variable $x^v(\cdot)$ cannot be observed directly, but we can observe a related process $Y(\cdot)$, which is described by

$$dY(t) = h(t, x^v(t), v(t))dt + dV^v(t), \qquad Y(0) = 0, \tag{2}$$

where $h : [0, T] \times \Re^n \times U \to \Re^r$ and $V^v(\cdot)$ denotes a stochastic process depending on the control variable $v(\cdot)$.

Let $\mathcal{Y}_t = \sigma\{Y(s); 0 \le s \le t\}$. For $m = 2, 3, 4, \ldots$, a control variable $v(t) : [0, T] \times \Omega \to U$ is called admissible, if it is $\mathcal{Y}_t$-adapted and satisfies $\sup_{0\le t\le T} \mathbb{E}|v(t)|^m < +\infty$, a.e., a.s. The set of all the admissible control variables is denoted by $\mathcal{U}_{ad}$.

We assume that the following hypothesis holds.

(H1) The functions b, σ, h are twice continuously differentiable in x. They and their partial derivatives b_x, b_{xx}, σ_x, σ_{xx}, h_x, h_{xx} are continuous in (x, v); b_x, b_{xx}, $\sigma_x, \sigma_{xx}, h, h_x, h_{xx}$ are bounded and there exists a constant $C > 0$ such that both b and σ are bounded by $C(1 + |x| + |v|)$. x_0 has finite moments of arbitrary order.

For any $v(\cdot) \in \mathcal{U}_{ad}$, (H1) implies that (1) admits a unique $\mathcal{F}_t$-adapted solution. Define $dP^v = Z^v(t)dP$ with

$$Z^v(t) = \exp\left\{\int_0^t h^*(s, x^v(s), v(s))dY(s) - \frac{1}{2}\int_0^t |h(s, x^v(s), v(s))|^2 ds\right\},$$

where h^* denotes the transpose of a matrix h and $|\cdot|$ denotes the square root of the sum of all squares of components in the underlying matrix. Obviously, $Z^v(\cdot)$ is a unique $\mathcal{F}_t$-adapted solution of

$$dZ^v(t) = Z^v(t)h^*(t, x^v(t), v(t))dY(t), \qquad Z^v(0) = 1. \tag{3}$$

Then, Girsanov's theorem and (H1) imply that P^v is a new probability measure and $(W(\cdot), V^v(\cdot))$ is an $\mathfrak{R}^{d+r}$-valued standard Brownian motion defined on the new probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P^v)$.

In terms of (H1), BDG inequality, Gronwall's inequality and an elementary inequality

$$|m_1 + m_2 + m_3|^n \le 3^n(|m_1|^n + |m_2|^n + |m_3|^n), \quad \forall n > 0,$$

we obtain easily the following result.

Lemma 1.1 *Let* (H1) *hold. For any* $v(\cdot) \in \mathcal{U}_{ad}$, *the solutions of* (3) *and* (1) *satisfy*

$$\sup_{0\le t\le T} \mathbb{E}|Z^v(t)|^m < +\infty, \qquad \sup_{0\le t\le T} \mathbb{E}|x^v(t)|^m \le C\Big(1 + \sup_{0\le t\le T} \mathbb{E}|v(t)|^m\Big),$$

with a constant $C > 0$ *and* $m = 2, 3, 4, \ldots$.

We introduce the following cost functional:

$$J(v(\cdot)) = \mathbb{E}^v \Psi\left[\int_0^T l(t, x^v(t), v(t))dt + \Phi(x^v(T))\right], \tag{4}$$

where $\mathbb{E}^v$ denotes expectation on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P^v)$ and $\Psi : \mathfrak{R} \to \mathfrak{R}$ is a monotonically increasing disutility function. Ψ, l and Φ satisfy some suitable conditions such that $J(v(\cdot)) > -\infty$ holds for any $v(\cdot) \in \mathcal{U}_{ad}$. The problem is to seek an admissible control $u(\cdot)$ to minimize $J(v(\cdot))$ subject to (1) and (3). If $u(\cdot)$ attains the minimum value (if it exists), then it is called optimal, the corresponding solutions $x(\cdot)$, $Z(\cdot)$ of (1) and (3) are called the optimal trajectories. For simplification, we also use the abbreviation $V(\cdot) = V^u(\cdot)$.

The cost functional (4) subject to (1) and (3) consists of a partially observed risk-sensitive optimal control problem. Let us now interpret the meaning of the word risk-sensitive by an intuitive argument. We also refer to Yong and Zhou (Ref. [1]) for more information on risk-sensitive. Define

$$X = \int_0^T l(t, x^v(t), v(t))dt + \Phi(x^v(T))$$

and suppose that Ψ is twice differentiable at $\mathbb{E}^v X$. By Taylor's expansion, we have

$$\mathbb{E}^v \Psi(X) \approx \Psi(\mathbb{E}^v X) + \frac{1}{2}\Psi''(\mathbb{E}^v X)\mathbb{E}^v(X - \mathbb{E}^v X)^2.$$

If Ψ is strictly concave near $\mathbb{E}^v X$, then $\Psi''(\mathbb{E}^v X) < 0$, which implies that the controller is risk-seeking from an economic point of view. If Ψ is strictly convex near $\mathbb{E}^v X$, then $\Psi''(\mathbb{E}^v X) > 0$, which implies that the controller is risk-averse. Finally, if $\Psi''(\mathbb{E}^v X) = 0$, the risk-sensitive optimal control problem reduces to the standard risk-neutral situation.

The aforementioned risk-sensitive optimal control problem has been discussed by many researchers, such as Bensoussan and Van Schuppen (Ref. [2]), Charalambous and Hibey (Ref. [3]) and references therein. They usually made the following two assumptions: (i) $\Psi(x) = \theta e^{\theta x}$, where $\theta \neq 0$, a fixed constant, is the so called risk-sensitive parameter; (ii) σ and h in (1) and (2) do not contain the control variable $v(\cdot)$. In fact, they only studied a special class of problem. Recently, risk-sensitive optimal control problems have attracted more research attention. One reason is that the theory in itself is interesting and challenging. Another is that the risk-sensitive parameter can describe the risk attitude of an investor, thus this kind of model can be used to study some financial problems. The related work can be found in Nagai (Ref. [4]), Nagai and Peng (Ref. [5]).

The other frequently used disutility function is the HARA utility

$$\Psi(x) = \frac{1}{\gamma}x^{\gamma}, \quad \gamma \neq 0,\ x > 0.$$

If $x(\cdot)$ represents the wealth process of an investor, then maximizing $\frac{1}{\gamma}\mathbb{E}[x(T)]^{\gamma}$ subject to (1) formulates an important risk-sensitive optimal portfolio problem arising from a financial market. Therefore, this kind of cost functional has practical sense. As a generalization of the expected HARA utility maximization problem, we consider the cost functional

$$J(v(\cdot)) = \frac{1}{\gamma}\mathbb{E}^{v}[\Phi(x^{v}(T))]^{\gamma}, \tag{5}$$

where γ, a fixed constant, is called the risk-sensitive parameter. The problem is to seek an appropriate admissible control variable $u(\cdot)$ such that $J(u(\cdot)) = \min_{v(\cdot)\in\mathcal{U}_{ad}} J(v(\cdot))$ subject to (1) and (3). Our main task is to find a necessary condition, the so called maximum principle, of the optimal control $u(\cdot)$. If $\gamma = 1$, it reduces to the risk-neutral case. There exist lots of references, such as Bensoussan (Ref. [6]), Haussmann (Ref. [7]), Baras, Elliott and Kohlmann (Ref. [8]), Zhou (Ref. [9]), Li and Tang (Ref. [10]), Baghery and Øksendal (Ref. [11]). $\gamma > 1$ and $\gamma < 1$ correspond to the risk-averse and the risk-seeking situations, respectively. To our best knowledge, there exists few literature on this topic in the situation $\gamma \neq 1$. In this paper, we are more interested in the situation of $\gamma > 0$. For the case $\gamma < 0$, we can obtain some similar conclusions. Obviously, (5) can be rewritten as

$$J(v(\cdot)) = \frac{1}{\gamma}\mathbb{E}\{Z^{v}(T)[\Phi(x^{v}(T))]^{\gamma}\}, \quad \gamma > 0. \tag{6}$$

Thus, our original problem (5) is equivalent to minimizing (6) subject to (1) and (3).

For any $x \in \mathfrak{R}^n$, we introduce the following hypotheses.

(H2) There exists a constant $C > 0$ such that

$$(1+|x|^2)^{-1}|\Phi(x)| + (1+|x|)^{-1}|\Phi_x(x)| + |\Phi_{xx}(x)| \leq C.$$
$$(\text{or } (1+|x|)^{-1}|\Phi(x)| + |\Phi_x(x)| + |\Phi_{xx}(x)| \leq C.)$$

(H3) When $0 < \gamma < 1$ or $1 < \gamma < 2$, we assume that $\mathbb{E}[\Phi(x^{v}(T))]^{2\gamma-4} < +\infty$ holds.

