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Quasiconvex Minimization on a Locally Finite Union of Convex Sets

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Abstract Extending the approach initiated in Aussel and Hadjisavvas (SIAM J. Optim. 16:358–367, 2005) and Aussel and Ye (Optimization 55:433–457, 2006), we obtain the existence of a local minimizer of a quasiconvex function on the locally finite union of closed convex subsets of a Banach space. We apply the existence result to some difficult nonconvex optimization problems such as the disjunctive programming problem and the bilevel programming problem.

Keywords Quasiconvex programming · Existence results · Nonconvex constraint set

1 Introduction

Let us consider the following mathematical programming problem:

$$\begin{aligned} \min \quad & f(x), \\ \text{s.t.} \quad & \min_{j \in \mathbf{J}} g_j(x) \le 0, \end{aligned}$$

Dedicated to Jean-Pierre Crouzeix on the occasion of his 65th birthday.

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where f is a quasiconvex function defined from a Banach space X to $\mathbb{R}$ and $\{g_j : j \in J\}$ is an infinite family of lower semicontinuous quasiconvex functions defined on X. The simplicity of the formulation of this problem hides an important difficulty: the constraint set is not convex in general. In fact, it is easy to observe that the constraint set is the infinite union of closed convex sets. But, as shown in the sequel, this problem, known in the literature as the disjunctive programming problem, is a particular case of a mathematical programming problem for which we prove the existence of a local minimizer, under reasonable assumptions.

In Aussel-Hadjisavvas [2], thanks to the normal operator approach of quasiconvex analysis, an existence result for the minimization of a quasiconvex function on a convex constraint set has been considered, while in Aussel-Ye [3] the case of a nonconvex constraint set has been considered for the particular case of the Mathematical Programming with Equilibrium Problem by using the special conical structure of the constraint set of such problems.

Our aim in this paper is to obtain the existence of local minimizers of another class of quasiconvex minimization problems involving a nonconvex constraint: those for which the constraint set is the locally finite union of closed convex sets. This class includes, as a particular case, the disjunctive problem proposed at the beginning of the section.

The paper is organized as follows. In Sect. 2, we define the basic concepts. Then, we prove our main existence result in Sect. 3, while Sect. 4 is devoted to the applications to the disjunctive programming problem and the bilevel problem.

2 Preliminaries and Definitions

Along the paper, X stands for a real Banach space, X^* for its topological dual equipped with the weak* topology (denoted by w^*), and $\langle \cdot, \cdot \rangle$ for the duality pairing. For any $x \in X$ and $\varepsilon > 0$, $B(x, \varepsilon)$ and $\overline{B}(x, \varepsilon)$ are the open ball and the closed ball of center x and radius ε. For any subset K of X, $\operatorname{cl} K$, $\operatorname{int}(K)$ and $\operatorname{bd}(K)$ stand respectively for the closure, interior and boundary of K.

Finally, given a function $f : X \to \mathbb{R} \cup \{+\infty\}$, we consider the following sublevel sets:

$$S_a(f) = \{x \in X : f(x) \leq a\}.$$

Let us recall that $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *quasiconvex* if the sublevel sets $S_a(f)$ are convex, for any $a \in \mathbb{R}$; or, in other words, for any $x, y \in \operatorname{dom} f$, we have

$$f(z) \leq \max\{f(x), f(y)\}, \quad \forall z \in [x, y],$$

where $[x, y]$ denotes the line segment joining x and y.

Definition 2.1 A subset C of X is said to be the locally finite union of (convex-closed) sets if there exists a (possibly infinite) family $\{C_\alpha : \alpha \in A\}$ of (convex-closed) subsets of X such that $C=\bigcup_{\alpha\in A} C_\alpha$ and for any $x\in C$, there exist $\rho>0$ and a finite subset $\tilde{A}_x$ of A such that

$$B(x,\rho)\cap C = B(x,\rho)\cap\left[\bigcup_{\alpha\in\tilde{A}_x} C_\alpha\right]. \tag{1}$$

In this case, we denote $C=\bigcup^{lf}_{\alpha\in A} C_\alpha$. The locally finite union of convex sets has been considered previously in the literature in numerous works (see e.g. [11, 15]) for different purposes (fixed points, differential geometry). In quasiconvex optimization, it provides a natural context to describe the constraint set of some interesting classes of problems (see Sect. 4.1).

As emphasized in the following proposition, the concept of locally finite union of subsets is dedicated to noncompact sets.

Proposition 2.1 *If a compact subset C of X is the locally finite union of subsets, then C is the union of a finite number of those subsets.*

Proof Let $C=\bigcup^{lf}_{\alpha\in A} C_\alpha$. By classical compactness arguments, we can extract a finite open covering $\{B(x_i,\rho_i)\}_{i=1}^k$ from the open covering provided by Definition 2.1. Thus, by setting $\tilde{A}=\bigcup_{i=1}^k \tilde{A}_{x_i}$, clearly we have $C=\bigcup_{\alpha\in\tilde{A}} C_\alpha$. □

Lemma 2.1 *If a subset C of X is the locally finite union of closed sets $C=\bigcup^{lf}_{\alpha\in A} C_\alpha$, then, for any $x\in C$, there exist $\rho_x>0$ and a finite subset A_x of A such that*

$$B(x,\rho_x)\cap C = B(x,\rho_x)\cap\left[\bigcup_{\alpha\in A_x} C_\alpha\right] \tag{2}$$

and

$$x\in C_\alpha,\quad \forall\alpha\in A_x\ . \tag{3}$$

Definition 2.2 For any subset C of X, the locally finite union of closed sets, a family $\mathcal{M}=\{(\rho_x,A_x): x\in C\}$, with $\rho_x>0$ and A_x a finite subset of A satisfying (2) and (3), is called a *local mapping* of C.

