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On a Class of Generalized Vector Quasiequilibrium Problems with Set-Valued Maps

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Abstract This paper considers the following generalized vector quasiequilibrium problem: find a point (z_0, x_0) of a set $E \times K$ such that $x_0 \in A(z_0, x_0)$ and

$$\forall \eta \in A(z_0, x_0), \quad \exists v \in B(z_0, x_0, \eta), \quad (F(v, x_0, \eta), C(v, x_0, \eta)) \in \alpha,$$

where α is a subset of $2^Y \times 2^Y$, $A : E \times K \to 2^K$, $B : E \times K \times K \to 2^E$, $C : E \times K \times K \to 2^Y$, $F : E \times K \times K \to 2^Y$ are set-valued maps and Y is a topological vector space. Existence theorems are established under suitable assumptions, one of which is the requirement of the openness of the lower sections of some set-valued maps which can be satisfied with maps B, C, F being discontinuous. It is shown that, in some special cases, this requirement can be verified easily by using the semicontinuity property of these maps. Another assumption in the obtained existence theorems is assured by appropriate notions of diagonal quasiconvexity.

Keywords Generalized vector quasiequilibrium problems · Set-valued maps · Open lower sections · Diagonal quasiconvexity · Semicontinuity

1 Introduction

Let X, Y, Z be topological vector spaces. Let K (resp. E) be a nonempty subset of X (resp. Z). Let $W = E \times K \times K$ and let $A : E \times K \to 2^K$, $B : W \to 2^E$, $C : W \to 2^Y$, $F : W \to 2^Y$ be set-valued maps with nonempty values. Let α be an arbitrary (binary) relation on 2^Y, i.e., a subset of $2^Y \times 2^Y$. We will be interested in the relations $\alpha = \alpha_i$,

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$i = 1, 2, 3, 4,$ where

$$\alpha_1 = \{(a, b) \in 2^Y \times 2^Y : a \not\subset b\},$$

$$\alpha_2 = \{(a, b) \in 2^Y \times 2^Y : a \subset b\},$$

$$\alpha_3 = \{(a, b) \in 2^Y \times 2^Y : a \cap b \neq \emptyset\},$$

$$\alpha_4 = \{(a, b) \in 2^Y \times 2^Y : a \cap b = \emptyset\},$$

$\emptyset$ being the empty set. For simplicity of notation, let us write $\alpha_{FC}(z, x, \eta)$ instead of $(F(z, x, \eta), C(z, x, \eta)) \in \alpha$, where $(z, x, \eta) \in W$. We will write $F(z, x, \eta) \equiv \mathbb{F}(x, \eta)$ (resp. $C(z, x, \eta) \equiv \mathbb{C}(x)$) if the map $F : E \times K \times K \to 2^Y$ (resp. $C : E \times K \times K \to 2^Y$) does not depend on the variable $z \in E$ (resp. $(z, \eta) \in E \times K$) and if its value at $(z, x, \eta) \in E \times K \times K$ equals $\mathbb{F}(x, \eta)$ (resp. $\mathbb{C}(x)$), where $\mathbb{F}(x, \eta)$ (resp. $\mathbb{C}(x)$) is some subset of Y. Similarly for $A(z, x) \equiv \mathbb{A}(x)$ and $B(z, x, \eta) \equiv \mathbb{B}(x)$. In this paper,we consider the following general problem.

Problem (P_α): Find $(z_0, x_0) \in E \times K$ such that $x_0 \in A(z_0, x_0)$ and

$$\forall \eta \in A(z_0, x_0), \quad \exists v \in B(z_0, x_0, \eta), \quad \alpha_{FC}(v, x_0, \eta).$$

Let us mention some special cases of Problem (P_α).

(i) For $\alpha = \alpha_1$, $A(z, x) \equiv K$, $F(z, x, \eta) \equiv \mathbb{F}(x, \eta)$ and $C(z, x, \eta) \equiv \mathbb{C}(x)$, Problem ($\mathrm{P}_\alpha$) is to find a point $x_0 \in K$ such that $\mathbb{F}(x_0, \eta) \not\subset \mathbb{C}(x_0)$ for all $\eta \in K$. This problem was extensively developed in recent years; see [1–7].

(ii) For $\alpha = \alpha_1$, $A(z, x) \equiv \mathbb{A}(x)$, $F(z, x, \eta) \equiv \mathbb{F}(x, \eta)$ and $C(z, x, \eta)$ a constant open cone, Problem (P_α) was considered in [8].

(iii) For $\alpha = \alpha_2$, $A(z, x) \equiv K$, $F(z, x, \eta) \equiv \mathbb{F}(x, \eta)$ and $C(z, x, \eta) \equiv \mathbb{C}(x)$, Problem ($\mathrm{P}_\alpha$) is to find $x_0 \in K$ such that $\mathbb{F}(x_0, \eta) \subset \mathbb{C}(x_0)$ for all $\eta \in K$. This problem was investigated in [2, 4].

(iv) For $\alpha = \alpha_1$, $A(z, x) \equiv K$, $B(z, x, \eta) \equiv \mathbb{B}(x)$ and $C(z, x, \eta) \equiv \operatorname{int} \mathbb{C}(x)$, Problem ($\mathrm{P}_\alpha$) is to find $x_0 \in K$ such that

$$\forall \eta \in K, \quad \exists v \in \mathbb{B}(x_0), \quad F(v, x_0, \eta) \not\subset \operatorname{int} \mathbb{C}(x_0),$$

where int denotes the interior. Such problem was studied in [3, 9–17].

In this paper, we give existence theorems for the general Problem (P_α) with α being an arbitrary relation on 2^Y. They are established by a unified approach based on the fixed-point theorem of [18]. We also use appropriate notions of diagonal quasi-convexity which generalize those given in [19].