The rest of this paper is organized as follows. In Sect. 2, we first introduce a standard spike variation to get first-order and second-order variational equations, then we derive the corresponding adjoint equations which are finite-dimensional backward stochastic differential equations (BSDEs). This is a standard method used to deal with the risk-neutral case. Please refer to Peng (Ref. [12]) and Li and Tang (Ref. [10]), where some fully observed and partially observed maximum principles were obtained, respectively. We also introduce an adjoint BSDE which depends on the risk-sensitive parameter γ to deal with the term produced by partial information. Then we derive some partially observed risk-sensitive maximum principles. Our method is different from Bensoussan (Ref. [6]), Haussmann (Ref. [7]), Baras et al. (Ref. [8]) and Zhou (Ref. [9]). To characterize an adjoint process which is necessary for a maximum principle, Bensoussan (Ref. [6]) adopted an infinite-dimensional BSDE, Baras et al. (Ref. [8]) used the theory of stochastic flows, Haussmann (Ref. [7]) and Zhou (Ref. [9]) needed the theory of stochastic partial differential equations.

Obviously, our maximum principle is a generalization of the one in [10]. However, even in the risk-neutral case (see e.g. [6–10]), few attention was paid to applications of maximum principles. One of main difficulties is that there is no general filtering estimate result for adjoint processes, which are characterized by BSDEs. So it is difficult to obtain explicit observable maximum principles, observable optimal controls and cost functionals. In Sect. 3, we focus on two interesting examples which are used to illustrate the applications of our theoretical results obtained in Sect. 2. One is a partially observed linear-quadratic (LQ) non-zero sum stochastic differential game problem. In [13], Hamadène studied a fully observed stochastic differential game problem. And then, Wu and Yu (Ref. [14]) generalized it to the case with random jump. In this paper, we will use the maximum principle to study a similar problem under partial information and give an explicit observable Nash equilibrium point. The other application is to a linear risk-sensitive optimal control problem, where the maximum principle can also be used.

In Sect. 4, we will derive a general maximum principle for a fully observed risk-sensitive optimal control problem. And then we apply this result to study a risk-sensitive optimal portfolio problem in Sect. 5. An explicit optimal investment strategy and an optimal cost functional are obtained. A numerical simulation is also used to show a specific influence of the risk-sensitive parameter γ on an optimal investment proportion in this section. The simulation result coincides with theoretical ones and the economic meaning of γ.

Finally in Sect. 6, we compare our results with the existing ones in other papers.

2 General Maximum Principles

In this section, combining Girsanov's theorem with a standard spike variational technique, we derive the general maximum principles for the aforementioned partially observed risk-sensitive (neutral) optimal control problems.

Let $u(\cdot)$ be optimal. Since the control set U is nonconvex, we introduce the spike variation

$$u^\varepsilon(t) = \begin{cases} v, & \text{if } \tau \le t \le \tau + \varepsilon, \\ u(t), & \text{otherwise,} \end{cases}$$

where $0 \le \tau < T$ is fixed, $\varepsilon > 0$ is sufficiently small and v is an arbitrary $\mathcal{Y}_\tau$-measurable random variable with values in U such that $\sup_{\omega\in\Omega} |v(\omega)| < +\infty$. Let $x^\varepsilon(\cdot)$ and $Z^\varepsilon(\cdot)$ be the trajectories corresponding to $u^\varepsilon(\cdot)$.

For simplification, we introduce the notations

$$\theta(u(t)) = \theta(t, x(t), u(t)), \qquad \theta(u^\varepsilon(t)) = \theta(t, x(t), u^\varepsilon(t)),$$

where $\theta = b$, σ, h, l as well as their partial derivatives with respect to the optimal trajectory x.

We now introduce the first-order variational equations

$$dx_1(t) = b_x(u(t))x_1(t)dt + [\sigma_x(u(t))x_1(t) + \sigma(u^\varepsilon(t)) - \sigma(u(t))]dW(t), \tag{7a}$$

$$x_1(0) = 0, \tag{7b}$$

$$\begin{aligned} dZ_1(t) = {} & [Z_1(t)h(u(t)) + Z(t)h_x(u(t))x_1(t) \\ & + Z(t)(h(u^\varepsilon(t)) - h(u(t)))]^* dY(t), \end{aligned} \tag{8a}$$

$$Z_1(0) = 0, \tag{8b}$$

and the second-order variational equations

$$\begin{aligned} dx_2(t) = {} & \left[b_x(u(t))x_2(t) + \frac{1}{2} b_{xx}(u(t))x_1(t)x_1(t) + b(u^\varepsilon(t)) - b(u(t)) \right] dt \\ & + \left[\sigma_x(u(t))x_2(t) + \frac{1}{2}\sigma_{xx}(u(t))x_1(t)x_1(t) \right. \\ & \left. + (\sigma(u^\varepsilon(t)) - \sigma(u(t)))x_1(t) \right] dW(t), \end{aligned} \tag{9a}$$

$$x_2(0) = 0, \tag{9b}$$

$$\begin{aligned} dZ_2(t) = {} & \left[Z_2(t)h(u(t)) + Z_1(t)h_x(u(t))x_1(t) + Z_1(t)(h(u^\varepsilon(t)) - h(u(t))) \right. \\ & + Z(t)h_x(u(t))x_2(t) + \frac{1}{2} Z(t)h_{xx}(u(t))x_1(t)x_1(t) \\ & \left. + Z(t)(h_x(u^\varepsilon(t)) - h_x(u(t)))x_1(t) \right]^* dY(t), \end{aligned} \tag{10a}$$

$$Z_2(0) = 0, \tag{10b}$$

where $f_{xx}yy = \sum_{i,j=1}^n f_{x_i x_j} y_i y_j$ for $f = b, \sigma, h$ and Φ. From (H1), (7), (8), (9) and (10) admit $\mathcal{F}_t$-adapted solutions, respectively.

The following Lemma 2.1 is due to Li and Tang (Ref. [10]).

Lemma 2.1 *Let* (H1) *hold. Then, we have*

$$\sup_{0\le t\le T} \mathbb{E}|x_1(t)|^m \le C\varepsilon^{\frac{m}{2}}, \qquad \sup_{0\le t\le T} \mathbb{E}|x_2(t)|^m \le C\varepsilon^m,$$

$$\sup_{0\le t\le T} \mathbb{E}|Z_1(t)|^m \le C\varepsilon^{\frac{m}{2}}, \qquad \sup_{0\le t\le T} \mathbb{E}|Z_2(t)|^m \le C\varepsilon^m,$$

$$\sup_{0\le t\le T}\mathbb{E}|x^\varepsilon(t)-x(t)-x_1(t)-x_2(t)|^m\le C_\varepsilon\varepsilon^m,$$
$$\sup_{0\le t\le T}\mathbb{E}|Z^\varepsilon(t)-Z(t)-Z_1(t)-Z_2(t)|^m\le C_\varepsilon\varepsilon^m,$$

where C *and* C_ε *are nonnegative constants and* $C_\varepsilon\to 0$ *when* $\varepsilon\to 0$.

Similarly, we obtain the following lemma.

Lemma 2.2 (Variational Inequality) *Let* (H1), (H2), (H3) *hold. Then, we get*

$$\begin{aligned}
&\frac{1}{\gamma}\mathbb{E}^u\{Z^{-1}(T)(Z_1(T)+Z_2(T))[\Phi(x(T))]^\gamma\}\\
&\quad+\mathbb{E}^u\{Z^{-1}(T)Z_1(T)[\Phi(x(T))]^{\gamma-1}\Phi_x^*(x(T))x_1(T)\}\\
&\quad+\mathbb{E}^u\{[\Phi(x(T))]^{\gamma-1}\Phi_x^*(x(T))(x_1(T)+x_2(T))\}\\
&\quad+\frac{1}{2}\mathbb{E}^u\{(\gamma-1)[\Phi(x(T))]^{\gamma-2}\Phi_x(x(T))\Phi_x^*(x(T))x_1(T)x_1(T)\}\\
&\quad+\frac{1}{2}\mathbb{E}^u\{[\Phi(x(T))]^{\gamma-1}\Phi_{xx}(x(T))x_1(T)x_1(T)\}\ge o(\varepsilon).
\end{aligned}\tag{11}$$

Proof Using the fact that $J(u^\varepsilon(\cdot))-J(u(\cdot))\ge 0$, the Taylor expansion and Lemma 2.1, we have

$$\begin{aligned}
0&\le\frac{1}{\gamma}\mathbb{E}\{Z^\varepsilon(T)[\Phi(x^\varepsilon(T))]^\gamma\}-\frac{1}{\gamma}\mathbb{E}\{Z(T)[\Phi(x(T))]^\gamma\}\\
&=\frac{1}{\gamma}\mathbb{E}\{(Z_1(T)+Z_2(T))[\Phi(x(T))]^\gamma\}\\
&\quad+\mathbb{E}\{Z_1(T)[\Phi(x(T))]^{\gamma-1}\Phi_x^*(x(T))x_1(T)\}\\
&\quad+\mathbb{E}\{Z(T)[\Phi(x(T))]^{\gamma-1}\Phi_x^*(x(T))(x_1(T)+x_2(T))\}\\
&\quad+\frac{1}{2}\mathbb{E}\{(\gamma-1)Z(T)[\Phi(x(T))]^{\gamma-2}\Phi_x(x(T))\Phi_x^*(x(T))x_1(T)x_1(T)\}\\
&\quad+\frac{1}{2}\mathbb{E}\{Z(T)[\Phi(x(T))]^{\gamma-1}\Phi_{xx}(x(T))x_1(T)x_1(T)\}+o(\varepsilon).
\end{aligned}$$

Thus, we draw the desired conclusion. □

We now focus on a necessary condition of the optimal control $u(\cdot)$. The method is similar to that of Peng (Ref. [12]) and Li and Tang (Ref. [10]), so we will omit similar proofs here.