It is clear that, in general, a given locally finite union of closed sets can admit more than one local mapping.

Proof of Lemma 2.1 For any x element of C, let $\rho>0$ and $\tilde{A}_x$ be a finite subset of A be such that (1) holds. Let us denote by I_x the subset of A defined by $I_x=\{\alpha\in A: x\in C_\alpha\}$. Then, for any $\alpha\in\tilde{A}_x\setminus I_x$, since C_α is closed, one can find a strictly positive real ρ_α such that $B(x,\rho_\alpha)\cap C_\alpha=\emptyset$. Thus, by setting $A_x=\tilde{A}_x\cap I_x$ and $\rho_x=\min\{\rho,\{\rho_\alpha:\alpha\in\tilde{A}_x\setminus A_x\}\}$, one obtains immediately (2) and (3). □

Remark 2.1

(a) Let us observe that the subset $I_x = \{\alpha \in A : x \in C_\alpha\}$, used in the above proof corresponding roughly speaking to the subsets C_α touched by x, can be infinite, even if C is the locally finite union of closed convex sets C_α. Indeed, (2) and (3) say only that $B(x,\rho) \cap C$ can be described by a finite number of subsets C_α which are touched by x; but x (and $B(x,\rho)$) can meet an infinite number of them.
(b) In the sequel, we assume sometimes that there exists a local mapping $\mathcal{M} = \{(\rho_x, A_x) : x \in C\}$ of a set C such that the subset $\{x \in C : \mathrm{card}(A_x) > 1\}$ is included in a weakly compact set. This assumption does not imply that other local mappings $\mathcal{M}' = \{(\rho'_x, A'_x) : x \in C\}$ share this property or even that the set $\{x \in C : \mathrm{card}(A'_x) > 1\}$ is bounded.
(c) A very simple example illustrating both remarks can be described in $\mathbb{R}^2$ by $C = C_0 \cup C'_0 \bigcup_{n \geq 1} C_n$ where $C_0 = [-1, 0] \times [-1, 0]$, $C'_0 = \mathbb{R}^+ \times \mathbb{R}^+$, and for $n \geq 1$, $C_n = \{(x, (1+n)x) : x \in \mathbb{R}\}$.

The concept of locally starshapedness of a subset, introduced in [3], is a natural extension of the notion of starshapedness, which is itself a natural extension of convexity. In [3], it has been shown that the constraint set of a quasiconvex-quasiaffine MPEC problem is locally starshaped (but not starshaped in general).

Definition 2.3 A subset C of X is said to be *starshaped* at $\bar{x} \in C$ (or with center $\bar{x}$) if $[\bar{x}, y] \subseteq C$, for any $y \in C$. A subset C of X is said to be *locally starshaped* at $\bar{x} \in C$ if there exists a positive real δ such that $C \cap B(\bar{x}, \delta)$ is starshaped at $\bar{x}$. Finally, C is said to be *locally starshaped* if it is locally starshaped at any element $\bar{x}$ of C.

Clearly, any convex set C is locally starshaped at any $\bar{x} \in C$. But the union of convex sets need not be locally starshaped as one can observe by simply considering in $\mathbb{R}^2$ the subset $C = \bigcup_{n \in \mathbb{N}} C_n$ with $C_0 = \mathbb{R} \times \{0\}$ and $C_n = \mathbb{R} \times \{1/n\}$ for $n \geq 1$. Obviously, the set C is not the locally finite union of convex sets and is not locally starshaped at $\bar{x} = (0, 0)$.

Proposition 2.2 *Any locally finite union of closed convex sets is locally starshaped.*

Proof This is a direct consequence of Lemma 2.1 and the definitions. Indeed, if x is an element of $C = \bigcup_{\alpha \in A}^{lf} C_\alpha$, then according to Lemma 2.1, there exists $\rho_x > 0$ and $A_x \subset A$ such that $B(x, \rho_x) \cap C = B(x, \rho_x) \cap [\bigcup_{A_x} C_\alpha]$ and $x \in C_\alpha$, for any $\alpha \in A_x$.

Now, for any $y \in B(x, \rho_x) \cap C$, there exists $\alpha \in A_x$ such that $y \in C_\alpha$, therefore, the segment $[x, y]$ is included in C_α, and hence in C, since C_α is convex. □

3 Main Existence Result

In [2], the existence of a global minimizer of a quasiconvex function over a convex, possibly noncompact, constraint set has been obtained thanks to the use of the normal operator. Then, in [3], an existence result was proved for a nonconvex and noncompact constraint set which satisfies the so-called "union of separated convex cones" property, in particular for quasiconvex-quasiaffine MPEC problems. In this section, our aim is to consider the case of constraint sets which are locally finite union of closed convex sets.

Let us denote by (P_C) the following optimization problem:

$$(\mathrm{P_C}) \quad \begin{array}{ll} \inf & f(x), \\ \text{s.t.} & x \in C, \end{array}$$

where $f : X \to \mathbb{R} \cup \{+\infty\}$ is a quasiconvex lower semicontinuous function and C is a locally finite union of closed convex sets.