We conclude this section by a brief comparison of the problem statement of this paper (Problem (P_α)) and that of [20]. A more detailed discussion of the relationship between the results of the present paper and the corresponding ones of [20] can be found in Remark 3.3 of Sect. 3. Problem (P'_α) considered in [20] is to find a point $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in \widetilde{B}(z_0, x_0) \times A(z_0, x_0)$ and $\forall \eta \in A(z_0, x_0)$, $\alpha_{FC}(z_0, x_0, \eta)$, where $\widetilde{B} : E \times K \to 2^E$ is a set-valued map. Thus, under the assumption that $B = \widetilde{B}$ (i.e., B does not depend on η), each solution of

(P'_α) is also a solution of (P_α). However, there are problems where strong solutions (i.e., solutions of Problem (P'_α)) do not exist, while weak solutions (i.e., solutions of Problem (P_α)) exist. As an example illustrating this remark, we take $X = Y = Z = \mathbb{R}$ (the real line), $\alpha = \alpha_4$, $E = [0, 2]$, $K = [0, 2]$ and, for each $(z, x, \eta) \in E \times K \times K$, we set $F(z, x, \eta) = \{(x + \eta - z)^2\}$, $\widetilde{B}(z, x) \equiv E$, $C(z, x, \eta) \equiv \text{int } R_+$ (the positive half-line) and

$$A(z, x) \equiv \begin{cases} [0, 1], & \text{if } x \in [0, 1], \\ [0, x), & \text{if } x \in (1, 2]. \end{cases}$$

Clearly, in this example, the solutions of Problem (P'_α) do not exist, while each point $(z_0, x_0) \in [0, 2] \times [0, 1]$ is a solution of Problem (P_α).

2 Preliminaries

Let X be a topological space. Each subset of X is also a topological space with a topology induced by the given topology of X. In this paper, neighborhoods of each point x of X, denoted by $U(x), U_1(x), \ldots,$ are assumed to be open. We will use the semicontinuity and continuity properties of set-valued maps in the usual sense of Definitions 1, 3 and 4 in [21, pp. 66–69]. If the graph of set-valued map $f : X \to 2^Y$ between topological spaces X and Y, denoted by gr f, is a closed (resp. open) set of $X \times Y$, then we say that f has closed (resp. open) graph. Recall that gr f is the set of all points $(x, y) \in X \times Y$ such that $y \in f(x)$. We say that f has open lower sections if, for all $y \in Y$, $f^{-1}(y) := \{x \in X : y \in f(x)\}$ is open in X.

The following result is a special case of Theorem 5 of [18].

Theorem 2.1 *Let $\mathcal{V}$ be a nonempty convex subset of a topological vector space $\mathcal{X}$ and let $\mathcal{V}_1$ be a nonempty compact subset of $\mathcal{V}$. Let $\Phi : \mathcal{V} \to 2^{\mathcal{V}}$ be a set-valued map satisfying the following conditions*:

(i) $\forall v \in \mathcal{V}$, $\Phi(v)$ *is nonempty and convex.*
(ii) $\forall v \in \mathcal{V}$, $\Phi^{-1}(v)$ *is open* (*in* $\mathcal{V}$).
(iii) *For each finite subset $\mathcal{N}$ of $\mathcal{V}$, there exists a nonempty compact convex set $\mathcal{V}_2$ of $\mathcal{V}$ such that $\mathcal{V}_2 \supset \mathcal{N}$ and*, $\forall v \in \mathcal{V}_2 \setminus \mathcal{V}_1$, $\Phi(v) \cap \mathcal{V}_2 \neq \emptyset$.

Then Φ has a fixed point.

Let $b : W \to 2^E$, $c : W \to 2^Y$ and $f : W \to 2^Y$ be set-valued maps with nonempty values, where $W = E \times K \times K$. Let β be a relation on 2^Y. Let us write $\beta_{fc}(z, x, \eta)$ instead of $(f(z, x, \eta), c(z, x, \eta)) \in \beta$. Denote by $\overline{\beta}$ the relation on 2^Y defined by $\overline{\beta} = [2^Y \times 2^Y] \setminus \beta$. Then the symbol $\overline{\beta}_{fc}(z, x, \eta)$ means that $(f(z, x, \eta), c(z, x, \eta)) \notin \beta$.

Let K be a convex set. We say that the pair (f, c) is β-diagonally quasiconvex (resp. strongly β-diagonally quasiconvex) in η with respect to b if, for each $z \in E$, each finite set $\{x_j, j = 1, 2, \ldots, n\} \subset K$ and each point $x \in \text{co}\,\{x_j, j = 1, 2, \ldots, n\}$, there exists an index $j \in \{1, 2, \ldots, n\}$ such that $\beta_{fc}(u, x, x_j)$ for some $u \in b(z, x, x_j)$

(resp. for each $u \in b(z, x, x_j)$). Obviously, strong β-diagonal quasiconvexity $\Rightarrow$ β-diagonal quasiconvexity. All notions of diagonal quasiconvexity of [19] are special cases of our notion of β-diagonal quasiconvexity.

We delete the easy proof of the following result.

Proposition 2.1 *Let $K \subset X$ be a convex set. For $(z, x) \in E \times K$, let*

$$l_{\overline{\beta}}(z, x) = \{\eta \in K : \forall u \in b(z, x, \eta), \overline{\beta}_{fc}(u, x, \eta)\},$$

$$\widehat{l_{\overline{\beta}}}(z, x) = \{\eta \in K : \exists u \in b(z, x, \eta), \overline{\beta}_{fc}(u, x, \eta)\}.$$

Then, the pair (f, c) is β-diagonally quasiconvex (resp. strongly β-diagonally quasiconvex) in η with respect to b if and only if, for all $(z, x) \in E \times K$, $x \notin \operatorname{co} l_{\overline{\beta}}(z, x)$ (resp. $x \notin \operatorname{co} \widehat{l_{\overline{\beta}}}(z, x)$).

Before giving an example of strong β-diagonal quasiconvexity, let us recall some known definitions. Let c' be a convex cone of a vector space Y' and $f' : W' \to 2^{Y'}$ be a set-valued map defined on a convex set W' of a vector space X'. We say that f' is convex on W' if its graph is a convex set. We say that f' is naturally c'-quasiconvex on W' if

$$f'(\lambda x_1 + (1 - \lambda)x_2) \subset \operatorname{co}\left\{\bigcup_{i=1}^{2} f'(x_i)\right\} - c',$$

for all $\lambda \in (0, 1)$ and $x_i \in W'$, $i = 1, 2$.