Define the Hamiltonian function

$$H(t,x^v,v,p,q,\bar z)=\langle p,b(t,x^v,v)\rangle+\sum_{i=1}^d\langle q_i,\sigma_i(t,x^v,v)\rangle+\langle\bar z,h(t,x^v,v)\rangle,\tag{12}$$

where $H : [0, T] \times \Re^n \times U \times \Re^n \times \Re^{n\times d} \times \Re^r \to \Re$ and $\langle \cdot, \cdot \rangle$ denotes the product of two vectors in an Euclidean space.

We introduce the adjoint equations which depend on the risk-sensitive parameter γ,

$$y(t) = \frac{1}{\gamma}[\Phi(x(T))]^\gamma - \int_t^T z(s)dW(s) - \int_t^T \bar{z}(s)dV(s), \tag{13}$$

$$-dp(t) = H_x^*(t, x(t), u(t), p(t), q(t), \bar{z}(t))dt - q(t)dW(t) - \bar{q}(t)dV(t), \tag{14a}$$

$$p(T) = [\Phi(x(T))]^{\gamma-1}\Phi_x^*(x(T)), \tag{14b}$$

$$\begin{aligned}
-dP(t) = \Bigg[& b_x^*(u(t))P(t) + P(t)b_x^*(u(t)) + \sum_{i=1}^d \sigma_{i,x}^*(u(t))P(t)\sigma_{i,x}(u(t)) \\
& + \sum_{i=1}^d \sigma_{i,x}^*(u(t))Q_i(t) + \sum_{i=1}^d Q_i(t)\sigma_{i,x}(u(t)) \\
& + H_{xx}(t, x(t), u(t), p(t), q(t), \bar{z}(t)) \\
& + \sum_{j=1}^r \bar{q}_j(t)h_{j,x}^*(u(t)) + \sum_{j=1}^r h_{j,x}(u(t))\bar{q}_j^*(t) \Bigg] dt \\
& - Q(t)dW(t) - \bar{Q}(t)dV(t),
\end{aligned} \tag{15a}$$

$$\begin{aligned}
P(T) = & (\gamma - 1)[\Phi(x(T))]^{\gamma-2}\Phi_x(x(T))\Phi_x^*(x(T)) \\
& + [\Phi(x(T))]^{\gamma-1}\Phi_{xx}(x(T)).
\end{aligned} \tag{15b}$$

Hereinafter, we use the following notations:

$$\sigma_{i,x}(u(\cdot)) = \frac{\partial}{\partial x}\sigma_i(u(\cdot)), \quad i = 1, 2, \ldots, d,$$

$$h_{j,x}(u(\cdot)) = \frac{\partial}{\partial x}h_j(u(\cdot)), \quad j = 1, 2, \ldots, r.$$

Obviously, (H1), (H2), (H3) imply that (13), (14), (15) admit $\mathcal{F}_t$-adapted solutions.

Applying Itô's formula to

$$\langle y(t), Z^{-1}(t)(Z_1(t) + Z_2(t))\rangle + \langle p(t), Z^{-1}(t)Z_1(t)x_1(t)\rangle,$$

we have

$$\begin{aligned}
&\mathbb{E}^u\left\{\frac{1}{\gamma}Z^{-1}(T)(Z_1(T) + Z_2(T))[\Phi(x(T))]^\gamma\right\} \\
&\quad + \mathbb{E}^u\{Z^{-1}(T)Z_1(T)[\Phi(x(T))]^{\gamma-1}\Phi_x^*(x(T))x_1(T)\}
\end{aligned}$$

$$= \mathbb{E}^u \int_0^T \langle \bar{z}(t), h_x(u(t))(x_1(t) + x_2(t)) + h(u^\varepsilon(t)) - h(u(t))\rangle dt$$
$$+ \mathbb{E}^u \int_0^T \mathrm{Tr}\left[\sum_{j=1}^r (h_{j,x}(u(t))\bar{q}_j(t))(x_1(t)x_1^*(t))\right] dt$$
$$+ \frac{1}{2}\mathbb{E}^u \int_0^T \mathrm{Tr}\langle \bar{z}(t), h_{xx}(u(t))(x_1(t)x_1^*(t))\rangle dt + o(\varepsilon). \tag{16}$$

Substituting (16) into (11), we obtain

$$\mathbb{E}^u \int_0^T \langle \bar{z}(t), h_x(u(t))(x_1(t) + x_2(t))\rangle dt$$
$$+ \mathbb{E}^u\{[\Phi(x(T))]^{\gamma-1}\Phi_x^*(x(T))(x_1(t) + x_2(t))\}$$
$$+ \mathbb{E}^u \int_0^T \langle \bar{z}(t), h(u^\varepsilon(t) - h(u(t)))\rangle dt$$
$$+ \mathbb{E}^u \int_0^T \mathrm{Tr}\left[\sum_{j=1}^r (h_{j,x}(u(t))\bar{q}_j(t))(x_1(t)x_1^*(t))\right] dt$$
$$+ \frac{1}{2}\mathbb{E}^u \int_0^T \mathrm{Tr}\langle \bar{z}(t), h_{xx}(u(t))(x_1(t)x_1^*(t))\rangle dt$$
$$+ \frac{1}{2}\mathbb{E}^u\{(\gamma - 1)Tr([\Phi(x(T))]^{\gamma-2}\Phi_x(x(T))\Phi_x^*(x(T))(x_1(T)x_1^*(T)))\}$$
$$+ \frac{1}{2}\mathbb{E}^u\{\mathrm{Tr}([\Phi(x(T))]^{\gamma-1}\Phi_{xx}(x(T))(x_1(T)x_1^*(T)))\} \geq o(\varepsilon). \tag{17}$$

Applying Itô's formula to $\langle p(t), x_1(t) + x_2(t)\rangle + P(t)[x_1(t)x_1^*(t)]$ and comparing it with (17), we get

$$\mathbb{E}^u \int_0^T \Big\{ H(t, x(t), u^\varepsilon(t), p(t), q(t), \bar{z}(t)) - H(t, x(t), u(t), p(t), q(t), \bar{z}(t))$$
$$+ \frac{1}{2}\mathrm{Tr}[(\sigma(u^\varepsilon(t)) - \sigma(u(t)))^* P(t)(\sigma(u^\varepsilon(t)) - \sigma(u(t)))]\Big\} dt \geq o(\varepsilon).$$

Therefore, we have the following theorem.

Theorem 2.1 (Risk-Sensitive Maximum Principle: I) *Assume that* (H1), (H2), (H3) *hold. Let* $u(\cdot)$ *be optimal. Then, the maximum principle*

$$\mathbb{E}^u\Big\{\Big[H(t, x(t), v, p(t), q(t), \bar{z}(t)) - H(t, x(t), u(t), p(t), q(t), \bar{z}(t))$$
$$+ \frac{1}{2}\mathrm{Tr}[(\sigma(v) - \sigma(u(t)))^* P(t)(\sigma(v) - \sigma(u(t)))]\Big] | \mathcal{Y}_t\Big\} \geq 0, \quad \forall v \in U, a.e., a.s.,$$

holds, where the Hamiltonian function H *is defined by* (12).

We now study a problem with a general running cost functional, i.e.,

$$\min_{v(\cdot)\in\mathcal{U}_{ad}} J(v(\cdot)),$$
$$J(v(\cdot)) = \frac{1}{\gamma}\mathbb{E}^v\left[\int_0^T l(t,x^v(t),v(t))dt + \Phi(x^v(T))\right]^\gamma, \quad \gamma\in(0,+\infty), \quad (18)$$

subject to (1) and (3). Here b, σ, h satisfy (H1) and Φ satisfies (H2). For any $(t,x,v)\in[0,T]\times\mathfrak{R}^n\times U$, we also assume that (H4) and (H5) below hold.

(H4) $l:[0,T]\times\mathfrak{R}^n\times U\to\mathfrak{R}$ is continuously differentiable in x; l, l_x, l_{xx} are continuous in (x,v). There exists a constant C such that

$$(1+|x|+|v|^2)^{-1}|l(t,x,v)| + |l_x(t,x,v)| + |l_{xx}(t,x,v)| \le C.$$

(H5) For $0<\gamma<1$ or $1<\gamma<2$, we suppose that

$$\mathbb{E}\left[\int_0^T l(t,x^v(t),v(t))dt + \Phi(x^v(T))\right]^{2\gamma-4} < +\infty.$$

Our target is to give a necessary condition for the optimal control $u(\cdot)$. The method is to combine the proof of Theorem 2.1 with a reformulation of the cost functional (18).

