J Optim Theory Appl (2008) 139: 361–378
DOI 10.1007/s10957-008-9419-x

Solutions of Fuzzy Multiobjective Programming Problems Based on the Concept of Scalarization

H.C. Wu

Published online: 25 April 2008

Abstract Scalarization of fuzzy multiobjective programming problems using the embedding theorem and the concept of convex cone (ordering cone) is proposed in this paper. Since the set of all fuzzy numbers can be embedded into a normed space, this motivation naturally inspires us to invoke the scalarization techniques in vector optimization problems to evaluate the a multiobjective programming problem. Two solution concepts are proposed in this paper by considering different convex cones.

Keywords Fuzzy numbes · Convex cones · Partial ordering · (Weakly) minimal elements · (Weak) Pareto optimal solutions

1 Introduction

Ever since Bellman and Zadeh [1] inspired the development of decision making under the fuzzy environment by providing the aggregation operators to combine the fuzzy goals and fuzzy decision space, the topic of fuzzy optimization has been widely investigated. The fuzzy optimization problems deal with fuzziness in optimization problems. In such cases, some useful tools in fuzzy sets theory are invoked to solve the optimization problems under the fuzzy environment. The collection of papers on fuzzy optimization edited by Slowiński [2] and Delgado et al. [3] give the main stream of this topic. The books by Zimmermann [4] and Lai and Hwang [5, 6] also give an insightful survey. Moreover, for the other variants of fuzzy multiobjective programming problems, we may refer to the literature review in Wu [7]. On the other hand, for the vector optimization problems with respect to cones, we may refer to Benson [8–10], Borwein [11] and Yu [12].

Communicated by H.P. Benson.

H.C. Wu (✉)
Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 802, Taiwan
e-mail: hcwu@nknucc.nknu.edu.tw

The technique for solving fuzzy optimization problems using embedding theorem was proposed by Wu [13]. The solution concept of fuzzy multiobjective programming problem based on convex cones was also proposed by Wu [7]. The purpose of this paper is to consider another viewpoint, i.e., the scalarization of fuzzy multiobjective programming problem based on the concept of convex cones and the embedding theorem simultaneously. However the embedding theorem used in this paper is different from that of Wu [7].

The set of all fuzzy numbers is not a vector space in general. However, Puri and Ralescu [14] and Kaleva [15] proved that the set of all fuzzy numbers can be embedded into a normed space. Under this motivation, the scalarization technique in vector optimization turns into a useful tool in solving the corresponding vector optimization problem that can be transformed from the original fuzzy multiobjective programming problem using the embedding theorem and a suitable linear defuzzification function.

In Sect. 2, we present the embedding theorem and prove the order preserving property under the embedding function (that is, the order does not change the direction under the embedding function). In Sect. 3, we formulate the fuzzy multiobjective programming problems using the concept of convex cones and introduce different notions of optimality. In Sect. 4, the scalarization methodology for fuzzy multiobjective programming problem is developed by following the essence of scalarization technique in vector optimization problems. Use of the methodology developed in this paper is illustrated by applying it to some practical problems in Sect. 5. Conclusions of the present study are finally drawn in Sect. 6.

2 Embedding and Order Preserving

Let A be a subset of $\mathbb{R}$. Then, the corresponding indicator function of A is given by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$. The fuzzy subset $\tilde{a}$ of $\mathbb{R}$ is defined by a function $\xi_{\tilde{a}} : \mathbb{R} \to [0, 1]$, which is an extension of indicator function and is called a *membership function* of $\tilde{a}$. The α-level set of $\tilde{a}$, denoted by $\tilde{a}_\alpha$, is defined by $\tilde{a}_\alpha = \{x \in \mathbb{R} : \xi_{\tilde{a}}(x) \geq \alpha\}$ for all $\alpha \in (0, 1]$. The 0-level set $\tilde{a}_0$ is defined as the closure of the set $\{x \in \mathbb{R} : \xi_{\tilde{a}}(x) > 0\}$, i.e., $\tilde{a}_0 = \mathrm{cl}(\{x \in \mathbb{R} : \xi_{\tilde{a}}(x) > 0\})$.

Definition 2.1 The fuzzy subset $\tilde{a}$ of $\mathbb{R}$ is said to be a *fuzzy number* if the following conditions are satisfied:

(i) $\tilde{a}$ is normal, i.e., there exists an $x \in \mathbb{R}$ such that $\xi_{\tilde{a}}(x) = 1$;
(ii) $\xi_{\tilde{a}}$ is quasiconcave, i.e., $\xi_{\tilde{a}}(tx + (1-t)y) \geq \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{a}}(y)\}$ for $t \in [0, 1]$;
(iii) $\xi_{\tilde{a}}$ is upper semicontinuous, i.e., $\{x \in \mathbb{R} : \xi_{\tilde{a}}(x) \geq \alpha\}$ is a closed subset of $\mathbb{R}$ for each $\alpha \in (0, 1]$;
(iv) the 0-level set $\tilde{a}_0$ is a closed and bounded subset of $\mathbb{R}$.

Since $\tilde{a}_\alpha \subset \tilde{a}_0$ for each $\alpha \in (0, 1]$, we see that the α-level sets $\tilde{a}_\alpha$ are bounded subsets of $\mathbb{R}$ for all $\alpha \in (0, 1]$. We denote by $\mathcal{F}(\mathbb{R})$ the set of all fuzzy numbers. It is well known that if $\tilde{a} \in \mathcal{F}(\mathbb{R})$, then the α-level set of $\tilde{a}$ is a closed, bounded and convex subset of $\mathbb{R}$, i.e., a closed interval in $\mathbb{R}$. Therefore, we denote by $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$. For convenience, the membership function of $\tilde{0}$ is defined by

$$\xi_{\tilde{0}}(r) = \begin{cases} 1, & \text{if } r = 0, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that $\tilde{0}^L_\alpha = 0 = \tilde{0}^U_\alpha$ for all $\alpha \in [0, 1]$.

Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. Using the extension principle in Zadeh [16–18] and referring to Puri and Ralescu [14], the membership function of the addition $\tilde{a} \oplus \tilde{b}$ is defined by

$$\xi_{\tilde{a}\oplus\tilde{b}}(z) = \sup_{\{(x,y):x+y=z\}} \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{b}}(y)\} \tag{1}$$

and the membership function of the scalar multiplication $\lambda\tilde{a}$, $\lambda \in \mathbb{R}$, is defined by

$$\xi_{\lambda\tilde{a}}(z) = \begin{cases} \xi_{\tilde{a}}(z/\lambda), & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0 \text{ and } z \neq 0, \\ 1, & \text{if } \lambda = 0 = z. \end{cases} \tag{2}$$

It also means that $\lambda\tilde{a} = \tilde{0}$ if $\lambda = 0$.

Proposition 2.1 *Let $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$. Then $\tilde{a} \oplus \tilde{b} \in \mathcal{F}(\mathbb{R})$ and $\lambda\tilde{a} \in \mathcal{F}(\mathbb{R})$ for $\lambda \in \mathbb{R}$ and $\lambda \neq 0$. Moreover, we also have the following useful results*:

(i) $(\tilde{a} \oplus \tilde{b})^L_\alpha = \tilde{a}^L_\alpha + \tilde{b}^L_\alpha$ *and* $(\tilde{a} \oplus \tilde{b})^U_\alpha = \tilde{a}^U_\alpha + \tilde{b}^U_\alpha$ *for* $\alpha \in [0, 1]$,
(ii) $(\lambda\tilde{a})^L_\alpha = \lambda \cdot \tilde{a}^L_\alpha$ *and* $(\lambda\tilde{a})^U_\alpha = \lambda \cdot \tilde{a}^U_\alpha$ *for* $\lambda > 0$ *and* $\alpha \in [0, 1]$,
(iii) $(\lambda\tilde{a})^L_\alpha = \lambda \cdot \tilde{a}^U_\alpha$ *and* $(\lambda\tilde{a})^U_\alpha = \lambda \cdot \tilde{a}^L_\alpha$ *for* $\lambda < 0$ *and* $\alpha \in [0, 1]$.

Definition 2.2 Let $\tilde{a}$ be a fuzzy number. We call $\tilde{a}$ a *canonical fuzzy number* if $\tilde{a}^L_\alpha$ and $\tilde{a}^U_\alpha$ are continuous with respect to α on $[0, 1]$, i.e., the functions $f(\alpha) = \tilde{a}^L_\alpha$ and $g(\alpha) = \tilde{a}^U_\alpha$ are continuous on $[0, 1]$.

We denote by $\mathcal{F}_c(\mathbb{R})$ the set of all canonical fuzzy numbers. In general, $\mathcal{F}_c(\mathbb{R})$ is not a vector space according to the addition and scalar multiplication described in (1) and (2), respectively. However, Puri and Ralescu [14] and Kaleva [15] proved that $\mathcal{F}_c(\mathbb{R})$ can be embedded into a normed space $(\mathcal{N}, \|\cdot\|)$ isometrically and isomorphically. In other words, if π is the embedding function $\pi : \mathcal{F}_c(\mathbb{R}) \to \mathcal{N}$, then

(i) $\pi(\tilde{a} \oplus \tilde{b}) = \pi(\tilde{a}) + \pi(\tilde{b})$,
(ii) $\pi(\lambda\tilde{a}) = \lambda\pi(\tilde{a})$ for $\lambda \geq 0$,
(iii) $d(\tilde{a}, \tilde{b}) = \|\pi(\tilde{a}) - \pi(\tilde{b})\|$,

where the metric $d(\cdot, \cdot)$ on $\mathcal{F}_c(\mathbb{R})$ is defined by

$$d(\tilde{a}, \tilde{b}) = \sup_{0\leq\alpha\leq1} d_H(\tilde{a}_\alpha, \tilde{b}_\alpha)$$

and the Hausdorff distance d_H is given by

$$d_H(A, B) = \max\left\{\sup_{a\in A} \inf_{b\in B} |a - b|, \sup_{b\in B} \inf_{a\in A} |a - b|\right\}.$$

More precisely, the normed space $(\mathcal{N}, \|\cdot\|)$ consists of the equivalence classes $[\![\tilde{a}, \tilde{b}]\!]$ induced by the equivalence relation $(\tilde{a}, \tilde{b}) \sim (\tilde{c}, \tilde{d})$ if and only if $\tilde{a} \oplus \tilde{d} = \tilde{b} \oplus \tilde{c}$, where $(\tilde{a}, \tilde{b}), (\tilde{c}, \tilde{d}) \in \mathcal{F}_c(\mathbb{R}) \times \mathcal{F}_c(\mathbb{R})$. The vector addition and scalar multiplication in $(\mathcal{N}, \|\cdot\|)$ is defined by

$$[\![\tilde{a}, \tilde{b}]\!] + [\![\tilde{c}, \tilde{d}]\!] = [\![\tilde{a} \oplus \tilde{c}, \tilde{b} \oplus \tilde{d}]\!] \tag{3}$$

and

$$\lambda[\![\tilde{a}, \tilde{b}]\!] = \begin{cases} [\![\lambda\tilde{a}, \lambda\tilde{b}]\!], & \text{if } \lambda \geq 0, \\ [\![(-\lambda)\tilde{b}, (-\lambda)\tilde{a}]\!], & \text{if } \lambda < 0. \end{cases}$$