It is well known that, whenever f is quasiconvex, the convexity of the constraint set C implies the convexity of $\arg\min_C f$. In the following proposition, we show that this link between the structure of the constraint set and the structure of the set of minimizers still exists for locally finite union of convex sets.

Proposition 3.1

(i) *The set of (global) minimizers of a quasiconvex function on the locally finite union of convex sets is the locally finite union of convex sets.*
(ii) *The set of (global) minimizers of a lower semicontinuous quasiconvex function on the locally finite union of closed convex sets is the locally finite union of closed convex sets.*

Proof We prove only (i) since (ii) follows from (i) easily. Let C be the locally finite union of the family $\{C_\alpha : \alpha \in A\}$ of convex subsets of X. Since f is quasiconvex, each subset $\arg\min_{C_\alpha} f$ is convex. Without loss of generality, we can assume that $\arg\min_C f$ is nonempty. Therefore, the set

$$A_{\mathrm{opt}} = \left\{\alpha \in A : \inf_C f = \inf_{C_\alpha} f\right\}$$

is nonempty. Then

$$\begin{aligned} \arg\min_C f &= \arg\min_C f \cap \left[\bigcup_{\alpha \in A} C_\alpha\right] \\ &= \bigcup_{\alpha \in A_{\mathrm{opt}}} \left[\arg\min_C f \cap C_\alpha\right] \\ &= \bigcup_{\alpha \in A_{\mathrm{opt}}} \arg\min_{C_\alpha} f. \end{aligned} \tag{4}$$

For any $x \in \arg\min_C f$, there exists $\rho > 0$ and a finite set A_x such that

$$B(x,\rho) \cap C = B(x,\rho) \cap \left[\bigcup_{\alpha \in A_x} C_\alpha\right].$$

Therefore,

$$\begin{aligned} B(x,\rho)\cap \mathop{\arg\min}_C f &= \mathop{\arg\min}_C f \cap B(x,\rho)\cap C \\ &= \mathop{\arg\min}_C f \cap B(x,\rho)\cap \Bigg[\bigcup_{\alpha\in A_x} C_\alpha\Bigg] \\ &= B(x,\rho)\cap \Bigg[\bigcup_{\alpha\in A_{\text{opt}}\cap A_x} \mathop{\arg\min}_{C_\alpha} f\Bigg]. \end{aligned}$$

Combining this with (4), we conclude that $\arg\min_C f$ is the locally finite union of convex sets. □

First, we recall a classical existence result. Let $f : X \to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous quasiconvex function. Let the following global coercivity condition holds on C:

> There exist $\rho > 0$ and $y \in C\cap \overline{B}(0,\rho)$ such that $C\cap \overline{B}(0,\rho)$ is weakly compact and $f(y) < f(x), \forall x \in C \setminus \overline{B}(0,\rho)$.

Then, f admits a global minimizer of f over C. Indeed, since f is lower semicontinuous and quasiconvex, f is weakly lower semicontinuous and thus attains its infimum (at $\bar{x}$) on the weakly compact set $C\cap \overline{B}(0,\rho)$. Therefore $\bar{x}$ is a global minimizer of f over C, since

$$f(\bar{x}) = \inf_{C\cap\overline{B}(0,\rho)} f \le f(y) \le \inf_{C\setminus\overline{B}(0,\rho)} f.$$

The above global coercivity condition is very strong for problem ($\mathrm{P_C}$) since the coercivity condition is imposed on the whole constraint set C. The main result of this section is the following existence result under local coercivity conditions imposed on the subsets C_α.

Theorem 3.1 *Let $f : X \to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous quasiconvex function and let C be the locally finite union of closed convex sets $\{C_\alpha : \alpha \in A\}$. Suppose that one of the following conditions holds*:

(i) *f is radially continuous on* $\operatorname{dom} f$, $C \subseteq \operatorname{int}(\operatorname{dom} f)$, *for every $\lambda > \inf_X f$*, $\operatorname{int}(S_\lambda(f)) \neq \emptyset$. *For any $\alpha \in A$, $C_\alpha \cap \overline{B}(0,n)$ is weakly compact for any $n \in \mathbb{N}$ and the following local coercivity condition holds on C_α*:

> *There exists $\rho > 0$ such that, $\forall x \in C_\alpha \setminus \overline{B}(0,\rho)$,*
> *$\exists y_x \in C_\alpha \cap B(0,\|x\|)$ with $f(y_x) < f(x)$.*

(ii) *For any $\alpha \in A$, the following global coercivity condition holds*:

> *There exist $\rho > 0$ and $y \in C_\alpha \cap \overline{B}(0,\rho)$ such that $C_\alpha \cap \overline{B}(0,\rho)$ is weakly compact and $f(y) < f(x), \forall x \in C_\alpha \setminus \overline{B}(0,\rho)$.*

Then, if A is finite, f has a global minimizer on C; if A is not finite, but there exists a local mapping $\mathcal{M} = \{(\rho_x, A_x) : x \in C\}$ of C such that the set $\{x \in C : \operatorname{card}(A_x) > 1\}$ is included in a weakly compact subset of C, problem ($\mathrm{P_C}$) *admits a local solution.*

Remark 3.1

(a) Note that in Theorem 3.1 the condition of weak compactness of $C_\alpha \cap \overline{B}(0,n)$ is satisfied automatically if C_α is weakly compact or X is reflexive and C_α is closed while the local (resp. global) coercivity conditions is automatically fulfilled if C_α is bounded (resp. weakly compact and bounded). The main difference between the coercivity conditions (i) and (ii) is that in (ii) the same y must work for any x, while in (i), the point y_x depends on x.