For a set-valued map $f : E \times K \times K \to 2^Y$ and points $z \in E$ and $x \in K$, we define $f_{z,x} : K \to 2^Y$ by setting $f_{z,x}(\eta) = f(z, x, \eta)$ for each $\eta \in K$. Similarly, $f_x : E \times K \to 2^Y$ is defined by setting $f_x(z, \eta) = f(z, x, \eta)$ for each $(z, \eta) \in E \times K$.

Example 2.1 Assume that $K \subset X$, $E \subset Z$ are convex sets and that $c' \subset Y$ is a convex cone with nonempty interior. Assume that:

(i) The set-valued map $b : E \times K \times K \to 2^E$ is such that, for each $(z, x) \in E \times K$, $b_{z,x} : K \to 2^E$ is convex on K.
(ii) The set-valued map $f : E \times K \times K \to 2^Y$ is such that, for each $x \in K$, $f_x : E \times K \to 2^Y$ is naturally c'-quasiconvex on $E \times K$.
(iii) The set-valued map $c : E \times K \times K \to 2^Y$ is given by $c(z, x, \eta) = -\operatorname{int} c'$ for each $(z, x, \eta) \in E \times K \times K$.
(iv) For each $(z, x) \in E \times K$ and $u \in b(z, x, x)$, we have

$$f(u, x, x) \not\subset -\operatorname{int} c'.$$

Then, the pair (f, c) is strongly α_1-diagonally quasiconvex in η with respect to b. Indeed, since $\beta = \alpha_1$, we have

$$\widehat{l_{\overline{\beta}}}(z, x) = \{\eta \in K : \exists u \in b(z, x, \eta), f(u, x, \eta) \subset -\operatorname{int} c'\},$$

for each $(z, x) \in E \times K$. Condition (iv) in Example 2.1 yields $x \notin \widehat{l_{\overline{\beta}}}(z, x)$. In addition,it is a simple matter to verify from conditions (i) and (ii) of Example 2.1 that $\widehat{l_{\overline{\beta}}}(z, x)$ is a convex set. So, Proposition 2.1 yields the desired conclusion.

Let us observe that all the assumptions of Example 2.1 hold if we take $X = Y = Z = \mathbb{R}$, $E = K = [0, 1]$, $c' = \mathbb{R}_+$ (the nonnegative half-line), $f(z, x, \eta) = \{z + \eta - x\}$ and $b(z, x, \eta) \equiv [0, 1]$ for each $(z, x, \eta) \in E \times K \times K$.

3 Main Results

In this paper, unless otherwise specified, we assume that X, Y, Z are topological vector spaces, $E \subset Z$ and $K \subset X$ are nonempty convex sets, and $A : E \times K \to 2^K$ is a set-valued map with nonempty convex values and open lower sections. We will assume that the set

$$M := \{(z, x) \in E \times K : x \in A(z, x)\}$$

is closed in $E \times K$. This assumption is automatically satisfied if $A(z, x) \equiv K$.

Let $L : E \times K \to 2^K$ and $A : E \times K \to 2^K$ be set-valued maps. We say that the pair (L, A) satisfies the coercivity condition if there exists a nonempty compact subset $\mathcal{V}_1 \subset E \times K$ with the following property: for each finite subset $\mathcal{N} \subset E \times K$, we can find a nonempty compact convex set $\mathcal{V}_2$ of $E \times K$ such that $\mathcal{V}_2 \supset \mathcal{N}$ and, for each $(z, x) \in \mathcal{V}_2 \setminus \mathcal{V}_1$, there exists $(z', x') \in \mathcal{V}_2$ such that $x' \in A(z, x) \cap \operatorname{co} L(z, x)$ if $(z, x) \in M$, and $x' \in A(z, x)$ if $(z, x) \notin M$.

Let $W := E \times K \times K$, and let $B : W \to 2^E$, $C : W \to 2^Y$, $F : W \to 2^Y$ be set-valued maps with nonempty values. Let α be a relation on 2^Y. We consider the set-valued map $L_{\overline{\alpha}} : E \times K \to 2^K$ defined by

$$L_{\overline{\alpha}}(z, x) = \{\eta \in K : \forall v \in B(z, x, \eta), \overline{\alpha}_{FC}(v, x, \eta)\},$$

where $(z, x) \in E \times K$. The following lemma gives sufficient conditions for the existence of a solution of (P_α).

Lemma 3.1 *Let M be closed in $E \times K$. Let $L : E \times K \to 2^K$ be a set-valued map such that*:

(i) $L_{\overline{\alpha}} \subset L$ (*i.e.*, $L_{\overline{\alpha}}(z, x) \subset L(z, x)$ *for all* $(z, x) \in E \times K$).
(ii) *L has open lower sections.*
(iii) $x \notin \operatorname{co} L(z, x)$, $\forall (z, x) \in M$.
(iv) *The pair (L, A) satisfies the coercivity condition.*

Then there exists a solution of Problem (P_α).

Proof It is enough to show that there exists a point $(z_0, x_0) \in M$ such that $A(z_0, x_0) \cap L_{\overline{\alpha}}(z_0, x_0) = \emptyset$. Indeed, assume to the contrary that $A(z, x) \cap L_{\overline{\alpha}}(z, x) \neq \emptyset$ for all $(z, x) \in M$. Since $L_{\overline{\alpha}}(z, x) \subset L(z, x) \subset \widetilde{L}(z, x) := \operatorname{co} L(z, x)$, this yields

$$A(z, x) \cap \widetilde{L}(z, x) \neq \emptyset,$$

for all $(z, x) \in M$. It is clear that the set

$$H(z,x) = \begin{cases} A(z,x) \cap \widetilde{L}(z,x), & \text{if } (z,x) \in M, \\ A(z,x), & \text{if } (z,x) \in [E \times K] \setminus M, \end{cases}$$

is nonempty and convex for each $(z, x) \in E \times K$.