The norm in $\mathcal{N}$ is defined by

$$\|[\![\tilde{a}, \tilde{b}]\!]\| = d(\tilde{a}, \tilde{b}).$$

We see that $[\![\tilde{0}, \tilde{0}]\!]$ is the zero element of the normed space $(\mathcal{N}, \|\cdot\|)$, since from (3),

$$[\![\tilde{a}, \tilde{b}]\!] + [\![\tilde{0}, \tilde{0}]\!] = [\![\tilde{a}, \tilde{b}]\!] = [\![\tilde{0}, \tilde{0}]\!] + [\![\tilde{a}, \tilde{b}]\!].$$

The embedding function $\pi : \mathcal{F}_c(\mathbb{R}) \to \mathcal{N}$ is then defined by

$$\pi(\tilde{a}) = [\![\tilde{a}, \tilde{0}]\!]. \tag{4}$$

Now suppose that $[\![\tilde{a}, \tilde{0}]\!] = [\![\tilde{b}, \tilde{0}]\!]$. Then $(\tilde{a}, \tilde{0}) \sim (\tilde{b}, \tilde{0})$, i.e., $\tilde{a} = \tilde{b}$. It says that the embedding function π is injective (one-to-one). We see that $\pi(\tilde{0}) = [\![\tilde{0}, \tilde{0}]\!]$ is the zero element of the normed space $(\mathcal{N}, \|\cdot\|)$.

Definition 2.3 The function $\eta : \mathcal{F}(\mathbb{R}) \to \mathbb{R}$ is called a *defuzzification function*. We say that the defuzzification function η is *linear* if the following conditions are satisfied:

$$\eta(\tilde{a} \oplus \tilde{b}) = \eta(\tilde{a}) + \eta(\tilde{b}) \quad \text{and} \quad \eta(\lambda\tilde{a}) = \lambda \cdot \eta(\tilde{a}), \tag{5}$$

for $\lambda \in \mathbb{R}$.

A fuzzy number $\tilde{a} \in \mathcal{F}(\mathbb{R})$ is defuzzified into a real number $\eta(\tilde{a})$. Therefore we call η as the defuzzication function.

Example 2.1 Let $\tilde{a} \in \mathcal{F}(\mathbb{R})$. We define

$$\eta(\tilde{a}) = \frac{1}{2}\int_0^1 (\tilde{a}_\alpha^L + \tilde{a}_\alpha^U)d\alpha.$$

From Proposition 2.1, it is not hard to prove that η is a linear defuzzification function.

By referring to Puri and Ralescu [14], we give the following definition.

Definition 2.4 Let $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$. If there exists a fuzzy number $\tilde{c} \in \mathcal{F}(\mathbb{R})$ such that $\tilde{b} = \tilde{a} \oplus \tilde{c}$ (this is well-defined since the addition "$\oplus$" is commutative), then $\tilde{c}$ is unique, and $\tilde{c}$ is called the *Hukuhara difference* between $\tilde{b}$ and $\tilde{a}$. We also write $\tilde{c} = \tilde{b} \ominus_H \tilde{a}$.

Proposition 2.2 *Let $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$. If the Hukuhara difference $\tilde{c} = \tilde{b} \ominus_H \tilde{a}$ exists, then $\tilde{c}^L_\alpha = \tilde{b}^L_\alpha - \tilde{a}^L_\alpha$ and $\tilde{c}^U_\alpha = \tilde{b}^U_\alpha - \tilde{a}^U_\alpha$ for all $\alpha \in [0, 1]$. If $\tilde{a}, \tilde{b} \in \mathcal{F}_c(\mathbb{R})$ and the Hukuhara difference $\tilde{c} = \tilde{b} \ominus_H \tilde{a}$ exists, then $\tilde{c} \in \mathcal{F}_c(\mathbb{R})$.*

Proof The results follow immediately from Proposition 2.1. □

Proposition 2.3 *The following statements hold true.*

(i) *Let $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$. Suppose that η is a linear defuzzification function on $\mathcal{F}(\mathbb{R})$. If the Hukuhara difference $\tilde{b} \ominus_H \tilde{a}$ exists, then $\eta(\tilde{b} \ominus_H \tilde{a}) = \eta(\tilde{b}) - \eta(\tilde{a})$.*
(ii) *Let $\tilde{a}, \tilde{b} \in \mathcal{F}_c(\mathbb{R})$. Suppose that π is the embedding function given in (4). If the Hukuhara difference $\tilde{b} \ominus_H \tilde{a}$ exists and belongs to $\mathcal{F}_c(\mathbb{R})$, then $\pi(\tilde{b} \ominus_H \tilde{a}) = \pi(\tilde{b}) - \pi(\tilde{a})$.*

Proof Let $\tilde{c} = \tilde{b} \ominus_H \tilde{a}$, i.e., $\tilde{b} = \tilde{a} \oplus \tilde{c}$. Then, we have $\eta(\tilde{b}) = \eta(\tilde{a}) + \eta(\tilde{c})$. On the other hand, we also have $\pi(\tilde{b}) = \pi(\tilde{a}) + \pi(\tilde{c})$. This completes the proof. □

Definition 2.5 Each binary relation $\leq$ on the real vector space $\mathcal{V}$ is called a partial ordering on $\mathcal{V}$ if the following axioms are satisfied:

(A1) $x \leq x$ for all $x \in \mathcal{V}$ (reflexive);
(A2) if $x \leq y$ and $y \leq z$ then $x \leq z$ for all $x, y, z \in \mathcal{V}$ (transitive);
(A3) if $x \leq y$ and $a \leq b$ then $x + a \leq y + b$ for all $x, y, a, b \in \mathcal{V}$;
(A4) if $x \leq y$ and λ is a positive real number then $\lambda x \leq \lambda y$ for all $x, y \in \mathcal{V}$.