(b) As observed in Proposition 2.1, in the case of a compact set C, the locally finite union reduces to a finite one. Let us observe that this is not the case if we only assume (like in Theorem 3.1), that there exists a compact (or weakly compact) subset D of C containing the subset $\{x \in C : \operatorname{card}(A_x) > 1\}$. As an example, one can simply consider, in $\mathbb{R}^2$, the set $C = \bigcup_{n \in \mathbb{N}} C_n$, where $C_0 = \overline{B}(0,1)$ and, for $n \geq 1$,

$$C_n = \{(\rho \cos(\pi/2n), \rho \sin(\pi/2n) : \rho \in [0,1] \cup [n, n+1]\}.$$

The set C is a locally finite union of closed sets, but cannot be described by the finite union of the sets C_n, although $\{x \in C : \operatorname{card}(A_x) > 1\}$ is included in $\overline{B}(0,1)$ for any chosen local mapping.

Example 3.2 Let us note that the conclusion of the above theorem may fail without the assumption on the set $\{x \in C : \operatorname{card}(A_x) > 1\}$. Indeed, consider the linear function f defined on $\mathbb{R}^2$ by $f(x,y) = -y$ and the subset $C = \bigcup_{k=1}^{\infty} \operatorname{conv}\{(0,k), (1,k-1), (-1,k-1)\}$, which is the locally finite union of closed convex sets. Then, $\{x \in C : \operatorname{card}(A_x) > 1\} = \{(0,k) : k \in \mathbb{N}\}$ is unbounded and f does not admit any local minimizer on C.

The proof of Theorem 3.1 under condition (i) relies on the following existence result from [2] with a slight modification in [3]. We sketch the proof for completeness.

Proposition 3.2 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous quasiconvex function, radially continuous on* $\operatorname{dom} f$. *Assume that*, $\operatorname{int}(S_\lambda(f)) \neq \emptyset$, *for every* $\lambda > \inf_X f$. *Let K be a nonempty convex subset with $K \subseteq \operatorname{int}(\operatorname{dom} f)$ such that $K \cap \overline{B}(0,n)$ is weakly compact for any $n \in \mathbb{N}$ and the following local coercivity condition holds*:

There exists $\rho > 0$ such that, $\forall x \in K \setminus \overline{B}(0,\rho)$,
$\exists y_x \in K \cap B(0,\|x\|)$ such that $f(y_x) < f(x)$.

Then, f admits a global minimizer on K.

Proof Let us observe that, in [2, Corollary 4.4] the assumption "$K^\perp = \{0\}$" can be replaced by "$K \subset \operatorname{int}(\operatorname{dom} f)$", if one replaces the use of [2, Proposition 4.1] by [3, Proposition 3.2] in the proof of [2, Theorem 4.3 and Corollary 4.4]. □

Proof of Theorem 3.1 The existence of a global minimizer of f over each C_α under condition (ii) follows from the classical existence theory stated before Theorem 3.1, while the one under condition (i) is a consequence of Proposition 3.2. Indeed, each C_α is closed convex and the required coercivity condition is satisfied for any α. Now, if A is finite, then let $\bar{x} \in C$ be such that $f(\bar{x}) = \min_{\alpha\in A}\{\min_{C_\alpha} f(x)\}$. Then, $\bar{x}$ is a global minimizer of f on C.

We suppose now that A is infinite but the set $\{x \in C : \mathrm{card}(A_x) > 1\}$ is included in D, which is a weakly compact subset of C. If $[C \setminus D] \cap [\bigcup_{\alpha\in A} \arg\min_{C_\alpha} f]$ is nonempty, then any point x of this intersection is a local minimizer of f over C. Indeed, according to Lemma 2.1, there exists $\rho_x > 0$ such that

$$B(x, \rho_x) \cap C = B(x, \rho_x) \cap C_{\alpha_x},$$

where α_x is the unique element of A_x; hence, by the optimality of f on C_{α_x},

$$f(x) \le f(y), \quad \forall y \in B(x, \rho_x) \cap C.$$

Now, let us suppose that $[\bigcup_{\alpha\in A} \arg\min_{C_\alpha} f] \subseteq D$. Since D is weakly compact, f attains its minimum on D, at a point $\bar{x}$. Then,

$$\inf_C f \le f(\bar{x}) = \min_D f \le \inf_{\bigcup_{\alpha\in A} \arg\min_{C_\alpha} f} f = \inf_{\bigcup_{\alpha\in A} C_\alpha} f = \inf_C f.$$

Therefore, $\bar{x}$ is a global minimizer of f on C in this case. □

4 Applications

4.1 Disjunctive Programming

In this section, we apply the existence result from Sect. 3 to the disjunctive quasiconvex programming problem having the following structure:

$$\begin{array}{lll} \text{(DP)} & \min & f(x), \\ & \text{s.t.} & h_i(x) \le 0, \quad i = 1, \dots, l, \\ & & \min\limits_{j\in \mathrm{J}} g_j(x) \le 0, \end{array}$$

where J is a possibly infinite index set and f, h_i, g_j are lower semicontinuous quasiconvex functions defined from X to $\mathbb{R} \cup \{+\infty\}$. For simplicity of notations, we include only one inequality of the form $\min_{j\in \mathrm{J}} g_j(x) \le 0$. The result can be extended easily to the case where there are finitely many inequalities of this kind.