Define the following stochastic differential equation (SDE):

$$dX^v(t) = l(t,x^v(t),v(t))dt, \qquad X^v(0)=0; \quad (19)$$

then, the corresponding first-order variational equation is

$$dX_1(t) = [l_x(u(t))X_1(t) + l(u^\varepsilon(t)) - l(u(t))]dt, \qquad X_1(0)=0.$$

For any $v(\cdot)\in\mathcal{U}_{ad}$, we can employ usual techniques to prove that

$$\sup_{0\le t\le T}\mathbb{E}|X^v(t)|^m \le C\Big(1+\sup_{0\le t\le T}\mathbb{E}|v(t)|^{2m}\Big), \qquad \sup_{0\le t\le T}\mathbb{E}|X_1(t)|^m \le C\varepsilon^m,$$
$$\sup_{0\le t\le T}\mathbb{E}|X^\varepsilon(t)-X(t)-X_1(t)|^m \le C_\varepsilon\varepsilon^m, \quad m=2,3,4,\ldots.$$

Thus, our original problem (18), subject to (1) and (3), is equivalent to minimizing

$$J(v(\cdot)) = \frac{1}{\gamma}\mathbb{E}^v[X^v(T)+\Phi(x^v(T)))]^\gamma \quad (20)$$

subject to (1), (3), (19). The fact that $J(u^\varepsilon(\cdot)) - J(u(\cdot)) \ge 0$ implies that

$$\frac{1}{\gamma}\mathbb{E}\{(Z_1(T)+Z_2(T))[X(T)+\Phi(x(T))]^\gamma\}$$
$$+\frac{1}{2}\mathbb{E}\{Z(T)[X(T)+\Phi(x(T))]^{\gamma-1}\Phi_{xx}(x(T))x_1(T)x_1(T)\}$$

$$+\frac{1}{2}\mathbb{E}\{(\gamma-1)Z(T)[X(T)+\Phi(x(T))]^{\gamma-2}\Phi_x(x(T))\Phi_x^*(x(T))x_1(T)x_1(T)\}$$
$$+\mathbb{E}\{Z(T)[X(T)+\Phi(x(T))]^{\gamma-1}\Phi_x^*(x(T))(x_1(T)+x_2(T))\}$$
$$+\mathbb{E}\{Z(T)[X(T)+\Phi(x(T))]^{\gamma-1}X_1(T)\}$$
$$+\mathbb{E}\{Z_1(T)[X(T)+\Phi(x(T))]^{\gamma-1}\Phi_x^*(x(T))x_1(T)\}\geq o(\varepsilon). \tag{21}$$

Since (21) is similar to (11), we introduce the following BSDEs:

$$\begin{aligned}
\Delta(t) &= [X(T)+\Phi(x(T))]^{\gamma-1}+\int_t^T l_x(u(s))\Delta(s)ds-\int_t^T \Lambda(s)dW(s),\\
\alpha(t) &= \frac{1}{\gamma}[X(T)+\Phi(x(T))]^{\gamma}-\int_t^T \beta(s)dW(s)-\int_t^T \bar\beta(s)dV(s),\\
-d\varphi(t) &= H_x^*(t,x(t),u(t),\varphi(t),\psi(t),\bar\beta(t))dt-\psi(t)dW(t)-\bar\psi(t)dV(t),\\
\varphi(T) &= [X(T)+\Phi(x(T))]^{\gamma-1}\Phi_x^*(x(T)),\\
-d\xi(t) &= \Bigg[b_x^*(u(t))\xi(t)+\xi(t)b_x^*(u(t))+\sum_{i=1}^d \sigma_{i,x}^*(u(t))\xi(t)\sigma_{i,x}(u(t))\\
&\quad+\sum_{i=1}^d \sigma_{i,x}^*(u(t))\eta_i(t)+\sum_{i=1}^d \eta_i(t)\sigma_{i,x}(u(t))\\
&\quad+H_{xx}(t,x(t),u(t),\varphi(t),\psi(t),\bar\beta(t))\\
&\quad+\sum_{j=1}^r \bar\psi_j(t)h_{j,x}^*(u(t))+\sum_{j=1}^r h_{j,x}(u(t))\bar\psi_j^*(t)\Bigg]dt\\
&\quad-\eta(t)dW(t)-\bar\eta(t)dV(t),\\
\xi(T) &= (\gamma-1)[X(T)+\Phi(x(T))]^{\gamma-2}\Phi_x(x(T))\Phi_x^*(x(T))\\
&\quad+[X(T)+\Phi(x(T))]^{\gamma-1}\Phi_{xx}(x(T)),
\end{aligned}$$

where the Hamiltonian function H is defined by (12).

Applying Itô's formula to $\langle\Delta(t),X_1(t)\rangle$, we have

$$\mathbb{E}^u\{[X(T)+\Phi(x(T))]^{\gamma-1}X_1(T)\}=\mathbb{E}^u\int_0^T (l(u^\varepsilon(t))-l(u(t)))\Delta(t)dt. \tag{22}$$

Define the Hamiltonian function

$$\mathcal{H}(t,x^v,v,\varphi,\psi,\bar\beta,\Delta)=H(t,x^v,v,\varphi,\psi,\bar\beta)+\langle\Delta,l(t,x^v,v)\rangle, \tag{23}$$

where $\mathcal{H}:[0,T]\times\mathfrak{R}^n\times U\times\mathfrak{R}^n\times\mathfrak{R}^{n\times d}\times\mathfrak{R}^r\times\mathfrak{R}\to\mathfrak{R}$.

From (21), (22), (23) and Theorem 2.1, we get

$$\mathbb{E}^u\bigg\{\Big[\mathcal{H}(t,x(t),v,\varphi(t),\psi(t),\bar\beta(t),\Delta(t))-\mathcal{H}(t,x(t),u(t),\varphi(t),\psi(t),\bar\beta(t),\Delta(t))$$

$$+\frac{1}{2}\mathrm{Tr}[(\sigma(v)-\sigma(u(t)))^*\xi(t)(\sigma(v)-\sigma(u(t)))]\Big]|\mathcal{Y}_t\Big\}\geq 0,\quad \forall v\in U,\ a.e., a.s. \tag{24}$$

Theorem 2.2 (Risk-Sensitive Maximum Principle: II) *Assume that* (H1), (H2), (H4), (H5) *hold. Let* $u(\cdot)$ *be optimal. Then, the maximum principle* (24) *holds.*

Obviously, the assumption conditions in Theorem 2.2 are rigorous. If we let $\gamma = 1$, the cost functional (18) reduces to the risk-neutral case. In the situation, the hypothesis on $l(\cdot, x^v(\cdot), v(\cdot))$ can be replaced by

(H6) $l : [0, T] \times \mathfrak{R}^n \times U \to \mathfrak{R}$ is continuously differentiable in x; l, l_x, l_{xx} are continuous in (x, v). For any $(t, x, v) \in [0, T] \times \mathfrak{R}^n \times U$, there exists a constant $C > 0$ such that

$$(1+|x|^2+|v|^2)^{-1}|l(t,x,v)|+(1+|x|+|v|)^{-1}|l_x(t,x,v)|+|l_{xx}(t,x,v)|\leq C.$$

Suppose that (H1), (H2), (H6) hold and the risk-sensitive parameter $\gamma = 1$. Using the same techniques as the proof of Theorem 2.1, we get the following Theorem 2.3, which coincides with Theorem 2.1 in Li and Tang (Ref. [10]).

Theorem 2.3 (Risk-Neutral Maximum Principle) *Let* $u(\cdot)$ *be optimal. Then, the maximum principle*

$$\mathbb{E}^u\Big\{\Big[\mathcal{H}(t,x(t),v,\varphi(t),\psi(t),\bar\beta(t))-\mathcal{H}(t,x(t),u(t),\varphi(t),\psi(t),\bar\beta(t))$$
$$+\frac{1}{2}\mathrm{Tr}[(\sigma(v)-\sigma(u(t)))^*\xi(t)(\sigma(v)-\sigma(u(t)))]\Big]|\mathcal{Y}_t\Big\}\geq 0,\quad \forall v\in U,\ a.e., a.s.,$$

holds, where $\bar\beta(\cdot)$, $\varphi(\cdot)$, $\psi(\cdot)$, $\xi(\cdot)$ *satisfy*

$$\alpha(t)=\Phi(x(T))+\int_t^T l(u(s))ds-\int_t^T\beta(s)dW(s)-\int_t^T\bar\beta(s)dV(s),$$
$$-d\varphi(t)=\mathcal{H}_x^*(t,x(t),u(t),\varphi(t),\psi(t),\bar\beta(t))dt-\psi(t)dW(t)-\bar\psi(t)dV(t),$$
$$\varphi(T)=\Phi_x^*(x(T)),$$
$$-d\xi(t)=\Big[b_x^*(u(t))\xi(t)+\xi(t)b_x^*(u(t))+\sum_{i=1}^d\sigma_{i,x}^*(u(t))\xi(t)\sigma_{i,x}(u(t))$$
$$+\sum_{i=1}^d\sigma_{i,x}^*(u(t))\eta_i(t)+\sum_{i=1}^d\eta_i(t)\sigma_{i,x}(u(t))$$
$$+\mathcal{H}_{xx}(t,x(t),u(t),\varphi(t),\psi(t),\bar\beta(t))$$
$$+\sum_{j=1}^r\bar\psi_j(t)h_{j,x}^*(u(t))+\sum_{j=1}^r h_{j,x}(u(t))\bar\psi_j^*(t)\Big]dt$$

$$
\begin{aligned}
&\qquad - \eta(t)dW(t) - \bar{\eta}(t)dV(t),\\
&\xi(T) = \Phi_{xx}(x(T))
\end{aligned}
$$

with the Hamiltonian function $\mathcal{H} : [0, T] \times \mathfrak{R}^n \times U \times \mathfrak{R}^n \times \mathfrak{R}^{n\times d} \times \mathfrak{R}^r \to \mathfrak{R}$ *defined by*

$$\mathcal{H}(t, x^v, v, \varphi, \psi, \bar{\beta}) = H(t, x^v, v, \varphi, \psi, \bar{\beta}) + l(t, x^v, v). \tag{25}$$

Let us now study an important case, i.e., $\gamma = 1$, $\sigma(\cdot, x^v(\cdot), v(\cdot)) \equiv \sigma(\cdot, x^v(\cdot))$ and $l(\cdot, x^v(\cdot), v(\cdot)) \equiv l(\cdot, v(\cdot))$. For any $(t, x, v) \in [0, T] \times \mathfrak{R}^n \times U$, we also suppose that the following hypothesis holds.