Observe that, for each $x' \in K$, we have from [22]

$$H^{-1}(x') = [A^{-1}(x') \cap \widetilde{L}^{-1}(x')] \cup [([E \times K] \setminus M) \cap A^{-1}(x')].$$

Since L has open lower sections, it follows from [23] that $\widetilde{L}$ has open lower sections. Since M is closed in $E \times K$, and since both the maps $\widetilde{L}$ and A have open lower sections, we see that, for each $x' \in K$, $H^{-1}(x')$ is open in K. Now, let us construct the set-valued map $\phi : E \times K \to 2^{E \times K}$ by setting $\phi(z, x) = E'(z, x) \times H(z, x)$, where $E' : E \times K \to 2^E$ is the constant set-valued map defined by $E'(z, x) \equiv E$. For $(z', x') \in E \times K$, it is clear that

$$\phi^{-1}(z', x') = \{(z, x) \in E \times K : z' \in E'(z, x), x' \in H(z, x)\} = H^{-1}(x').$$

As we have shown above that $H^{-1}(x')$ is open in $E \times K$, we conclude that $\phi^{-1}(z', x')$ is open in $E \times K$. Now, let us set $\mathcal{X} = Z \times X$, $\mathcal{V} = E \times K$ and $\Phi(v) = \phi(z, x)$ for each $v = (z, x) \in \mathcal{V}$. From the coercivity condition it follows that condition (iii) of Theorem 2.1 holds. By this theorem, the set-valued map Φ has a fixed point, denoted by $v_0 = (z_0, x_0)$. Thus, $v_0 = (z_0, x_0) \in \mathcal{V} = E \times K$ and $(z_0, x_0) \in \phi(z_0, x_0)$. This proves that $z_0 \in E'(z_0, x_0) = E$ and $x_0 \in H(z_0, x_0)$. Since $H(z_0, x_0) \subset A(z_0, x_0)$, it follows that $x_0 \in A(z_0, x_0)$. Therefore, $(z_0, x_0) \in M$. By the very definition of H, this yields

$$H(z_0, x_0) = A(z_0, x_0) \cap \operatorname{co} L(z_0, x_0) \subset \operatorname{co} L(z_0, x_0).$$

Hence, $x_0 \in \operatorname{co} L(z_0, x_0)$, a contradiction to condition (iii) of Lemma 3.1. □

Let us set

$$\begin{aligned} W_1 &= \{w = (z, x, \eta) \in W : (z, x) \in M, \eta \in A(z, x)\} \\ &= \{w = (z, x, \eta) \in W : x \in A(z, x), \eta \in A(z, x)\}. \end{aligned}$$

We say that condition (ps) (resp. condition (wps)) holds if there exist a relation β on 2^Y and set-valued maps $b : W \to 2^E$, $f : W \to 2^Y$, $c : W \to 2^Y$ with nonempty values such that, for all $(z, x, \eta) \in W_1$,

$$[\exists u \in b(z, x, \eta), \beta_{fc}(u, x, \eta)] \Rightarrow [\forall v \in B(z, x, \eta), \alpha_{FC}(v, x, \eta)]$$

$$(\text{resp. } [\exists u \in b(z, x, \eta), \beta_{fc}(u, x, \eta)] \Rightarrow [\exists v \in B(z, x, \eta), \alpha_{FC}(v, x, \eta)]).$$

Obviously, condition (ps) $\Rightarrow$ condition (wps), and the converse implication is no longer true. Condition (ps) (resp. condition (wps)) generalizes the pseudomonotonicity (resp. weak pseudomonotonicity) property of [17].

From now on, we assume that β, b, f, c are the objects appearing in the definition of condition (ps) or condition (wps).

To formulate Lemma 3.2, whose proof is immediate, we consider the set-valued maps $\widehat{L}_{\overline{\alpha}} : E \times K \to 2^K$, $l_{\overline{\beta}} : E \times K \to 2^K$ defined by

$$\widehat{L}_{\overline{\alpha}}(z, x) = \{\eta \in K : \exists v \in B(z, x, \eta), \overline{\alpha}_{FC}(v, x, \eta)\},$$

$$l_{\overline{\beta}}(z, x) = \{\eta \in K : \forall u \in b(z, x, \eta), \overline{\beta}_{fc}(u, x, \eta)\}.$$

We also introduce the following conditions:

(a) $x \notin \operatorname{co} L_{\overline{\alpha}}(z, x), \ \forall (z, x) \in M.$
(b) $x \notin \operatorname{co} \widehat{L}_{\overline{\alpha}}(z, x), \ \forall (z, x) \in M.$
(c) Condition (ps) holds and $x \notin \operatorname{co} l_{\overline{\beta}}(z, x), \ \forall (z, x) \in M.$
(d) Condition (wps) holds and $x \notin \operatorname{co} l_{\overline{\beta}}(z, x), \ \forall (z, x) \in M.$

Lemma 3.2 *We have* (c) $\Rightarrow$ (b) $\Rightarrow$ (a) *and* (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

Theorem 3.1 *Let M be closed in $E \times K$. Under the following assumptions, there exists a solution of Problem (P_α):*

(i) *$L_{\overline{\alpha}}$ has open lower sections.*
(ii) *The pair $(L_{\overline{\alpha}}, A)$ satisfies the coercivity condition.*
(iii) *At least one of the conditions* (a), (b), (c), (d) *holds.*

Proof By Lemma 3.2, it suffices to consider Problem (P_α) under conditions (i), (ii), (a). A solution of this problem exists by Lemma 3.1 with $L = L_{\overline{\alpha}}$. □

Corollary 3.1 *Let $E \subset Z$, $K \subset X$ be nonempty compact convex sets. Let $A : E \times K \to 2^K$ be a set-valued map with nonempty convex values and open lower sections. Let the set M be closed in $E \times K$. Assume that:*