Starting from the pioneering work of Balas [4, 5], disjunctive optimization has been studied extensively (see e.g. [6, 7, 14, 16] for theoretical aspect and [8, 12, 13] for the computational point of view). Many important optimization problems can be reformulated as disjunctive programming problems. For example, the mathematical program with equilibrium constraints (MPECs) (see e.g. [3, 17, 18, 21] for details), which has received a lot of attention in the last decade, the mathematical program with vanishing constraints (see [1]), which has interesting applications and many of the generalized semi-infinite programming (see [16]) can be reformulated as disjunctive programming problems.

From the formulation of (DP), one can observe easily that the constraint set C of this problem is the union of closed convex sets, namely,

$$C = \left[\bigcap_{i=1}^{l} S_0(h_i)\right] \cap \left[\bigcup_{j\in \mathbf{J}} S_0(g_j)\right].$$

In order to apply the existence result obtained in Sect. 3, in the following proposition, we give conditions ensuring that the union defining this constraint set is locally finite. To simplify the notations, let us define the function $g : X \to \mathbb{R}$ by $g(x) = \min_{j\in \mathbf{J}} g_j(x)$. Let us observe that, as usual, this definition implies that this minimum is attained for any $x \in X$.

Proposition 4.1 *Let $(g_j)_{j\in \mathbf{J}}$ be a family of lower semicontinuous quasiconvex functions and let $(h_i)_{i=1}^{l}$ be a finite family of lower semicontinuous quasiconvex functions defined on X. Let the following hypothesis holds*:

$$\text{(H)} \quad \forall x \in C, \textit{ there exist } \rho > 0 \textit{ and } J_x \textit{ finite } \subset J \textit{ such that}$$
$$\{j \in J : g_j(u) = g(u)\} \subset J_x, \quad \forall u \in B(x,\rho).$$

Then, the constraint set C of problem (DP) *is the locally finite union of closed convex sets, namely,*

$$C = \left[\bigcap_{i=1}^{l} S_0(h_i)\right] \cap \left[\bigcup_{j\in \mathbf{J}}^{lf} S_0(g_j)\right].$$

Proof Without loss of generality, one can assume that C is nonempty. Since there is a finite number of functions h_i and since each of them is assumed to be lower semicontinuous quasiconvex, it is sufficient to prove that the union $\tilde{C} = \bigcup_{j\in \mathbf{J}} S_0(g_j)$ is locally finite.

Let x be any element of $\tilde{C}$. From Hypothesis (H) there exist $\rho > 0$ and a finite subset J_x of J such that, for any $u \in B(x,\rho)$, we have

$$J_u = \{j \in J : g_j(u) = g(u)\} \subset J_x. \tag{5}$$

We claim that $\tilde{C} \cap B(x,\rho) = [\bigcup_{j\in J_x} S_0(g_j)] \cap B(x,\rho)$. First, it is clear that $[\bigcup_{j\in J_x} S_0(g_j)] \cap B(x,\rho)$ is included in $\tilde{C} \cap B(x,\rho)$. Now, let u be any element of $\tilde{C} \cap B(x,\rho)$. This implies that there exists $j_0 \in J$ such that $g_{j_0}(u) \le 0$; therefore, $g_j(u) = g(u) \le 0$, for any $j \in J_u$. Thus, together with (5), we have

$$u \in \left[\bigcup_{j\in J_u} S_0(g_j)\right] \subset \left[\bigcup_{j\in J_x} S_0(g_j)\right]$$

and the claim is proved, showing at the same time that C is the locally finite union of closed convex sets. □

Let us recall that a function $\varphi : X \to \mathbb{R}$ is said to be coercive if we have $\lim_{\|x\|\to\infty} \varphi(x) = +\infty$.

Theorem 4.1 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous quasiconvex function, radially continuous on* $\operatorname{dom} f$. *Assume that the constraint set C of problem* (DP) *is nonempty, included in* $\operatorname{int}(\operatorname{dom} f)$ *and that*:

- (i) *Hypothesis (H) holds and* $\operatorname{int}(S_\lambda(f)) \neq \emptyset$, *for every $\lambda > \inf_X f$.*
- (ii) *Let $(g_j)_{j\in \mathrm{J}}$ be a family of lower semicontinuous quasiconvex functions and, for any $j \in J$, let g_j be either coercive or satisfies the following coercivity condition*:

  There exists $\rho_j > 0$ such that, $\forall x \in S_0(g_j) \cap [\bigcap_{i=1}^{l} S_0(h_i)] \setminus \overline{B}(0, \rho_j)$,
  $\exists y_x \in S_0(g_j) \cap [\bigcap_{i=1}^{l} S_0(h_i)] \cap B(0, \|x\|)$ with $f(y_x) < f(x)$.
- (iii) *For any j and any $n \in \mathbb{N}$, the subset $S_0(g_j) \cap [\bigcap_{i=1}^{l} S_0(h_i)] \cap \overline{B}(0, n)$ is weakly compact.*
- (iv) *For any i, h_i is lower semicontinuous quasiconvex.*

Then, if J is finite, (DP) *admits a global solution. If J is not finite, but there exists a local mapping $\mathcal{M} = \{(\rho_x, A_x) : x \in C\}$ of C such that the set* $\{x \in C : \operatorname{card}(A_x) > 1\}$ *is included in a weakly compact subset of C, problem* (DP) *admits a local solution.*