Remark 2.1

(i) Let $\leq$ be a partial ordering on the real vector space $\mathcal{V}$. It is well known that the set $\mathcal{C}_\mathcal{V} = \{x \in \mathcal{V} : 0 \leq x\}$ is a convex cone. In this case, we say that $\mathcal{C}_\mathcal{V}$ is induced by $\leq$. Conversely, if $\mathcal{C}_\mathcal{V}$ is a convex cone in $\mathcal{V}$, then the binary relation $\leq$ defined by $x \leq y$ if and only if $y - x \in \mathcal{C}_\mathcal{V}$ is a partial ordering on $\mathcal{V}$. In this case, we say that $\leq$ is induced by $\mathcal{C}_\mathcal{V}$.
(ii) The convex cone $\mathcal{C}_\mathcal{V}$ is called *pointed* if and only if $\mathcal{C}_\mathcal{V} \cap (-\mathcal{C}_\mathcal{V}) = 0$. We see that if the ordering cone $\mathcal{C}_\mathcal{V}$ is pointed then the partial ordering $\leq$ induced by $\mathcal{C}_\mathcal{V}$ is antisymmetric. Conversely, if the partial ordering $\leq$ is antisymmetric, then the convex cone $\mathcal{C}_\mathcal{V}$ induced by $\leq$ is pointed.

Definition 2.6 Let $\tilde{a} \in \mathcal{F}(\mathbb{R})$. We say that $\tilde{a}$ is *nonnegative* if and only if $\tilde{a}^L_\alpha \geq 0$ for all $\alpha \in [0, 1]$. We say that $\tilde{a}$ is *positive* if and only if $\tilde{a}^L_\alpha > 0$ for all $\alpha \in [0, 1]$.

Remark 2.2 Let $\tilde{a} \in \mathcal{F}(\mathbb{R})$. Since $\tilde{a}^L_\alpha \leq \tilde{a}^U_\alpha$ for all $\alpha \in [0, 1]$, we see that $\tilde{a}$ is nonnegative if and only if $\tilde{a}^L_\alpha \geq 0$ and $\tilde{a}^U_\alpha \geq 0$ for all $\alpha \in [0, 1]$, and $\tilde{a}$ is positive if and only if $\tilde{a}^L_\alpha > 0$ and $\tilde{a}^U_\alpha > 0$ for all $\alpha \in [0, 1]$.

We write $\mathcal{F}_c^n(\mathbb{R}) = \mathcal{F}_c(\mathbb{R}) \times \cdots \times \mathcal{F}_c(\mathbb{R})$ (n times). For $\underline{\tilde{u}} \in \mathcal{F}_c^n(\mathbb{R})$, we also write $\underline{\tilde{u}} = (\tilde{u}^1, \ldots, \tilde{u}^n)$, where $\tilde{u}^j \in \mathcal{F}_c(\mathbb{R})$ for all $j = 1, \ldots, n$. The addition and scalar multiplication in $\mathcal{F}_c^n(\mathbb{R})$ can be defined componentwise; that is, for $\underline{\tilde{u}}, \underline{\tilde{v}} \in \mathcal{F}_c^n(\mathbb{R})$, we have

$$\underline{\tilde{u}} \oplus \underline{\tilde{v}} = (\tilde{u}^1 \oplus \tilde{v}^1, \ldots, \tilde{u}^n \oplus \tilde{v}^n) \quad \text{and} \quad \lambda\underline{\tilde{u}} = (\lambda\tilde{u}^1, \ldots, \lambda\tilde{u}^n)$$

for $\lambda \in \mathbb{R}$. We also say that the Hukuhara difference $\underline{\tilde{v}} \ominus_H \underline{\tilde{u}}$ exists if $\tilde{v}^j \ominus_H \tilde{u}^j$ exist for all $j = 1, \ldots, n$. In this case, $\underline{\tilde{v}} \ominus_H \underline{\tilde{u}}$ means

$$\underline{\tilde{v}} \ominus_H \underline{\tilde{u}} = (\tilde{v}^1 \ominus_H \tilde{u}^1, \ldots, \tilde{v}^n \ominus_H \tilde{u}^n).$$

Let η be a linear defuzzification function. In this paper, we are going to consider two solution concepts. Therefore, we consider the following two sets:

$$\mathcal{C}^1 = \{\underline{\tilde{u}} = (\tilde{u}^1, \ldots, \tilde{u}^n) : \eta(\tilde{u}^j) \geq 0 \text{ for all } j = 1, \ldots, n \text{ and } \underline{\tilde{u}} \in \mathcal{F}_c^n(\mathbb{R})\}$$

and

$$\mathcal{C}^2 = \{\underline{\tilde{u}} = (\tilde{u}^1, \ldots, \tilde{u}^n) : \tilde{u}^j \text{ are nonnegative for all } j = 1, \ldots, n \text{ and } \underline{\tilde{u}} \in \mathcal{F}_c^n(\mathbb{R})\}.$$

Two binary relations $\preceq^1$ and $\preceq^2$ on $\mathcal{F}_c^n(\mathbb{R})$ are defined below.