(H7) The functions b, σ, h satisfy (H1). There exists a constant $C > 0$ such that

$$(1 + |v|^2)^{-1}|l(t, v)| + (1 + |x|^2)^{-1}|\Phi(x)| + (1 + |x|)^{-1}|\Phi_x(x)| \le C.$$

In this case, the corresponding second-order variational equations reduce to the usual first-order ones, which are

$$
\begin{aligned}
&\alpha(t) = \Phi(x(T)) + \int_t^T l(u(s))dt - \int_t^T \beta(s)dV(s),\\
&-d\varphi(t) = [b_x^*(u(t))\varphi(t) + \sigma_x^*(u(t))\psi(t) + h_x^*(u(t))\beta(t)]dt - \psi(t)dW(t),\\
&\varphi(T) = \Phi_x^*(x(T)).
\end{aligned}
$$

From Theorem 2.3, we get easily the following result.

Corollary 2.1 *Assume that* $\gamma = 1$, $\sigma(\cdot, x^v(\cdot), v(\cdot)) \equiv \sigma(\cdot, x^v(\cdot))$, $l(\cdot, x^v(\cdot), v(\cdot)) \equiv l(\cdot, v(\cdot))$ *and* (H7) *hold. Let* $u(\cdot)$ *be optimal. Then, the maximum principle*

$$\mathbb{E}^u[\mathcal{H}(t, x(t), v, \varphi(t), \psi(t), \beta(t))|\mathcal{Y}_t] \ge \mathbb{E}^u[\mathcal{H}(t, x(t), u(t), \varphi(t), \psi(t), \beta(t))|\mathcal{Y}_t],$$

$\forall v \in U, a.e., a.s.$ *holds, where the Hamiltonian function* $\mathcal{H}$ *is defined by* (25).

This result will be applied to study a partially observed LQ nonzero sum differential game problem in the next section.

3 Two Interesting Examples

From Theorem 2.1, 2.2, 2.3 and Corollary 2.1, we notice that maximum principles depend strongly on adjoint processes. To get an observable optimal control, it is necessary to investigate the filtering estimate for adjoint processes which satisfy BSDEs. Since there is no general filtering results for BSDEs, it is difficult to get explicitly observable maximum principles. However, we will try to give some applications of partially observed maximum principles to differential game and optimal control problems.

As before, we always assume that $(W(\cdot), Y(\cdot))$ is a 2-dimensional standard Brownian motion defined on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. x_0 is an $\mathcal{F}_0$-measurable random variable with the mean m_0 and is independent of $(W(\cdot), Y(\cdot))$.

We first study a partially observed LQ nonzero sum differential game problem.

Example 3.1 For simplification, let us consider only the case of two players. The 1-dimensional state and observation equations are as follows:

$$dx(t) = (Ax(t) + B_1 v_1(t) + B_2 v_2(t))dt + CdW(t), \qquad x(0) = x_0, \tag{26}$$

$$dY(t) = D(t)dt + dV^v(t), \qquad Y(0) = 0. \tag{27}$$

Here $N_i > 0$, $Q_i \geq 0$, B_i, $i = 1, 2$; A and C are constants and $D(\cdot)$ is bounded deterministic in $[0, T]$. Then, (26) admits a unique solution denoted by $x^v(\cdot)$. The cost functionals of the two players are described as

$$J_i(v_1(\cdot), v_2(\cdot)) = \frac{1}{2}\mathbb{E}^v\left\{\int_0^T N_i v_i^2(t)dt + Q_i[x^v(T)]^2\right\}, \quad i = 1, 2. \tag{28}$$

Our problem is to seek a pair of $(u_1(\cdot), u_2(\cdot))$ such that

$$J_1(u_1(\cdot), u_2(\cdot)) = \min_{v_1(\cdot)\in\mathcal{U}_{ad}} J_1(v_1(\cdot), u_2(\cdot)),$$

$$J_2(u_1(\cdot), u_2(\cdot)) = \min_{v_2(\cdot)\in\mathcal{U}_{ad}} J_2(u_1(\cdot), v_2(\cdot)).$$

Such a pair $(u_1(\cdot), u_2(\cdot))$ (if it exists) is called a Nash equilibrium point of the game problem. The corresponding trajectory is denoted by $x(\cdot)$. We use three steps to solve this problem.

Step 1 The Hamiltonian functions are

$$\begin{aligned}
&\mathcal{H}_1(t, x^{v_1}(t), v_1(t), u_2(t), \varphi_1(t), \psi_1(t), \beta_1(t)) \\
&\quad = (Ax^{v_1}(t) + B_1 v_1(t) + B_2 u_2(t))\varphi_1(t) + C\psi_1(t) + D(t)\beta_1(t) + \frac{1}{2}N_1 v_1^2(t), \\
&\mathcal{H}_2(t, x^{v_2}(t), u_1(t), v_2(t), \varphi_2(t), \psi_2(t), \beta_2(t)) \\
&\quad = (Ax^{v_2}(t) + B_1 u_1(t) + B_2 v_2(t))\varphi_2(t) + C\psi_2(t) + D(t)\beta_2(t) + \frac{1}{2}N_2 v_2^2(t),
\end{aligned}$$

where $(\varphi_i(\cdot), \psi_i(\cdot))$, $i = 1, 2$, and $(\alpha_i(\cdot), \beta_i(\cdot))$, $i = 1, 2$, are the solutions of

$$\varphi_i(t) = Q_i x(T) + \int_t^T A\varphi_i(s)ds - \int_t^T \psi_i(s)dW(s), \quad i = 1, 2,$$

$$\alpha_i(t) = \frac{1}{2}Q_i x^2(T) + \frac{1}{2}\int_t^T N_i u_i^2(s)ds - \int_t^T \beta_i(s)dV(s), \quad i = 1, 2.$$

Here, let $x^{v_1}(\cdot)$ and $x^{v_2}(\cdot)$ be the trajectories under the controls $(v_1(\cdot), u_2(\cdot))$ and $(u_1(\cdot), v_2(\cdot))$, respectively. Noticing the aforementioned Hamiltonian functions, if

$(u_1(\cdot), u_2(\cdot))$ is a Nash equilibrium point, then Corollary 2.1 implies that

$$u_i(t) = -N_i^{-1} B_i \mathbb{E}^u[\varphi_i(t)|\mathcal{Y}_t], \quad i = 1, 2, \tag{29}$$

where

$$\varphi_i(t) = e^{A(T-t)} Q_i \mathbb{E}^u[x(T)|\mathcal{F}_t], \quad i = 1, 2.$$

Step 2 We will use filtering theory to find a more explicit representation of $(u_1(\cdot), u_2(\cdot))$ defined by (29). It is worthwhile pointing out that the most successful result of filtering theory was obtained for linear systems by Kalman (Ref. [15]) in 1960. However, Kalman's filtering result cannot be applied directly to the state and observation systems (26)–(27). Now, we will use the filtering results developed in Liptser and Shiryayev (Ref. [16]). Let $\hat{x}(t) = \mathbb{E}^u[x(t)|\mathcal{Y}_t]$ be the filtering estimate of the state $x(t)$ depending on the observable filtration $\mathcal{Y}_t$. From Theorem 12.1 in Liptser and Shiryayev (Ref. [16]), we get

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B_1 u_1(t) + B_2 u_2(t), \qquad \hat{x}(0) = m_0. \tag{30}$$

In light of Theorem 2.5 in Hamadène (Ref. [13]) or Theorem 2 in Wu and Yu (Ref. [14]), it is natural to conjecture that $u_i(\cdot)$, $i = 1, 2$, are the linear feedbacks of the state filtering estimate, i.e.,

$$u_i(t) = -N_i^{-1} B_i \pi_i(t)\hat{x}(t), \quad i = 1, 2. \tag{31}$$

Here, $\pi_i(\cdot)$, $i = 1, 2$, are determined by

$$\dot{\pi}_1(t) + 2A\pi_1(t) - N_2^{-1} B_2^2 \pi_1(t)\pi_2(t) - N_1^{-1} B_1^2 \pi_1^2(t) = 0, \tag{32a}$$

$$\pi_1(T) = Q_1, \tag{32b}$$

$$\dot{\pi}_2(t) + 2A\pi_2(t) - N_1^{-1} B_1^2 \pi_1(t)\pi_2(t) - N_2^{-1} B_2^2 \pi_2^2(t) = 0, \tag{33a}$$

$$\pi_2(T) = Q_2, \tag{33b}$$

and $\hat{x}(\cdot)$ is the solution of (30).