Proof This is direct consequence of Theorem 3.1, Remark 3.1 and Proposition 4.1 since, if g_j is coercive, then $S_0(g_j)$ is bounded and hence the local coercivity condition holds. □

4.2 Bilevel Programming

Let us now turn our attention to the so-called bilevel programming problem:

$$
\begin{aligned}
\text{(BL)}\quad & \inf && f_u(x, y),\\
& \text{s.t.} && y \in S(x),\\
& && g_k(x, y) \le 0, \quad k = 1, \dots, p,
\end{aligned}
$$

where $S(x)$ is the solution set of the *lower level problem*

$$
\begin{aligned}
(\text{PL}_x)\quad & \inf_{y'} && f_l(x, y'),\\
& \text{s.t.} && (x, y') \in C,\\
& && h_j(x, y') \le 0, \quad j = 1, \dots, q,
\end{aligned}
$$

where $f_u(x, y)$, $f_l(x, y)$, $g_k(x, y)$, $h_j(x, y)$ are extended-valued functions on $X \times Y$ and C is the locally finite union of a family $\{C_\alpha : \alpha \in A\}$ of closed convex sets of $X \times Y$. The bilevel programming problem (also called a Stackelberg game) was introduced first in an economic model by Von Stackelberg [19]. The reader is referred to the recent monograph [9] and the recent papers [10, 20, 22] for applications of bilevel programming and the recent developments on the subject of bilevel programming.

Based on the previous notations, let us consider the set-valued map $S : X \to 2^Y$ which associates to any point x the (possibly empty) solution set of the lower level problem (PL_x) defined by x, that is,

$$S(x) = \operatorname*{arg\,min}_{\Omega(x)} f_l(x, \cdot), \tag{6}$$

where $\Omega(x)$ is the feasible region of the lower level problem (PL_x), namely,

$$\Omega(x) = \{(x, y) \in X \times Y : (x, y) \in C \text{ and } h_j(x, y) \leq 0,\ \ j = 1, \dots, q\}.$$

In the following theorem, we obtain a first existence result for the bilevel problem assuming that the graph of the set-valued map S, that is,

$$\mathrm{Gr}(S) = \{(x, y) \in X \times Y : y \in S(x)\},$$

is the locally finite union of closed convex sets.

Theorem 4.2 *Let $f_u : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous quasiconvex function, radially continuous on* dom f_u. *Assume that*:

(a) *For every $\lambda > \inf_{X \times Y} f_u$, $\mathrm{int}(S_\lambda(f_u)) \neq \emptyset$.*
(b) $\mathrm{Gr}(S)$ *is the locally finite union of a family $\{K_\alpha : \alpha \in A\}$ of closed convex sets of $X \times Y$.*
(c) *The functions g_k, $k = 1, \dots, p$, are lower semicontinuous quasiconvex on $X \times Y$.*
(d) *For any $\alpha \in A$, the set $\tilde{K}_\alpha := K_\alpha \cap (\bigcap_{k=1}^p S_0(g_k))$ is weakly compact and bounded.*
(e) *The feasible region of problem* (BL) *is nonempty.*

Then, if A is finite the bilevel programming problem (BL) *admits a global solution*; *if A is not finite, but there exists a local mapping $\mathcal{M} = \{(\rho_w, A_w) : w \in \mathrm{Gr}(S)\}$ of* $\mathrm{Gr}(S)$ *such that the set $\{w \in \mathrm{Gr}(S) : \mathrm{card}(A_w) > 1\}$ is included in a weakly compact subset of* $\mathrm{Gr}(S)$, *the bilevel programming problem* (BL) *admits a local solution.*

Remark 4.1 Again, Hypothesis (d) can be replaced either by the following local coercivity condition:

For any $\alpha \in A$, $\tilde{K}_\alpha \cap \overline{B}(0, n)$ is weakly compact for any $n \in \mathbb{N}$
and there exists $\rho > 0$ such that, $\forall x \in \tilde{K}_\alpha \setminus \overline{B}(0, \rho)$,
$\exists y_x \in \tilde{K}_\alpha \cap B(0, \|x\|)$ with $f_u(y_x) < f_u(x)$,

or the following global coercivity condition:

For any $\alpha \in A$, there exists $\rho > 0$ and $y \in \tilde{K}_\alpha \cap \overline{B}(0, \rho)$ such that $\tilde{K}_\alpha \cap \overline{B}(0, \rho)$ is weakly compact and $f_u(y) < f_u(x)$, $\forall x \in \tilde{K}_\alpha \setminus \overline{B}(0, \rho)$.

Proof This is a consequence of Theorem 3.1 and Remark 3.1. Indeed, due to Hypothesis (b) and (c), the constraint set

$$\Omega_u = \mathrm{Gr}(S) \cap \left(\bigcap_{k=1}^{p} S_0(g_k) \right)$$

of the upper level problem is a locally finite union (finite union if A is finite) of closed convex subsets, that is, $\Omega_u = \bigcup_{\alpha \in A} \tilde{K}_\alpha$. Moreover, by assumption (d), for any $\alpha \in A$, $\tilde{K}_\alpha$ is bounded and, for any n, $\tilde{K}_\alpha \cap \overline{B}(0, n)$ is weakly compact as a weakly closed subset of the weakly compact subset $\tilde{K}_\alpha$. Hence, if A is finite, then (BL) admits a global solution by virtue of Theorem 3.1.