Definition 2.7 Let $\underline{\tilde{u}}, \underline{\tilde{v}} \in \mathcal{F}_c^n(\mathbb{R})$. We write $\underline{\tilde{u}} \preceq^1 \underline{\tilde{v}}$ (resp. $\underline{\tilde{u}} \preceq^2 \underline{\tilde{v}}$) if the Hukuhara difference $\underline{\tilde{v}} \ominus_H \underline{\tilde{u}}$ exists and $\underline{\tilde{v}} \ominus_H \underline{\tilde{u}} \in \mathcal{C}^1$ (resp. $\underline{\tilde{v}} \ominus_H \underline{\tilde{u}} \in \mathcal{C}^2$).

Proposition 2.4 *Let $\underline{\tilde{u}}, \underline{\tilde{v}} \in \mathcal{F}_c^n(\mathbb{R})$ and η be a linear defuzzification function on $\mathcal{F}_c(\mathbb{R}) \subset \mathcal{F}(\mathbb{R})$.*

(i) *If $\underline{\tilde{u}} \preceq^1 \underline{\tilde{v}}$ then $\eta(\tilde{u}^j) \leq \eta(\tilde{v}^j)$ for all $j = 1, \ldots, n$, and if $\underline{\tilde{u}} \preceq^2 \underline{\tilde{v}}$ then $(\tilde{u}^j)_\alpha^L \leq (\tilde{v}^j)_\alpha^L$ and $(\tilde{u}^j)_\alpha^U \leq (\tilde{v}^j)_\alpha^U$ for all $j = 1, \ldots, n$ and all $\alpha \in [0, 1]$.*
(ii) *The binary relations $\preceq^1$ and $\preceq^2$ defined on $\mathcal{F}_c^n(\mathbb{R})$ satisfy axioms* (1)–(4) *of Definition* 2.5.

Proof The results follow immediately from Propositions 2.1, 2.2 and 2.3. □

Remark 2.3 Although the binary relations $\preceq^1$ and $\preceq^2$ satisfy axioms (1)–(4) of Definition 2.5, we cannot say that $\preceq^1$ and $\preceq^2$ are partial orderings on $\mathcal{F}_c^n(\mathbb{R})$, since $\mathcal{F}_c^n(\mathbb{R})$ is not a real vector space in general. However, if we regard $\mathcal{F}_c^n(\mathbb{R})$ as a set, then $\preceq^1$ and $\preceq^2$ are partial orderings on $\mathcal{F}_c^n(\mathbb{R})$. On the other hand, if the real vector space $\mathcal{V}$ in Definition 2.5 is relaxed (replaced) as just saying that $\mathcal{V}$ is a set instead of a real vector space with some defined addition and scalar multiplication, then we can conclude that $\preceq^1$ and $\preceq^2$ are partial orderings on $\mathcal{F}_c^n(\mathbb{R})$ under this new definition for partial ordering. Sometimes, if $\mathcal{V}$ is a set, then we say that $\preceq$ is a partial ordering on $\mathcal{V}$ if conditions (1) and (2) in Definition 2.5 are satisfied. We don't have to check conditions (3) and (4), since $\mathcal{V}$ is not a vector space.

Proposition 2.5 *Let $\underline{\tilde{u}}, \underline{\tilde{v}} \in \mathcal{C}^1$. Then, we have the following results*:

(i) $\lambda\underline{\tilde{u}} \in \mathcal{C}^1$ *for* $\lambda > 0$;
(ii) $\lambda\underline{\tilde{u}} \oplus (1-\lambda)\underline{\tilde{v}} \in \mathcal{C}^1$ *for* $\lambda \in (0,1)$.

Proof It is easy to see $\lambda u^j, \lambda\tilde{u}^j \oplus (1-\lambda)\tilde{v}^j \in \mathcal{F}_c(\mathbb{R})$ for all $j = 1,\dots,n$. Since η is a linear defuzzification function, we have $\eta(\lambda\tilde{u}^j) = \lambda \cdot \eta(\tilde{u}^j) \geq 0$ and $\eta(\lambda\tilde{u}^j \oplus (1-\lambda)\tilde{v}^j) = \lambda \cdot \eta(\tilde{u}^j) + (1-\lambda)\cdot\eta(\tilde{v}^j) \geq 0$ for all $j = 1,\dots,n$. This completes the proof. □

Proposition 2.6 *Let* $\underline{\tilde{u}}, \underline{\tilde{v}} \in \mathcal{C}^2$. *Then, we have the following results*:

(i) $\lambda\underline{\tilde{u}} \in \mathcal{C}^2$ *for* $\lambda > 0$;
(ii) $\lambda\underline{\tilde{u}} \oplus (1-\lambda)\underline{\tilde{v}} \in \mathcal{C}^2$ *for* $\lambda \in (0,1)$.

Proof The results follow immediately from Proposition 2.1 and Remark 2.2. □

Remark 2.4 Propositions 2.5 and 2.6 show that $\mathcal{C}^1$ and $\mathcal{C}^2$ have the structure of convex cone in some sense. However, we cannot say that $\mathcal{C}^1$ and $\mathcal{C}^2$ are convex cones, since $\mathcal{F}_c^n(\mathbb{R})$ is not a vector space. Of course, we may say that $\mathcal{C}^1$ and $\mathcal{C}^2$ are convex cones in $\mathcal{F}_c^n(\mathbb{R})$ if the definition of convex cone is taken in a set instead of a real vector space.