We notice that (32) and (33) are coupled. To prove the existence of solutions, we need an additional assumption, $N_1^{-1} B_1^2 = N_2^{-1} B_2^2$. Introduce the following ordinary differential equations (ODEs):

$$\dot{\pi}(t) + 2A\pi(t) - N_1^{-1} B_1^2 \pi^2(t) = 0, \qquad \pi(T) = Q_1 + Q_2, \tag{34}$$

$$\dot{\bar{\pi}}_1(t) + (2A - N_2^{-1} B_2^2 \pi(t))\bar{\pi}_1(t) = 0, \qquad \bar{\pi}_1(T) = Q_1, \tag{35}$$

$$\dot{\bar{\pi}}_2(t) + (2A - N_1^{-1} B_1^2 \pi(t))\bar{\pi}_2(t) = 0, \qquad \bar{\pi}_2(T) = Q_2. \tag{36}$$

It is well known that (34) is a standard Riccati differential equation, which admits a unique nonnegative solution. Therefore (35) and (36) also admit unique solutions, respectively. If we let $\bar{\pi}(\cdot) = \bar{\pi}_1(\cdot) + \bar{\pi}_2(\cdot)$, it is easy to check that $\bar{\pi}(\cdot)$ satisfies (33), i.e., $\bar{\pi}(\cdot) = \pi(\cdot)$. Substituting $\pi(\cdot) = \bar{\pi}_1(\cdot) + \bar{\pi}_2(\cdot)$ into (35) and (36), we know that (32) and (33) admit unique solutions, respectively.

The remaining task is to verify that our conjecture (31) holds. We define

$$u_i(t) = -K_i(t)\hat{x}(t), \qquad i = 1, 2, \tag{37}$$

where $K_i(\cdot)$, $i = 1, 2$, are deterministic functions defined later on. Let $\Psi(s, t)$ be the fundamental matrix solution of

$$\dot{X}_s = \begin{pmatrix} A - B_1K_1(s) - B_2K_2(s) & 0 \\ -B_1K_1(s) - B_2K_2(s) & A \end{pmatrix} X_s. \tag{38}$$

From (26), (30), (37), (38), we derive

$$\begin{pmatrix} \hat{x}(T) \\ x(T) \end{pmatrix} = \Psi(T, t)\begin{pmatrix} \hat{x}(t) \\ x(t) \end{pmatrix} + \int_t^T \Psi(T, s)\begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} d\begin{pmatrix} V(s) \\ W(s) \end{pmatrix}.$$

By a property of conditional expectation, it follows that

$$\begin{aligned} \mathbb{E}^u[\varphi_i(t)|\mathcal{Y}_t] &= e^{A(T-t)}Q_i\mathbb{E}^u[x(T)|\mathcal{Y}_t] \\ &= e^{A(T-t)}Q_i\begin{pmatrix} 0 & 1 \end{pmatrix}\Psi(T, t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\hat{x}(t) \\ &= Q_i e^{\int_t^T (2A - B_1K_1(s) - B_2K_2(s))ds}\hat{x}(t), \quad i = 1, 2. \end{aligned}$$

Set $K_i(t) = N_i^{-1}B_i\pi_i(t)$, $i = 1, 2$. In light of (32) and (33), it is clear that

$$\mathbb{E}^u[\varphi_i(t)|\mathcal{Y}_t] = \pi_i(t)\hat{x}(t), \quad i = 1, 2.$$

From (29) and (37), we know that our conjecture (31) holds.

Step 3 Now, we prove that $(u_1(\cdot), u_2(\cdot))$ defined by (31) is an observable Nash equilibrium point. Combining Wohnam's separation theorem (Ref. [17]) with usual techniques for LQ problems (see e.g. [13] or [14]), we can prove that $(u_1(\cdot), u_2(\cdot))$ defined by (31) satisfies

$$\begin{aligned} J_1(u_1(\cdot), u_2(\cdot)) &\le J_1(v_1(\cdot), u_2(\cdot)), \quad \forall v_1(\cdot) \in \mathcal{U}_{ad}, \\ J_2(u_1(\cdot), u_2(\cdot)) &\le J_1(u_1(\cdot), v_2(\cdot)), \quad \forall v_2(\cdot) \in \mathcal{U}_{ad}. \end{aligned}$$

Therefore, we have the following proposition.

Proposition 3.1 *Let all the hypotheses in Example 3.1 hold. Then, the pair $(u_1(\cdot), u_2(\cdot))$ defined by (31) is an observable Nash equilibrium point of the partially observed game problem.*

Remark 3.1 To guarantee that (32) and (33) admit solutions, we need an additional assumption, $N_1^{-1}B_1^2 = N_2^{-1}B_2^2$. But, if $A = 0$ in (26), the restriction condition can be eliminated. In fact, let us introduce the following ODEs:

$$\dot{\Sigma}(t) - \Sigma^2(t) = 0, \qquad \Sigma(T) = N_1^{-1}B_1^2Q_1 + N_2^{-1}B_2^2Q_2, \tag{39}$$

$$\dot{\Sigma}_1(t) - \Sigma_1(t)\Sigma(t) = 0, \qquad \Sigma_1(T) = Q_1, \tag{40}$$

$$\dot{\Sigma}_2(t) - \Sigma_2(t)\Sigma(t) = 0, \qquad \Sigma_2(T) = Q_2. \tag{41}$$

Clearly, there exist solutions to (39)–(41). If we define $\bar{\Sigma}(\cdot) = N_1^{-1}B_1^2\Sigma_1(\cdot) + N_2^{-1}B_2^2\Sigma_2(\cdot)$, then $\bar{\Sigma}(\cdot)$ is the solution of (39). Noticing (40) and (41), we easily get the desired result.

The rest of this section is related to a partially observed risk-sensitive optimization problem. Since the control has entered into the diffusion, Wonham's separation theorem does not work in this setting. However, the maximum principle (Theorem 2.1) developed in Sect. 2 is still an alternative tool. From the maximum principle, an optimal control of the aforementioned optimization problem can be obtained. But we will try to use another direct construction technique to solve this problem.

Example 3.2 Consider the 1-dimensional risk-sensitive optimal control problem

$$J(u(\cdot)) = \min_{v(\cdot)\in\mathcal{U}_{ad}} J(v(\cdot)), \qquad J(v(\cdot)) = \frac{1}{\gamma}\mathbb{E}^v[x^v(T)]^\gamma, \quad \gamma \geq 2, \tag{42}$$

subject to

$$dx^v(t) = (A(t)x^v(t) + B(t)v(t))dt + (C(t)x^v(t) + F(t)v(t))dW(t), \tag{43a}$$

$$x^v(0) = x_0, \tag{43b}$$

and the observation (27). Here $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $F(\cdot)$, $F^{-1}(\cdot)$ are bounded deterministic in $[0, T]$.

Applying Itô's formula, we get easily

$$\begin{aligned} d[x^v(t)]^\gamma = \gamma\bigg\{&\Big[A(t) + \frac{1}{2}(\gamma-1)C^2(t)\Big][x^v(t)]^\gamma \\ &+ [B(t) + (\gamma-1)C(t)F(t)][x^v(t)]^{\gamma-1}v(t) \\ &+ \frac{1}{2}(\gamma-1)F^2(t)[x^v(t)]^{\gamma-2}v^2(t)\bigg\}dt \\ &+ \gamma(C(t)x^v(t) + D(t)v(t))[x^v(t)]^{\gamma-1}dW(t). \end{aligned}$$

For convenience, we set $\hat{X}(t, v; \gamma) = \mathbb{E}^v\{[x^v(t)]^\gamma|\mathcal{Y}_t\}$. By Theorem 8.1 in Liptser and Shiryayev (Ref. [16]), it follows that

$$\begin{aligned} \hat{X}(t, v; \gamma) = x_0^\gamma + \gamma\int_0^t\bigg\{&[A(s) + \frac{1}{2}(\gamma-1)C^2(s)]\hat{X}(s, v; \gamma) \\ &+ [B(s) + (\gamma-1)C(s)F(s)]\hat{X}(s, v; \gamma-1)v(s) \\ &+ \frac{1}{2}(\gamma-1)F^2(s)\hat{X}(s, v; \gamma-2)v^2(s)\bigg\}ds. \end{aligned} \tag{44}$$

On the other hand, it is clear that

$$J(v(\cdot)) = \frac{1}{\gamma}\mathbb{E}^v\{\mathbb{E}^v\{[x^v(T)]^\gamma|\mathcal{Y}_T\}\} = \frac{1}{\gamma}\mathbb{E}^v[\hat{X}(T, v; \gamma)]. \tag{45}$$

Thus, (45) subject to (44) yields a fully observed optimization problem. To obtain an optimal control, we need an additional assumption, $(\gamma-1)\hat{X}(T,v;\gamma-2)>0$. Noticing that the integrand in (44) is quadratic with respect to $v(\cdot)$, we get easily the following result.

Proposition 3.2 *The optimal control of* (42) *subject to* (43) *and* (27) *is given by*

$$u(t)=-\frac{B(t)+(\gamma-1)C(t)F(t)}{(\gamma-1)F^2(t)\hat{X}(t,u;\gamma-2)}\hat{X}(t,u;\gamma-1).$$

Remark 3.2 Generally, it is difficult to get an explicitly observable optimal control for the partially observed risk-sensitive problem. To our knowledge, this is still an open problem. But, if we let $\gamma=2$, the cost functional (42) reduces to an indefinite control weight cost. From (44) and Proposition 3.2, the following corollary is obvious.