Now, we consider the case where A is infinite. Let us denote,

$$J_{(x,y)} := \{\alpha \in A_{(x,y)} : (x, y) \in \tilde{K}_\alpha\},$$
$$D := \{(x, y) \in \mathrm{Gr}(S) : \mathrm{card}(A_{(x,y)}) > 1\},$$
$$D_u := \{(x, y) \in \Omega_u : \mathrm{card}(J_{(x,y)}) > 1\}.$$

Then, since $\Omega_u \subseteq \mathrm{Gr}(S)$ and since, $J_{(x,y)} \subseteq A_{(x,y)}$, for any $(x, y) \in \mathrm{Gr}(S)$, we have $D_u \subseteq D \cap \Omega_u$. Moreover, since $A_{(x,y)}$ is finite and each $\tilde{K}_\alpha$ is closed, there exists $\rho'_{(x,y)} \in]0, \rho_{(x,y)}[$ such that

$$B((x, y), \rho'_{(x,y)}) \cap \Omega_u = B((x, y), \rho'_{(x,y)}) \cap \left[\bigcup_{\alpha \in J_{(x,y)}} \tilde{K}_\alpha \right].$$

On the other hand, by assumption, D is included in a weakly compact subset K of $\mathrm{Gr}(S)$. The collection $\{B((x, y), \rho'_{(x,y)}) : (x, y) \in K\}$ is an open cover of K from which we can extract a finite subcovering $\{B((x_i, y_i), \rho'_{(x_i,y_i)}) : i = 1, \dots, n\}$. Thus, we have

$$\begin{aligned} D_u &\subseteq (D \cap \Omega_u) \subseteq (K \cap \Omega_u) \\ &= \bigcup_{i=1}^{n} [K \cap \overline{B}((x_i, y_i), \rho'_{(x_i,y_i)}) \cap \Omega_u] \\ &= \bigcup_{i=1}^{n} \left[K \cap \overline{B}((x_i, y_i), \rho'_{(x_i,y_i)}) \cap \left(\bigcup_{\alpha \in J(x_i,y_i)} \tilde{K}_\alpha \right) \right]. \end{aligned}$$

The subset $K \cap \Omega_u$ is a closed set as the finite union of closed sets. Thus D_u is included in a weakly compact subset of Ω_u. By Theorem 3.1, the bilevel programming problem (BL) admits a local solution in this case. □

Let us consider the following particular case of the bilevel problem:

$$\begin{aligned} (\mathrm{BL\_Lin}) \quad \inf \quad & f_u(x, y), \\ \text{s.t.} \quad & y \in S(x), \\ & g_k(x, y) \le 0, \quad k = 1, \dots, p, \end{aligned}$$

where $S(x)$ is the solution set of the linear lower level problem

$$\begin{aligned} (\mathrm{PL}_x\mathrm{\_Lin}) \quad \inf_{y'} \quad & L_1(x, y'), \\ \text{s.t.} \quad & L_2(x, y') \le 0, \end{aligned}$$

where L_1 and L_2 are affine functions defined on $\mathbb{R}^n \times \mathbb{R}^m$ with values, respectively, in $\mathbb{R}$ and $\mathbb{R}^q$.

It is interesting to notice that, even in this very particular linear case, the graph of the set-valued map S is not convex in general. But, as observed in [9, Theorem 3.1], the set-valued map S is always polyhedral, i.e. the graph of S is the finite union of closed convex sets.

Therefore, as an immediate consequence of Theorem 4.2 and Remark 4.1, we obtain the following existence result for problem (BL_Lin) which slightly extends (f_u quasiconvex, possibly unbounded constraint set, g_k quasiconvex) the classical results for linear bilevel problems (see e.g. [9]).

Set $M = \{(x, y) : L_2(x, y) \le 0, g_k(x, y) \le 0, k = 1, \dots, p\}$.

Corollary 4.1 *Let $f_u : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous quasiconvex function, radially continuous on* dom f_u. *Assume that*:

(a) *For every $\lambda > \inf_{\mathbb{R}^{n+m}} f_u$, $\operatorname{int}(S_\lambda(f_u)) \neq \emptyset$.*
(b) *The functions g_k are lower semicontinuous quasiconvex, L_1 and L_2 are affine functions.*
(c) *Either the set M is bounded or f_u satisfies the following global coercivity condition on M: there exists $\rho > 0$ and $y \in M \cap \overline{B}(0, \rho)$ such that $f_u(y) < f_u(x)$, $\forall x \in M \setminus \overline{B}(0, \rho)$.*
(d) *The feasible region of problem* (BL_Lin) *is nonempty.*

Then, the bilevel programming problem (BL_Lin) *admits a global solution.*

Remark 4.2 Thanks to Remark 4.1, the global coercivity condition in (c) can be replaced by the following local condition:

For each $i = 1, \dots, l$, there exists $\rho > 0$ such that, $\forall x \in K_i \setminus \overline{B}(0, \rho)$,
$\exists y_x \in K_i \cap B(0, \|x\|)$ with $f_u(y_x) < f_u(x)$,

where K_i, $i = 1, \dots, l$, are closed convex sets which form the polyhedral set $\mathrm{Gr}(S)$, i.e. $\mathrm{Gr}(S) = \bigcup_{i=1}^{l} K_i$.

Let us now turn our attention back to the general bilevel programming problem (BL). For any $\alpha \in A$, the marginal function $l_\alpha : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ of the lower level subproblem on C_α is defined by

$$\begin{aligned} l_\alpha(x) = \inf_{y'} & f_l(x, y'), \\ \text{s.t.} \quad & (x, y') \in C_\alpha, \\ & h_j(x, y') \le 0, \quad j = 1, \dots, q. \end{aligned}$$

In the following theorem, using the marginal functions l_α, we provide another case of bilevel problem for which the graph of the solution map S is the locally finite union of closed convex sets.