(i) *The following set-valued map has open lower sections:*

$$(z, x) \in E \times K \mapsto L'_{\overline{\alpha}}(z, x) := \{\eta \in K : \overline{\alpha}_{FC}(z, x, \eta)\}.$$

(ii) $\forall (z, x) \in M, \ x \notin \operatorname{co} L'_{\overline{\alpha}}(z, x).$

Then, there exists a point $(z_0, x_0) \in E \times K$ such that $x_0 \in A(z_0, x_0)$ and

$$\alpha_{FC}(z_0, x_0, \eta), \quad \forall \eta \in A(z_0, x_0).$$

Proof This is a special case of Theorem 3.1 with $B(z, x, \eta) \equiv \{z\}$. □

Corollary 3.2 *Let $K \subset X$ be a nonempty compact convex set. Let $\mathbb{A} : K \to 2^K$ be a set-valued map with nonempty convex values and open lower sections. Let the set*

$$M' = \{x \in K : x \in \mathbb{A}(x)\}$$

be closed in K. Let $\mathbb{F} : K \times K \to 2^Y$, $\mathbb{C} : K \times K \to 2^Y$ be set-valued maps with nonempty values. Assume that:

(i) *For each* $\eta \in K$, *the following set is closed in* K:

$$\{x \in K : (\mathbb{F}(x,\eta), \mathbb{C}(x,\eta)) \in \alpha\}.$$

(ii) *For each* $x \in K$, *the following set is convex*:

$$d(x) = \{\eta \in K : (\mathbb{F}(x,\eta), \mathbb{C}(x,\eta)) \in \overline{\alpha}\}.$$

(iii) *For each* $x \in K$, $(\mathbb{F}(x,x), \mathbb{C}(x,x)) \in \alpha$.

Then, there exists a point $x_0 \in K$ *such that* $x_0 \in \mathbb{A}(x_0)$ *and*

$$\forall \eta \in \mathbb{A}(x_0), \quad (\mathbb{F}(x_0,\eta), \mathbb{C}(x_0,\eta)) \in \alpha.$$

Proof This is a special case of Corollary 3.1, where $E = K$, $A(x,\eta) \equiv \mathbb{A}(x)$, $F(z,x,\eta) \equiv \mathbb{F}(x,\eta)$, $C(z,x,\eta) \equiv \mathbb{C}(x,\eta)$ and $L'_{\overline{\alpha}}(z,x) \equiv d(x)$. □

Remark 3.1 When $\alpha = \alpha_1$, the conclusion of Corollary 3.2 is established in Corollary 3.1 of [8] for a class of maps A which is more general than that used in Corollary 3.2.

Corollary 3.3 *Let* $K \subset X$ *be a nonempty compact convex set. Let* $\mathbb{A} : K \to 2^K$ *be a set-valued map with nonempty convex values and open lower sections. Let the set* M' *defined in Corollary* 3.2 *be closed in* K. *Let* $T : K \to 2^{X^*}$ *be a set-valued map with nonempty values, where* X^* *is the topological dual of* X *with the duality pairing* $\langle \cdot, \cdot \rangle : X^* \times X \to \mathbb{R}$. *Let*

$$\zeta(x,\eta) := \inf_{v \in T(x)} \langle v, x - \eta \rangle \in \mathbb{R}$$

for each $(x,\eta) \in E \times K$. *If, for each* $\eta \in K$, *the set* $\{x \in K : \zeta(x,\eta) \leq 0\}$ *is closed in* K, *then there exists a point* $x_0 \in K$ *such that* $x_0 \in \mathbb{A}(x_0)$ *and* $\forall \eta \in \mathbb{A}(x_0)$, $\zeta(x_0,\eta) \leq 0$.

Proof This is a consequence of Corollary 3.2, where $Y = \mathbb{R}$, $\alpha = \alpha_2$, $\mathbb{F}(x,\eta) \equiv \{\zeta(x,\eta)\}$ and $\mathbb{C}(x,\eta) \equiv -\mathbb{R}_+$ (the nonpositive half-line). Indeed, since

$$(\mathbb{F}(x,\eta), \mathbb{C}(x,\eta)) \in \alpha_2 \Leftrightarrow \zeta(x,\eta) \leq 0,$$

we see that condition (i) of Corollary 3.2 holds. Condition (ii) of this corollary is assured by the concavity of $\zeta(x,\cdot)$. Condition (iii) is trivially satisfied since $\mathbb{F}(x,x) = \zeta(x,x) \equiv 0$. □

Remark 3.2 A special case of Corollary 3.3 with $\mathbb{A}(x) \equiv K$ is considered in Theorem 3.2 of [24]. Corollary 3.3 is established in Lemma 3.1 of [25] under an assumption different from that used in Corollary 3.3 for the set-valued map $\mathbb{A}$.

Theorem 3.2 *Let* M *be closed in* $E \times K$. *Under the following assumptions, there exists a solution of Problem* (P_α) *such that*:

(i) $\widehat{L}_{\overline{\alpha}}$ *has open lower sections.*
(ii) *One of the pairs* $(L_{\overline{\alpha}}, A)$ *and* $(\widehat{L}_{\overline{\alpha}}, A)$ *satisfies the coercivity condition.*
(iii) *One of the conditions* (b) *and* (c) *holds.*

Proof If the pair $(L_{\overline{\alpha}}, A)$ satisfies the coercivity condition, then the pair $(\widehat{L}_{\overline{\alpha}}, A)$ also satisfies this condition. On the other hand, by Lemma 3.2, (c) ⇒ (b). So, it suffices to prove Theorem 3.2 under the conditions (i), (b) and the condition that the pair $(\widehat{L}_{\overline{\alpha}}, A)$ satisfies the coercivity condition. To complete our proof, it remains to apply Lemma 3.1 with $L = \widehat{L}_{\overline{\alpha}}$. □

The proof of Theorems 3.3 and 3.4 below is similar to that of Theorems 3.1 and 3.2, and therefore is deleted.