Corollary 3.1 *The optimal control is*

$$u(t)=-(B(t)+C(t)F(t))F^{-2}(t)\hat{x}(t),$$

where $\hat{x}(\cdot)=\hat{X}(\cdot,u;1)$ *is the solution of*

$$\dot{\hat{x}}(t)=A(t)\hat{x}(t)+B(t)u(t),\qquad \hat{x}(0)=m_0.$$

Remark 3.3 The optimal control obtained in Corollary 3.1 is a linear feedback of the state filtering estimate. If the diffusion $C(\cdot)x^v(\cdot)+F(\cdot)v(\cdot)$ in (43) is displaced by the aforementioned $F(\cdot)$, the corresponding optimal control problem reduces to the classical one in Liptser and Shiryayev (Ref. [16]).

4 Fully Observed Maximum Principle

As a natural deduction of the results in Sect. 2, we now desire to get a maximum principle for a fully observed risk-sensitive optimal control problem. Then, we apply the maximum principle to study a risk-sensitive optimal portfolio problem in next section.

Set $\mathcal{F}_t^W=\sigma\{W(s);0\le s\le t\}$. Suppose that we can fully observe the filtration $\mathcal{F}_t^W$ at time t, $0\le t\le T$. For $m=2,3,4,\dots$, a control variable $v(\cdot)$ is called admissible if $v(t):[0,T]\times\Omega\to U\subseteq\mathfrak{R}^k$ is $\mathcal{F}_t^W$-adapted and satisfies $\sup_{0\le t\le T}\mathbb{E}|v(t)|^m<+\infty$, a.e., a.s. The set of admissible controls is denoted by $\mathcal{A}_{ad}$. For any $v(\cdot)\in\mathcal{A}_{ad}$, we let $x^v(\cdot)$ be the trajectory corresponding to (1).

In this setting, (6) subject to (1) and (3) reduces to minimizing the cost functional

$$J(v(\cdot))=\frac{1}{\gamma}\mathbb{E}[\Phi(x^v(T))]^\gamma,\quad \gamma>0,\tag{46}$$

subject to (1) and $v(\cdot)\in\mathcal{A}_{ad}$.

Combining Lemma 2.2 with Theorem 2.1, we get the following corollary.

Corollary 4.1 *Let* (H1), (H2), (H3) *hold. If* $u(\cdot)$ *is optimal, the maximum principle,*

$$\begin{aligned}&\bar{H}(t,x(t),v,\bar{p}(t),\bar{q}(t))-\bar{H}(t,x(t),u(t),\bar{p}(t),\bar{q}(t))\\&\quad+\frac{1}{2}\mathrm{Tr}[(\sigma(v)-\sigma(u(t)))^*\bar{P}(t)(\sigma(v)-\sigma(u(t)))]\geq 0,\quad \forall v\in U,\ a.e., a.s.,\end{aligned}$$

holds, where $(\bar{p}(\cdot),\bar{q}(\cdot))$, $(\bar{P}(\cdot),\bar{Q}(\cdot))$ *are the solutions of the following BSDEs*:

$$\begin{aligned}-d\bar{p}(t)&=\bar{H}_x^*(t,x(t),u(t),\bar{p}(t),\bar{q}(t))dt-\bar{q}(t)dW(t),\\ \bar{p}(T)&=[\Phi(x(T))]^{\gamma-1}\Phi_x^*(x(T)),\\ -d\bar{P}(t)&=\Bigg[b_x^*(u(t))\bar{P}(t)+\bar{P}(t)b_x^*(u(t))+\sum_{i=1}^d\sigma_{i,x}^*(u(t))\bar{P}(t)\sigma_{i,x}(u(t))\\&\quad+\sum_{i=1}^d\sigma_{i,x}^*(u(t))\bar{Q}_i(t)+\sum_{i=1}^d\bar{Q}_i(t)\sigma_{i,x}(u(t))\\&\quad+\bar{H}_{xx}(t,x(t),u(t),\bar{p}(t),\bar{q}(t))\Bigg]dt-\bar{Q}(t)dW(t),\\ \bar{P}(T)&=(\gamma-1)[\Phi(x(T))]^{\gamma-2}\Phi_x(x(T))\Phi_x^*(x(T))+[\Phi(x(T))]^{\gamma-1}\Phi_{xx}(x(T)),\end{aligned}$$

and the Hamiltonian function $\bar{H}:[0,T]\times\mathfrak{R}^n\times U\times\mathfrak{R}^n\times\mathfrak{R}^{n\times d}\to\mathfrak{R}$ *is defined by*

$$\bar{H}(t,x^v,v,\bar{p},\bar{q})=\langle\bar{p},b(t,x^v,v)\rangle+\sum_{i=1}^d\langle\bar{q}_i,\sigma_i(t,x^v,v)\rangle.$$

Remark 4.1 If $\mathcal{F}_t^W$ is fully observable at time t, the general cost functional (18) subject to (1) and (3) reduces to the fully observed case. Using the techniques presented in Theorem 2.2 and Corollary 4.1, a similar maximum principle can also be derived.

5 Applications to Finance

In this section, we will apply Corollary 4.1 to study a fully observed risk-sensitive optimal portfolio problem.

Let us consider a financial market in which two securities can be continuously traded. One is a risk-free asset (bond), whose price is defined by the ODE

$$dS_0(t)=r(t)S_0(t)dt,$$

where $r(t)$ is the risk-free interest rate at time t. Without loss of generality, we set $S_0(0)=1$. Generally speaking, a change of the risk-free interest rate can affect the price of a risky security (stock). For simplification, we assume that the riskysecurity price satisfies the following SDE:

$$dS_1(t) = S_1(t)[(r(t) + \mu(t))dt + \sigma(t)dW(t)].$$

Here, $r(\cdot) + \mu(\cdot)$ is the instantaneous expected rate of return and $\sigma(\cdot)$ is the instantaneous volatility rate. $W(\cdot)$ is a standard 1-dimensional Brownian motion defined on $(\Omega, \mathcal{F}^W, (\mathcal{F}_t^W), P)$ equipped with the filtration $\mathcal{F}_t^W = \sigma\{W(s); 0 \le s \le t\}$ and $\mathcal{F}^W = \mathcal{F}_T^W$. The coefficients $r(\cdot)$, $\mu(\cdot)$, $\sigma(\cdot)$, $\sigma^{-1}(\cdot)$ are bounded and deterministic in $[0, T]$.

We denote by $x(t)$ the wealth of an investor at time t, by $\pi(t)$ the amount that she/he invests in the stock. It is well known that, under a so called self-financing portfolio, the wealth process of the investor, starting with some initial endowment $x_0 > 0$, satisfies the following wealth equation (see e.g. Karatzas and Shreve [18])

$$\begin{aligned} dx(t) &= (r(t)x(t) + \mu(t)\pi(t))dt + \sigma(t)\pi(t)dW(t), \\ x(0) &= x_0. \end{aligned}$$

Define $\bar{x}(t) = x(t)e^{-\int_0^t r(s)ds}$. It follows from the Itô's formula that

$$d\bar{x}(t) = \mu(t)\pi(t)e^{-\int_0^t r(s)ds}dt + \sigma(t)\pi(t)e^{-\int_0^t r(s)ds}dW(t), \tag{47a}$$

$$\bar{x}(0) = x_0. \tag{47b}$$

Let

$$\bar{\mathcal{A}}_{ad} = \Big\{\pi(\cdot)|\pi(t) : [0, T] \times \Omega \to U \subseteq \mathfrak{R} \text{ is } \mathcal{F}_t^W\text{-adapted and satisfies} \\ \sup_{0\le t\le T} \mathbb{E}\pi^2(t)dt < +\infty \text{ and } \bar{x}(t) \ge 0, \text{ a.e., a.s.}\Big\}.$$

An element of $\bar{\mathcal{A}}_{ad}$ is called an admissible portfolio. Here, we must point out that U is not necessarily convex. For any $\pi(\cdot) \in \bar{\mathcal{A}}_{ad}$, (47) admits a unique solution denoted by $\bar{x}^{x_0;\pi}(\cdot)$. The target of the investor is to choose an appropriate $\pi^*(\cdot) \in \bar{\mathcal{A}}_{ad}$ such that

$$J(\pi^*(\cdot); x_0) = \max_{\pi(\cdot)\in\bar{\mathcal{A}}_{ad}} \frac{1}{\gamma}\mathbb{E}[\bar{x}^{x_0;\pi}(T)]^\gamma, \quad 0 < \gamma < 1. \tag{48}$$

To be mathematically rigorous, we assume also that $\mathbb{E}[\bar{x}^{x_0;\pi}(T)]^{2\gamma-4} < +\infty$ holds. Here, $0 < \gamma < 1$ is a constant. If we let $\beta = 1 - \gamma$, then $0 < \beta < 1$ and it is the so called Arrow-Pratt risk aversion index (see e.g. Karatzas and Shreve [18]). $\pi^*(\cdot)$ is called an optimal portfolio and $\bar{x}^{x_0;\pi^*}(\cdot)$ is the corresponding optimal wealth process. Clearly, (48) subject to (47) is equivalent to

$$J(\pi^*(\cdot); x_0) = -\min_{\pi(\cdot)\in\bar{\mathcal{A}}_{ad}} \frac{1}{\gamma}\mathbb{E}[\bar{x}^{x_0;\pi}(T)]^\gamma. \tag{49}$$