Theorem 4.3 *Let $f_u : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous quasiconvex function, radially continuous on* dom f_u *and let $f_l : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous quasiconvex function. Assume that*:

(a) *For every* $\lambda > \inf_{X\times Y} f_u$, $\mathrm{int}(S_\lambda(f_u)) \neq \emptyset$.
(b) C *is the locally finite union of a family* $\{C_\alpha : \alpha \in A\}$ *of closed convex sets of* $X \times Y$.
(c) *The functions* g_k, $k = 1, \dots, p$, *and* h_j, $j = 1, \dots, q$, *are lower semicontinuous quasiconvex on* $X \times Y$.
(d) *For any* $\alpha \in A$, C_α *is weakly compact and bounded.*
(e) *For any* $\alpha \in A$, $l_\alpha(x) = l_\alpha(x')$ *for any* x, x' *such that* $(\{x\} \times Y) \cap C_\alpha \neq \emptyset$ *and* $(\{x'\} \times Y) \cap C_\alpha \neq \emptyset$.
(f) *The feasible region of problem* (BL) *is nonempty.*

Then, if A is finite, the bilevel programming problem (BL) *admits a global solution; if A is infinite but there exists a local mapping* $\mathcal{M} = \{(\rho_{(x,y)}, A_{(x,y)}) : (x, y) \in C\}$ *of* C *such that the set* $\{(x, y) \in C : \mathrm{card}(A_{(x,y)}) > 1\}$ *is included in a weakly compact subset of* C, *then the bilevel programming problem* (BL) *admits a local solution.*

Remark 4.3 Roughly speaking, Hypothesis (e) says that the minimal value of the lower level problem reduced to C_α not depending on x. But of course the corresponding optimal solution set still depends on x, in general.

Proof In order to apply Theorem 4.2, let us first show that $\mathrm{Gr}(S)$, the graph of the solution map for the lower level problem, is the locally finite union of closed convex sets. Indeed, recall that f_l is lower semicontinuous and that, by weak compactness of the convex subsets C_α, f_l is bounded below on C_α for any $\alpha \in A$. Therefore, according to Hypothesis (e), for any α, there exists $\beta_\alpha \in \mathbb{R}$ such that, for any x for which $C_\alpha \cap \Omega(x) \neq \emptyset$, we have

$$l_\alpha(x) = \inf_{C_\alpha \cap \Omega(x)} f_l = \beta_\alpha. \tag{7}$$

If we set, for any α, $\tilde{K}_\alpha = C_\alpha \cap \mathrm{Gr}(S)$, then by (7) we have

$$\begin{aligned}
(x, y) \in \tilde{K}_\alpha \quad &\Longleftrightarrow \quad (x, y) \in C_\alpha \quad \text{and} \quad y \in S(x) \\
&\Longleftrightarrow \quad (x, y) \in C_\alpha \cap \left(\bigcap_{j=1}^{q} S_0(h_j) \right) \quad \text{and} \quad f_l(x, y) = \beta_\alpha \\
&\Longleftrightarrow \quad (x, y) \in C_\alpha \cap \left(\bigcap_{j=1}^{q} S_0(h_j) \right) \cap S_{\beta_\alpha}(f_l).
\end{aligned}$$

Consequently, since f_l is lower semicontinuous quasiconvex, each function h_j is lower semicontinuous quasiconvex and, for any $\alpha \in A$, C_α is closed convex, the subsets $\tilde{K}_\alpha$ are closed convex. Since C is the locally finite union of the family $\{C_\alpha : \alpha \in A\}$, there exists $\rho > 0$ and a finite subset $\tilde{A}_x$ of A such that

$$B(x, \rho) \cap C = B(x, \rho) \cap \left(\bigcup_{\alpha \in \tilde{A}_x} C_\alpha \right).$$

It follows that Gr(S) is the locally finite union of the family $\{\tilde{K}_\alpha : \alpha \in A\}$, since

$$\begin{aligned} B(x,\rho) \cap Gr(S) &= B(x,\rho) \cap C \cap Gr(S) \\ &= B(x,\rho) \cap \left(\bigcup_{\alpha \in \tilde{A}_x} C_\alpha \right) \cap Gr(S) \\ &= B(x,\rho) \cap \bigcup_{\alpha \in \tilde{A}_x} [C_\alpha \cap Gr(S)] \\ &= B(x,\rho) \cap \bigcup_{\alpha \in \tilde{A}_x} \tilde{K}_\alpha . \end{aligned}$$

Moreover, for any $\alpha \in A$, the subsets $\tilde{K}_\alpha$ are weakly compact and bounded as weakly closed subsets of the weakly compact and bounded set C_α. Therefore, by Theorem 4.2, if A is finite, (BL) has a global solution.

Now, assume that A is infinite. Using the same arguments as in the last part of the proof of Theorem 4.2, but with Ω_u and Gr(S) replaced by Gr(S) and C respectively, we can prove that the set $\{(x,y) \in \mathrm{Gr}(S) : \mathrm{card}(J_{(x,y)}) > 1\}$ is included in a weakly compact subset of Gr(S), where $J_{(x,y)} = \{\alpha \in A_{(x,y)} : (x,y) \in \tilde{K}_\alpha\}$. □

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