Theorem 3.3 *Let M be closed in $E \times K$. Under the following assumptions there exists a solution of Problem* (P_α):

(i) $l_{\overline{\beta}}$ *has open lower sections.*
(ii) *One of the pairs* $(L_{\overline{\alpha}}, A)$, $(\widehat{L}_{\overline{\alpha}}, A)$, $(l_{\overline{\beta}}, A)$ *satisfies the coercivity condition.*
(iii) *Condition* (c) *holds.*

Theorem 3.4 *Let M be closed in $E \times K$. Under the following assumptions, there exists a solution of Problem* (P_α):

(i) $l_{\overline{\beta}}$ *has open lower sections.*
(ii) *One of the pairs* $(L_{\overline{\alpha}}, A)$ *and* $(l_{\overline{\beta}}, A)$ *satisfies the coercivity condition.*
(iii) *Condition* (d) *holds.*

The following conditions are useful for checking (a), (b), (c) or (d):

(a′) The pair (F, C) is α-diagonally quasiconvex in η with respect to B.
(b′) The pair (F, C) is strongly α-diagonally quasiconvex in η with respect to B.
(c′) Condition (ps) holds and the pair (f, c) is β-diagonally quasiconvex in η with respect to b.
(d′) Condition (wps) holds and the pair (f, c) is β-diagonally quasiconvex in η with respect to b.

(a″) For each $(z, x) \in E \times K$, $L_{\overline{\alpha}}(z, x)$ is convex, and $\alpha_{FC}(v, x, x)$ for some $v \in B(z, x, x)$.
(b″) For each $(z, x) \in E \times K$, $\widehat{L}_{\overline{\alpha}}(z, x)$ is convex, and $\alpha_{FC}(v, x, x)$ for all $v \in B(z, x, x)$.
(c″) Condition (ps) holds and, for each $(z, x) \in E \times K$, $l_{\overline{\beta}}(z, x)$ is convex, and $\beta_{fc}(u, x, x)$ for some $u \in b(z, x, x)$.
(d″) Condition (wps) holds and, for each $(z, x) \in E \times K$, $l_{\overline{\beta}}(z, x)$ is convex, and $\beta_{fc}(u, x, x)$ for some $u \in b(z, x, x)$.

Proposition 3.1 *The following implications are true*: (a″) ⇒ (a′) ⇒ (a); (b″) ⇒ (b′) ⇒ (b); (c″) ⇒ (c′) ⇒ (c) *and* (d″) ⇒ (d′) ⇒ (d).

Proof This is a consequence of Proposition 2.1. □

Remark 3.3 Results similar to those of Sect. 3 are given for Problem (P'_α) of [20]. (For the formulation of (P'_α), see Sect. 1.) It should be noted that some assumptions used in Sect. 3 of the present paper are weaker than or different from the corresponding ones of [20]. Let us compare the difference between some corresponding assumptions of these papers. A common feature for each existence result of [20] is that the set-valued map B is independent of η and B is always assumed to be a compact acyclic map, while all these requirements are not needed for the validity of the results of the present paper. Another difference is that the set-valued maps $L_{\overline{\alpha}}$, $\widehat{L}_{\overline{\alpha}}$, $l_{\overline{\beta}}$, used to formulate the main results of this paper, are defined with the participation of set-valued map B or b, while the corresponding set-valued maps N_β and N'_β, used to formulate the main results of [20] (see Theorems 3.1 and 4.1 of [20]), are introduced without the participation of B and b. Similar remark can be made when we deal with the pseudomonotonicity type conditions (see conditions (ps) and (wps) of this paper, and conditions (PS) and (ps) of [20]) or the β-diagonal quasiconvexity type properties. The above discussions show that the results obtained in this paper are different from the corresponding ones of [20]. It is a simple mater to verify that the existence of a solution of Problem (P_α) with the data given in the example at the end of Sect. 1 can be derived from our Theorem 3.1.

4 Sufficient Conditions for the Existence of Open Lower Sections

As we have seen in Theorems 3.1–3.4 of the previous section, the existence of a solution of Problem (P_α) requires that one of the set-valued maps $L_{\overline{\alpha}}$, $\widehat{L}_{\overline{\alpha}}$ and $l_{\overline{\beta}}$ has open lower sections. Obviously, this requirement may be satisfied even when the set-valued maps involving in the definition of $L_{\overline{\alpha}}$, $\widehat{L}_{\overline{\alpha}}$ or $l_{\overline{\beta}}$ are discontinuous. However, checking this requirement in the general case is not an easy task. It is then natural to ask if this requirement can be discovered under suitable continuity assumptions. This section is devoted to an answer to this question for the case $\alpha = \alpha_i,\ i = 1, 2, 3, 4$. First observe from the formulas defining $l_{\overline{\beta}}(z, x)$ and $L_{\overline{\alpha}}(z, x)$ that the construction of $l_{\overline{\beta}}(z, x)$ is exactly that of $L_{\overline{\alpha}}(z, x)$ with β, f, c, b in place of α, F, C, B respectively. So, we can restrict ourselves to the existence of open lower sections of $L_{\overline{\alpha}}$ and $\widehat{L}_{\overline{\alpha}}$ whose constructions are quite different.

Proposition 4.1

(i) *Let $\alpha = \alpha_1$ (resp. $\alpha = \alpha_3$) and, for each $\eta \in K$, let $F(\cdot, \cdot, \eta)$ and $B(\cdot, \cdot, \eta)$ be usc and compact-valued; let $C(\cdot, \cdot, \eta)$ have an open graph (resp. a closed graph). Then, the set-valued map $L_{\overline{\alpha}}$ has open lower sections.*
(ii) *Let $\alpha = \alpha_2$ (resp. $\alpha = \alpha_4$) and, for each $\eta \in K$, let $F(\cdot, \cdot, \eta)$ be lsc; let $B(\cdot, \cdot, \eta)$ be usc and compact-valued; let $C(\cdot, \cdot, \eta)$ have a closed graph (resp. an open graph). Then, the set-valued map $L_{\overline{\alpha}}$ has open lower sections.*