To solve this problem, we write the Hamiltonian function, which is similar to that in Peng (Ref. [12]),

$$\begin{aligned}\bar{H}(t, \bar{x}^{x_0;\pi^*}(t), \pi(t), \bar{p}(t), \bar{q}(t) - \bar{P}(t)\pi^*(t)\sigma(t)e^{-\int_0^t r(s)ds})\\ + \frac{1}{2}\pi^2(t)\sigma^2(t)e^{-2\int_0^t r(s)ds}\bar{P}(t)\\ = \pi(t)\mu(t)\bar{p}(t)e^{-\int_0^t r(s)ds} + \pi(t)\sigma(t)e^{-\int_0^t r(s)ds}(\bar{q}(t)\\ - \bar{P}(t)\pi^*(t)\sigma(t)e^{-\int_0^t r(s)ds}) + \frac{1}{2}\pi^2(t)\sigma^2(t)e^{-2\int_0^t r(s)ds}\bar{P}(t),\end{aligned}$$

where $(\bar{p}(\cdot), \bar{q}(\cdot))$, $(\bar{P}(\cdot), \bar{Q}(\cdot))$ are the solutions of

$$\bar{p}(t) = -[\bar{x}^{x_0;\pi^*}(T)]^{\gamma-1} - \int_t^T \bar{q}(s)dW(s), \tag{50a}$$

$$\bar{P}(t) = (1-\gamma)[\bar{x}^{x_0;\pi^*}(T)]^{\gamma-2} - \int_t^T \bar{Q}(s)dW(s). \tag{50b}$$

Under the hypotheses above, from Corollary 4.1 we get easily

$$\mu(t)\bar{p}(t) + \sigma(t)\bar{q}(t) = 0. \tag{51}$$

We try to conjecture $\pi^*(\cdot) = m^*(\cdot)\bar{x}^{x_0;\pi^*}(\cdot)$. Here and below, $m^*(\cdot)$ and $n(\cdot)$ are bounded and deterministic functions, which are defined later on. Substituting $\pi^*(\cdot)$ into (47) and noticing the terminal condition of (50), it is natural to set

$$\bar{p}(t) = -[x^{x_0;\pi^*}(t)]^{\gamma-1}e^{\int_t^T n(s)ds}.$$

Applying Itô's formula to $\bar{p}(t)$, we have

$$\begin{aligned}-d\bar{p}(t) = \left[(1-\gamma)m^*(t)\mu(t) - \frac{1}{2}(\gamma-1)(\gamma-2)(m^*(t))^2\sigma^2(t) + n(t)\right]\bar{p}(t)dt\\ - (\gamma-1)m^*(t)\sigma(t)\bar{p}(t)dW(t).\end{aligned} \tag{52}$$

Comparing the drift and the diffusion terms of (52) with (50), we derive

$$\bar{q}(t) = (\gamma-1)m^*(t)\sigma(t)\bar{p}(t), \tag{53a}$$

$$(1-\gamma)m^*(t)\mu(t) - \frac{1}{2}(\gamma-1)(\gamma-2)(m^*(t))^2\sigma^2(t) + n(t) = 0. \tag{53b}$$

Noticing (51) and (53), we get

$$m^*(t) = \frac{\mu(t)}{\sigma^2(t)(1-\gamma)}, \qquad n(t) = \frac{\gamma\mu^2(t)}{2(1-\gamma)\sigma^2(t)}.$$

Therefore, we have

$$\pi^*(t) = \frac{\mu(t)}{\sigma^2(t)(1-\gamma)}x^{x_0;\pi^*}(t), \tag{54}$$

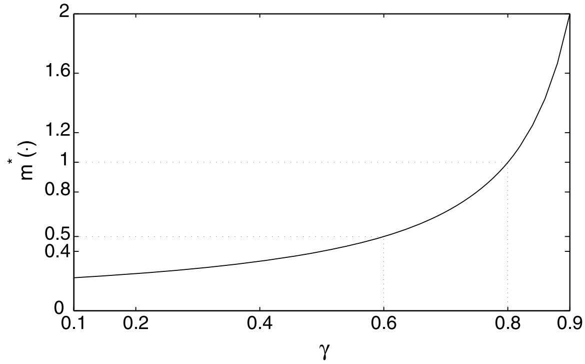

Fig. 1 The relationship between $m(\cdot)$ and γ

where $x^{x_0;\pi^*}(t)$ is the solution of the following optimal wealth equation

$$dx^{x_0;\pi^*}(t) = \left(r(t) + \frac{\mu^2(t)}{\sigma^2(t)(1-\gamma)}\right)x^{x_0;\pi^*}(t)dt + \frac{\mu(t)}{\sigma(t)(1-\gamma)}x^{x_0;\pi^*}(t)dW(t), \tag{55a}$$

$$x^{x_0;\pi^*}(0) = x_0. \tag{55b}$$

Clearly, $x^{x_0;\pi^*}(t) > 0$ and $\mathbb{E}[x^{x_0;\pi^*}(t)]^{2\gamma-4} < +\infty$, $0 \le t \le T$. The optimal cost functional is

$$J(\pi^*(\cdot); x_0) = \frac{1}{\gamma}x_0^\gamma e^{\gamma \int_0^T [r(s)+\frac{\mu^2(s)}{2\sigma^2(s)(1-\gamma)}]ds}. \tag{56}$$

We conclude the discussion above with the following proposition.

Proposition 5.1 *The optimal investment amount in the stock, the corresponding wealth equation and the optimal cost functional are given by* (54), (55), (56), *respectively.*

From (54), it is easy to see that $m^*(\cdot) = \frac{\pi^*(\cdot)}{x^{x_0;\pi^*}(\cdot)}$, which is called the optimal investment proportion in the stock. Obviously, $m^*(\cdot)$ does not depend on $r(\cdot)$ and it is increasing with respect to $\mu(\cdot)$ and decreasing with respect to $\sigma(\cdot)$ and $\beta = 1-\gamma$. Generally speaking, we are more interested in the influence of the risk-sensitive parameter γ to the optimal investment proportion $m^*(\cdot)$. To show explicitly the relationship between them, we give a numerical simulation example and plot the following Fig. 1, where the time unit is one year, $\sigma = 0.25$, $\mu = 0.0125$ and $0.1 \le \gamma \le 0.9$.

In Fig. 1, we notice that $m^*(\cdot)$ is larger than 0.2 and it is increasing with respect to γ. The optimal investment proportion curve is divided into three parts according to the different γ value intervals. When $0.1 \le \gamma \le 0.6$, we have $0.2 < m^*(\cdot) \le 0.5$. This means that the investor invests the most part of her/his wealth in the bond and smaller part in the stock. When $0.6 < \gamma \le 0.8$, we get $0.5 < m^*(\cdot) \le 1$. This means that the investor invests the most part of her/his wealth in the stock and smaller part in the bond. When $0.8 < \gamma \le 0.9$, we obtain $1 < m^*(\cdot) \le 2$. This means that the investor always borrows money from a bank and invests all her/his wealth in the stock.

Those illustrations mentioned above also interpret the meaning of the risk-sensitive parameter γ. In details, we assume $0<\gamma<1$, then $\beta=1-\gamma$ denotes the Arrow-Pratt risk aversion index. From Fig. 1, we know that the higher the risk-sensitive parameter γ, the bigger the optimal investment proportion $m^*(\cdot)$, the smaller the risk aversion index β. This means that, if the risk-seeking degree of the investor increases, then the risk-aversion degree decreases. Reversely, the lower the risk-sensitive parameter γ, the smaller the optimal investment proportion $m^*(\cdot)$, the bigger the risk aversion index β. This means that, if the risk-aversion degree of the investor increases, then the risk-seeking degree decreases. These coincide with our theoretical results and the economic meaning of the risk-sensitive parameter γ.

6 Comparison with Existing Results

The subject of stochastic maximum principles for optimal control problems has been discussed by many researchers, such as Bensoussan (Ref. [6]), Haussmann (Ref. [7]), Baras et al. (Ref. [8]), Zhou (Ref. [9]), Li and Tang (Ref. [10]), etc. Compared with the papers above, our work is a generalization of their results. Similar to Li and Tang (Ref. [10]), a boundedness condition on the observation h is imposed. We hope that the condition can be improved in our future works.

Three points are worth while pointing out:

(a) In our context, by a conventional method used in the fully observed risk-neutral situation, we derive some stochastic maximum principles for the partially observed risk-sensitive (neutral) optimal control problems. The related adjoint processes are characterized by solutions of some finite-dimensional BSDEs. Our method does not involve the Zakai equations; thus we can get rid of lots of complicated stochastic calculus in infinite dimensional spaces, in contrast with Bensoussan (Ref. [6]), Haussmann (Ref. [7]) and Zhou (Ref. [9]).

(b) Our Theorem 2.3 coincides with Theorem 2.1 of Li and Tang (Ref. [10]). If we let $\gamma=1$ in our Corollary 2.1, it includes the result of Baras et al. (Ref. [8]) as a special case, where the control variable $v(\cdot)$ is not in the diffusion and the observation. Even in the risk-neutral case (see e.g. [6–10], etc.), some theoretical results of maximum principles were obtained by different methods. However, little attention was paid to applications of these theoretical results. In Sect. 3 of this paper, we work out two examples to illustrate the applications of our theoretical results.

(c) Under our framework, the form of the maximum principles is similar to its risk-neutral counterpart, but the corresponding variational inequalities and the adjoint equations depend strongly on the risk-sensitive parameter γ. In Sect. 5, we apply the fully observed maximum principle (Corollary 4.1) to study a risk-sensitive optimal portfolio problem and get an explicit optimal solution. A numerical simulation result clearly illustrates the influence of the risk-sensitive parameter γ on the optimal investment proportion, which coincides with its economic meaning and theoretical results.

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