Now we consider the product vector space $\mathcal{N}^n = \mathcal{N} \times \cdots \times \mathcal{N}$ (n times). Then, from Kreyszig [19], we see that $\mathcal{N}^n$ is a normed space with norm given by

$$\|\underline{s}\| = \max\{\|s^1\|,\dots,\|s^n\|\},$$

where $\underline{s} = (s^1,\dots,s^n) \in \mathcal{N}^n$. Let π be the embedding function given in (4). We define a function $\Pi : \mathcal{F}_c^n(\mathbb{R}) \to \mathcal{N}^n$ by

$$\Pi(\underline{\tilde{u}}) = (\pi(\tilde{u}^1),\dots,\pi(\tilde{u}^n)) \tag{6}$$

for $\underline{\tilde{u}} \in \mathcal{F}_c^n(\mathbb{R})$.

Proposition 2.7 *The sets* $\Pi(\mathcal{C}^1)$ *and* $\Pi(\mathcal{C}^2)$ *are convex cones in* $\mathcal{N}^n$.

Proof Let $\underline{s}, \underline{t} \in \Pi(\mathcal{C}^1)$. Then, there exist $\underline{\tilde{u}}, \underline{\tilde{v}} \in \mathcal{C}^1$ such that $\pi(\tilde{u}^j) = s^j$ and $\pi(\tilde{v}^j) = t^j$ for all $j = 1,\dots,n$. We have $\lambda s^j + (1-\lambda)t^j = \lambda\cdot\pi(\tilde{u}^j) + (1-\lambda)\cdot\pi(\tilde{v}^j) = \pi(\lambda\tilde{u}^j \oplus (1-\lambda)\tilde{v}^j)$. From Proposition 2.5, we see that $\lambda\underline{s} + (1-\lambda)\underline{t} \in \Pi(\mathcal{C}^1)$. It shows that $\Pi(\mathcal{C}^1)$ is a convex subset of $\mathcal{N}^n$. We also see that $\lambda\underline{s} \in \Pi(\mathcal{C}^1)$ for $\lambda > 0$. Therefore, $\Pi(\mathcal{C}^1)$ is a convex cone in $\mathcal{N}^n$. Similarly, from Proposition 2.6, we see that $\Pi(\mathcal{C}^2)$ is a convex cone in $\mathcal{N}^n$. □

Using Proposition 2.7 and Remark 2.1, we can induce two partial orderings $\preceq^1$ and $\preceq^2$ on $\mathcal{N}^n$ from $\Pi(\mathcal{C}^1)$ and $\Pi(\mathcal{C}^2)$, respectively. Now we are going to present an order preserving property under the function Π.

Proposition 2.8 (Order Preserving) *Let* $\underline{\tilde{u}}, \underline{\tilde{v}} \in \mathcal{F}_c^n(\mathbb{R})$. *Then,* $\underline{\tilde{u}} \preceq^1 \underline{\tilde{v}}$ *if and only if* $\Pi(\underline{\tilde{u}}) \preceq^1 \Pi(\underline{\tilde{v}})$, *and* $\underline{\tilde{u}} \preceq^2 \underline{\tilde{v}}$ *if and only if* $\Pi(\underline{\tilde{u}}) \preceq^2 \Pi(\underline{\tilde{v}})$.

Proof By Proposition 2.3, we see that $\pi(\tilde{v}^j) - \pi(\tilde{u}^j) = \pi(\tilde{v}^j \ominus_H \tilde{u}^j)$ for all $j = 1, \ldots, n$. It says that $\Pi(\underline{\tilde{v}}) - \Pi(\underline{\tilde{u}}) = \Pi(\underline{\tilde{v}} \ominus_H \underline{\tilde{u}}) \in \Pi(\mathcal{C}^1)$, i.e., $\Pi(\underline{\tilde{u}}) \preceq^1 \Pi(\underline{\tilde{v}})$. Conversely, if $\Pi(\underline{\tilde{u}}) \preceq^1 \Pi(\underline{\tilde{v}})$, i.e., $\Pi(\underline{\tilde{v}}) - \Pi(\underline{\tilde{u}}) \in \Pi(\mathcal{C}^1)$, then there exists a $\underline{\tilde{w}} \in \mathcal{C}^1$ such that $\Pi(\underline{\tilde{v}}) - \Pi(\underline{\tilde{u}}) = \Pi(\underline{\tilde{w}})$. It says that $\pi(\tilde{v}^j) = \pi(\tilde{u}^j) + \pi(\tilde{w}^j) = \pi(\tilde{u}^j \oplus \tilde{w}^j)$ for all $j = 1, \ldots, n$. Since π is one-to-one, we have $\tilde{v}^j = \tilde{u}^j \oplus \tilde{w}^j$. This shows that $\tilde{w}^j = \tilde{v}^j \ominus_H \tilde{u}^j$ exists for all $j = 1, \ldots, n$, i.e., $\underline{\tilde{v}} \ominus_H \underline{\tilde{u}} = \underline{\tilde{w}} \in \mathcal{C}^1$. It also means that $\underline{\tilde{u}} \preceq^1 \underline{\tilde{v}}$. Similarly for the case of "$\preceq^2$", this completes the proof. □

In order to interpret the ordering concept for fuzzy constraint function values, we consider the following two sets:

$$\mathcal{C}_\pi^1 = \{\tilde{a} : \eta(\tilde{a}) \geq 0 \text{ and } \tilde{a} \in \mathcal{F}_c(\mathbb{R})\}$$

and

$$\mathcal{C}_\pi^2 = \{\tilde{a} : \tilde{a} \text{ is nonnegative and } \tilde{a} \in \mathcal{F}_c(\mathbb{R})\}.$$

Remark 2.5 Let $\underline{s} \in \mathcal{N}^n$. We see that $\underline{s} \in \Pi(\mathcal{C}^1)$ if and only if $s^j \in \pi(\mathcal{C}_\pi^1)$ for all $j = 1, \ldots, n$, and $\underline{s} \in \Pi(\mathcal{C}^2)$ if and only if $s^j \in \pi(\mathcal{C}_\pi^2)$ for all $j = 1, \ldots, n$.