Proof

(i) To prove that the set-valued map $L_{\overline{\alpha}}$ with $\alpha = \alpha_1$ has open lower sections, we need to show that, for fixed $\eta \in K$, the set

$$Q_1(\eta) := \{(z,x) \in E \times K : \forall v \in B_\eta(z,x),\ F_\eta(v,x) \subset C_\eta(v,x)\}$$

is open in $E \times K$, where $B_\eta(\cdot,\cdot) = B(\cdot,\cdot,\eta)$ and similarly for $F_\eta(\cdot,\cdot)$ and $C_\eta(\cdot,\cdot)$. Indeed, assuming that $(\widetilde{z},\widetilde{x}) \in Q_1(\eta)$, we have to find a neighborhood $U(\widetilde{z},\widetilde{x})$ in the topological space $E \times K$ such that

$$U(\widetilde{z},\widetilde{x}) \subset Q_1(\eta). \tag{1}$$

Indeed, $(\widetilde{z},\widetilde{x}) \in Q_1(\eta)$ means that, for each $\widetilde{v} \in B_\eta(\widetilde{z},\widetilde{x})$,

$$\varphi_\eta(\widetilde{v},\widetilde{x}) \subset \operatorname{gr} C_\eta, \tag{2}$$

where

$$\varphi_\eta(v,x) = (v,x,F_\eta(v,x)) \subset E \times K \times Y. \tag{3}$$

In other words,

$$p_\eta(\widetilde{z},\widetilde{x}) := \bigcup_{v \in B_\eta(\widetilde{z},\widetilde{x})} \varphi_\eta(v,\widetilde{x}) \subset \operatorname{gr} C_\eta. \tag{4}$$

Setting $\sigma = (z,x) \in X' := E \times K$, $\mu = (v,\xi) \in Z' := E \times K$ and $Y' := E \times K \times Y$, we can write

$$p_\eta(\sigma) = \bigcup_{\mu \in \psi_\eta(\sigma)} \varphi_\eta(\mu),$$

where

$$\psi_\eta(\sigma) = (B_\eta(z,x),x) \subset Z'. \tag{5}$$

Since $B_\eta : E \times K \to 2^E$ is usc and compact-valued, it follows from [21, Proposition 7, p. 73] that $\psi_\eta : X' \to 2^{Z'}$ is usc. In addition, since F_η is usc and compact-valued, it follows again from [21, Proposition 7, p. 73] that $\varphi_\eta : Z' \to 2^{Y'}$ is usc. Therefore, the set-valued map p_η is usc on X' (see [21, Proposition 6, p. 73]). On the other hand, by assumption gr C_η is an open set in Y', and by (4) $p_\eta(\widetilde{\sigma}) \subset \operatorname{gr} C_\eta$, where $\widetilde{\sigma} := (\widetilde{z},\widetilde{x})$. Therefore, by the upper semicontinuity of the map p_η, there exists a neighborhood $U(\widetilde{\sigma})$ in X' such that $p_\eta(\sigma) \subset \operatorname{gr} C_\eta$, for all $\sigma \in U(\widetilde{\sigma})$. In other words, there exists a neighborhood $U(\widetilde{z},\widetilde{x})$ in $E \times K$ such that

$$p_\eta(z,x) \subset \operatorname{gr} C_\eta, \quad \forall (z,x) \in U(\widetilde{z},\widetilde{x}). \tag{6}$$

Since $p_\eta(z,x) \subset \operatorname{gr} C_\eta$ means that $F_\eta(v,x) \subset C_\eta(v,x)$ for each $v \in B_\eta(z,x)$, we conclude from (6) that (1) holds, as desired.

Part (i) of Proposition 4.1 is thus established for $\alpha = \alpha_1$. Consider now the case $\alpha = \alpha_3$. Setting $C'(z,x,\eta) = Y \setminus C(z,x,\eta)$, we can verify that $C'(\cdot,\cdot,\eta) : E \times K \to 2^Y$ has an open graph if and only if $C(\cdot,\cdot,\eta)$ has a closed graph. On the other hand,

$$L_{\overline{\alpha}_3}(z,x) = \{\eta \in K : \forall v \in B(z,x,\eta),\ F(z,x,\eta) \subset C'(z,x,\eta)\},$$

i.e., $L_{\overline{\alpha}_3}(z,x)$ is exactly $L_{\overline{\alpha}_1}(z,x)$ with $C'(z,x,\eta)$ in place of $C(z,x,\eta)$. So, applying the result of part (i) for $\alpha=\alpha_1$, with $C'(z,x,\eta)$ instead of $C(z,x,\eta)$, we can conclude that $L_{\overline{\alpha}_4}$ has open lower sections.

(ii) To prove that the set-valued map $L_{\overline{\alpha}}$ with $\alpha=\alpha_2$ has open lower sections, we need to show that, for fixed $\eta\in K$, the set

$$Q_2(\eta):=\{(z,x)\in E\times K:\forall v\in B_\eta(z,x),\ F_\eta(v,x)\not\subset C_\eta(v,x)\}$$

is open in $E\times K$, where B_η, F_η, C_η are as above. Indeed, assuming that $(\widetilde{z},\widetilde{x})\in Q_2(\eta)$, we have to find a neighborhood $U(\widetilde{z},\widetilde{x})$ in the topological space $E\times K$ such that $U(\widetilde{z},\widetilde{x})\subset Q_2(\eta)$. Indeed, condition $(\widetilde{z},\widetilde{x})\in Q_2(\eta)$ means that, for each $\widetilde{v}\in B_\eta(\widetilde{z},\widetilde{x})$, $\varphi_\eta(\widetilde{v},\widetilde{x})\not\subset \operatorname{gr} C_\eta$, where $\varphi_\eta(v,x)$ is defined by (3). Since $F_\eta(\cdot,\cdot)$ is lsc, it can be verified that $\varphi_\eta(\cdot,\cdot)$ is lsc.