Using the similar arguments as in Proposition 2.7, we can show that $\pi(\mathcal{C}_\pi^1)$ and $\pi(\mathcal{C}_\pi^2)$ are convex cones in $\mathcal{N}$, where π is the embedding function given in (4). Therefore, we can induce two partial orderings "$\leq_\pi^1$" and "$\leq_\pi^2$" on $\mathcal{N}$ from $\pi(\mathcal{C}_\pi^1)$ and $\pi(\mathcal{C}_\pi^2)$, respectively. According to Definition 2.7, for $\tilde{a}, \tilde{b} \in \mathcal{F}_c(\mathbb{R})$, we can define $\tilde{a} \preceq_\pi^1 \tilde{b}$ (resp. $\tilde{a} \preceq_\pi^2 \tilde{b}$) if the Hukuhara difference $\tilde{b} \ominus_H \tilde{a}$ exists and $\tilde{b} \ominus_H \tilde{a} \in \mathcal{C}_\pi^1$ (resp. $\tilde{b} \ominus_H \tilde{a} \in \mathcal{C}_\pi^2$). We also have the order preserving property under the function π.

Proposition 2.9 (Order Preserving) *Let $\tilde{a}, \tilde{b} \in \mathcal{F}_c(\mathbb{R})$. Then, $\tilde{a} \preceq_\pi^1 \tilde{b}$ if and only if $\pi(\tilde{a}) \leq_\pi^1 \pi(\tilde{b})$, and $\tilde{a} \preceq_\pi^2 \tilde{b}$ if and only if $\pi(\tilde{a}) \leq_\pi^2 \pi(\tilde{b})$.*

Proof The results follow from the similar arguments as in Proposition 2.8. □

3 Fuzzy Multiobjective Programming Problems

Let X be a real vector space. The function $\tilde{f} : X \to \mathcal{F}_c(\mathbb{R})$ is called a canonical fuzzy-valued function defined on X. Now, we consider the following two fuzzy multiobjective programming problems:

$$\begin{array}{lll} \text{(FMOP1)} & \min & (\tilde{f}_1(x), \ldots, \tilde{f}_n(x)), \\ & \text{s.t.} & \tilde{g}_i(x) \preceq_\pi^1 \tilde{0}, \quad i = 1, 2, \ldots, m, \\ & & x \in X, \end{array}$$

and

$$\begin{array}{lll} \text{(FMOP2)} & \min & (\tilde{f}_1(x), \ldots, \tilde{f}_n(x)), \\ & \text{s.t.} & \tilde{g}_i(x) \preceq_\pi^2 \tilde{0}, \quad i = 1, 2, \ldots, m, \\ & & x \in X, \end{array}$$

where $\tilde{f}_j$ and $\tilde{g}_i$ are canonical fuzzy-valued functions defined on X for all $j = 1, \ldots, n$ and all $i = 1, \ldots, m$. Each problem will be solved with respect to its corresponding solution concept. The partial orderings $\preceq^1$ and $\preceq^2$ in Definition 2.7 will be used to tackle the fuzzy multiobjective function values $(\tilde{f}_1(x), \ldots, \tilde{f}_n(x))$ in problems (FMOP1) and (FMOP2), respectively.

Let us write $\underline{\tilde{f}}(x) = (\tilde{f}_1(x), \ldots, \tilde{f}_n(x))$. Then, we have

$$(\Pi \circ \underline{\tilde{f}})(x) = \Pi(\underline{\tilde{f}}(x)) = (\pi(\tilde{f}_1(x)), \ldots, \pi(\tilde{f}_n(x)) = ((\pi \circ \tilde{f}_1)(x), \ldots, (\pi \circ \tilde{f}_n)(x)).$$

Applying the embedding function π to problems (FMOP1) and (FMOP2) and using Propositions 2.8 and 2.9, it is reasonable to consider the following two corresponding multiobjective programming problems (MOP1) and (MOP2):

$$\begin{array}{lll} \text{(MOP1)} & \min & (\Pi \circ \underline{\tilde{f}})(x) = ((\pi \circ \tilde{f}_1)(x), \ldots, (\pi \circ \tilde{f}_n)(x)), \\ & \text{s.t.} & (\pi \circ \tilde{g}_i)(x) \leq^1_\pi \pi(\tilde{0}), \quad i = 1, 2, \ldots, m, \\ & & x \in X, \end{array}$$

and

$$\begin{array}{lll} \text{(MOP2)} & \min & (\Pi \circ \underline{\tilde{f}})(x) = ((\pi \circ \tilde{f}_1)(x), \ldots, (\pi \circ \tilde{f}_n)(x)), \\ & \text{s.t.} & (\pi \circ \tilde{g}_i)(x) \leq^2_\pi \pi(\tilde{0}), \quad i = 1, 2, \ldots, m, \\ & & x \in X, \end{array}$$

where $\pi(\tilde{0}) = [\![\tilde{0}, \tilde{0}]\!]$ is the zero element in the normed space $\mathcal{N}$. Since $(\Pi \circ \underline{\tilde{f}})(x) \in \mathcal{N}^n$ for $x \in X$, the partial orderings $\leq^1$ and $\leq^2$ induced from the convex cones $\Pi(\mathcal{C}^1)$ and $\Pi(\mathcal{C}^2)$, respectively, will be used to tackle the multiobjective function values in problems (MOP1) and (MOP2).