Let us fix $\widetilde{v}\in B_\eta(\widetilde{z},\widetilde{x})$. Since by assumption $\operatorname{gr} C_\eta$ is closed in $E\times K\times Y$, and since $\varphi_\eta(\widetilde{v},\widetilde{x})\not\subset \operatorname{gr} C_\eta$, we derive from the lower semicontinuity of $\varphi_\eta(\cdot,\cdot)$ that there exists a neighborhood $U(\widetilde{v})$ (resp. $U_{\widetilde{v}}(\widetilde{x})$) in the topological space E (resp. K) such that

$$\forall (v,x)\in U(\widetilde{v})\times U_{\widetilde{v}}(\widetilde{x}),\quad \varphi_\eta(v,x)\not\subset \operatorname{gr} C_\eta. \tag{7}$$

(The subscript $\widetilde{v}$ in $U_{\widetilde{v}}(\widetilde{x})$ means that this neighborhood of $\widetilde{x}$ depends on $\widetilde{v}$.) Since $B_\eta(\widetilde{z},\widetilde{x})$ is compact, there exist n neighborhoods $U(\widetilde{v}_i)$, $i=1,2,\dots,n$, such that

$$\bigcup_{i=1}^n U(\widetilde{v}_i)\supset B_\eta(\widetilde{z},\widetilde{x}).$$

Since $B_\eta(\cdot,\cdot)$ is usc, we find a neighborhood $U(\widetilde{z})$ (resp. $U(\widetilde{x})$) in the topological space E (resp. K) such that

$$\forall (z,x)\in U(\widetilde{z})\times U(\widetilde{x}),\quad \bigcup_{i=1}^n U(\widetilde{v}_i)\supset B_\eta(z,x). \tag{8}$$

Without loss of generality, we may assume that

$$U(\widetilde{x})\subset\bigcap_{i=1}^n U_{\widetilde{v}_i}(\widetilde{x}).$$

We now prove that the inclusion $U(\widetilde{z},\widetilde{x})\subset Q_2(\eta)$ holds, with $U(\widetilde{z},\widetilde{x})=U(\widetilde{z})\times U(\widetilde{x})$. Indeed, let us take an arbitrary point $(z,x)\in U(\widetilde{z})\times U(\widetilde{x})$. By (8), for each $v\in B_\eta(z,x)$, we can find an index i such that $v\in U(\widetilde{v}_i)$. Since $(v,x)\in U(\widetilde{v}_i)\times U_{\widetilde{v}_i}(\widetilde{x})$, we conclude from (7) with $\widetilde{v}=\widetilde{v}_i$ that $\varphi_\eta(v,x)\not\subset \operatorname{gr} C_\eta$, i.e., $F_\eta(v,x)\not\subset C_\eta(v,x)$. This proves that $(z,x)\in Q_2(\eta)$. Since this is true for each $(z,x)\in U(\widetilde{z})\times U(\widetilde{x})=:U(\widetilde{z},\widetilde{x})$, we obtain $U(\widetilde{z},\widetilde{x})\subset Q_2(\eta)$, as desired.

The part (ii) of Proposition 4.1 is thus established for the case $\alpha=\alpha_2$. Defining $C'(z,x,\eta)=Y\setminus C(z,x,\eta)$, we see that

$$L_{\overline{\alpha}_4}(z,x)=\{\eta\in K:\forall v\in B(z,x,\eta),\ F(z,x,\eta)\not\subset C'(z,x,\eta)\},$$

i.e., $L_{\overline{\alpha}_4}(z,x)$ is exactly $L_{\overline{\alpha}_2}(z,x)$ with $C'(z,x,\eta)$ in place of $C(z,x,\eta)$. Hence, we can apply the result of the part (ii) for $\alpha=\alpha_2$, with $C'(z,x,\eta)$ instead of $C(z,x,\eta)$, to derive that $L_{\overline{\alpha}_4}$ has open lower sections.

□

We delete the detailed proof of Propositions 4.2 and 4.3 below, observing that it is quite similar to that of Proposition 4.1.

Proposition 4.2

(i) *Let* $\alpha=\alpha_1$ (*resp.* $\alpha=\alpha_3$) *and*, *for each* $\eta\in K$, *let* $F(\cdot,\cdot,\eta)$ *be usc and compact-valued*, *let* $B(\cdot,\cdot,\eta)$ *be lsc*, *and let* $C(\cdot,\cdot,\eta)$ *have an open graph* (*resp. a closed graph*). *Then*, *the map* $\widehat{L_{\overline{\alpha}}}$ *has open lower sections.*

(ii) *Let* $\alpha=\alpha_2$ (*resp.* $\alpha=\alpha_4$) *and*, *for each* $\eta\in K$, *let* $F(\cdot,\cdot,\eta)$, $B(\cdot,\cdot,\eta)$ *be lsc*, *and let* $C(\cdot,\cdot,\eta)$ *have a closed graph* (*resp. an open graph*). *Then*, *the map* $\widehat{L_{\overline{\alpha}}}$ *has open lower sections.*

When $B(z,x,\eta)$ does not depend on z and x, the requirement of F and C in Proposition 4.2 can be weakened. Namely, we have the following result.

Proposition 4.3 *Let* $B(z,x,\eta)\equiv\mathbb{B}(\eta)$ *for each* $(z,x,\eta)\in E\times K\times K$.

(i) *Let* $\alpha=\alpha_1$ (*resp.* $\alpha=\alpha_3$) *and*, *for each* $(z,\eta)\in E\times K$, *let* $F(z,\cdot,\eta)$ *be usc and compact-valued*, *and let* $C(z,\cdot,\eta)$ *have an open graph* (*resp. a closed graph*). *Then*, *the map* $\widehat{L_{\overline{\alpha}}}$ *has open lower sections.*

(ii) *Let* $\alpha=\alpha_2$ (*resp.* $\alpha=\alpha_4$) *and*, *for each* $(z,\eta)\in E\times K$, *let* $F(z,\cdot,\eta)$ *be lsc*, *and let* $C(z,\cdot,\eta)$ *have a closed graph* (*resp. an open graph*). *Then*, *the map* $\widehat{L_{\overline{\alpha}}}$ *has open lower sections.*

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