Although the minimization problems for fuzzy multiobjective programming problems are considered only in this paper, the maximization problems for fuzzy multiobjective programming problems can also be similarly treated including the solution concepts that will be described below.

Let us recall some solution concepts. A convex cone $\mathcal{C}_\mathcal{V}$ defining a partial ordering as described before in the real vector space $\mathcal{V}$ is also called an *ordering cone*. Let $\mathcal{S}$ be any subset of $\mathcal{V}$ endowed with a partial ordering $\leq$. Referring to Jahn [20], an element $x^* \in \mathcal{S}$ is called a *minimal element* of $\mathcal{S}$ if $x \leq x^*$ for $x \in \mathcal{S}$ then $x^* \leq x$. If the partial ordering $\leq$ is regarded as an ordering cone $\mathcal{C}_\mathcal{V}$, then an element $x^* \in \mathcal{S}$ is a minimal element of the set $\mathcal{S}$ if $(\{x^*\} + (-\mathcal{C}_\mathcal{V})) \cap \mathcal{S} \subseteq \{x^*\} + \mathcal{C}_\mathcal{V}$. Similarly, an element $x^* \in \mathcal{S}$ is called a *maximal element* of $\mathcal{S}$ if $x^* \leq x$ for $x \in \mathcal{S}$ then $x \leq x^*$. Equivalently, an element $x^* \in \mathcal{S}$ is a maximal element of the set $\mathcal{S}$ if $(\{x^*\} + \mathcal{C}_\mathcal{V}) \cap \mathcal{S} \subseteq \{x^*\} + (-\mathcal{C}_\mathcal{V})$.

Now we say that $x^* \in \mathcal{S}$ is a *strongly minimal element* of $\mathcal{S}$ if $x^* \leq x$ for all $x \in \mathcal{S}$, and x^* is a *strongly maximal element* of $\mathcal{S}$ if $x \leq x^*$ for all $x \in \mathcal{S}$. Equivalently, an element $x^* \in \mathcal{S}$ is a strongly minimal element of $\mathcal{S}$ if $\mathcal{S} \subseteq \{x^*\} + \mathcal{C}_\mathcal{V}$, and an element $x^* \in \mathcal{S}$ is a strongly maximal element of $\mathcal{S}$ if $\mathcal{S} \subseteq \{x^*\} + (-\mathcal{C}_\mathcal{V})$.

Let $\mathcal{S}$ be a nonempty subset of the real vector space $\mathcal{V}$. The set

$$\begin{aligned} \operatorname{int}(\mathcal{S}) = \{x \in \mathcal{S}\colon & \text{ for each } y \in \mathcal{V} \text{ there exists some } \zeta > 0 \\ & \text{with } x + \lambda y \in \mathcal{S} \text{ for all } \lambda \in [0, \zeta]\} \end{aligned}$$

is called an algebraic interior of $\mathcal{S}$. By referring to Jahn [20], an element $x^* \in \mathcal{S}$ is called a *weakly minimal element* of $\mathcal{S}$ if $\text{int}(\mathcal{C}_{\mathcal{V}}) \neq \emptyset$ and there does not exist an $x \in \mathcal{S}$ such that $x^* - x \in \text{int}(\mathcal{C}_{\mathcal{V}})$ or, equivalently, $(\{x^*\} + (-\text{int}(\mathcal{C}_{\mathcal{V}}))) \cap \mathcal{S} = \emptyset$. Similarly, an element $x^* \in \mathcal{S}$ is called a *weakly maximal element* of $\mathcal{S}$ if $(\{x^*\} + \text{int}(\mathcal{C}_{\mathcal{V}})) \cap \mathcal{S} = \emptyset$.

Definition 3.1 Let η be a linear defuzzification function on $\mathcal{F}_c(\mathbb{R}) \subset \mathcal{F}(\mathbb{R})$. We say that η is a *canonical linear defuzzification function* on $\mathcal{F}_c(\mathbb{R})$ if $\tilde{a} \in \mathcal{F}_c(\mathbb{R})$ and $\eta(\tilde{a}) = 0$ imply $\tilde{a} = \tilde{0}$.

Proposition 3.1 *Let* Π *be the function given in* (6).

(i) *If* η *is a canonical linear defuzzification function on* $\mathcal{F}_c(\mathbb{R})$, *then the set* $\Pi(\mathcal{C}^1)$ *is a pointed convex cone in* $\mathcal{N}^n$.
(ii) *The set* $\Pi(\mathcal{C}^2)$ *is a pointed convex cone in* $\mathcal{N}^n$.

Proof From Proposition 2.7, it suffices to show that

$$\Pi(\mathcal{C}^1) \cap (-\Pi(\mathcal{C}^1)) = (\pi(\tilde{0}), \ldots, \pi(\tilde{0})) = \Pi(\mathcal{C}^2) \cap (-\Pi(\mathcal{C}^2)),$$

where $(\pi(\tilde{0}), \ldots, \pi(\tilde{0}))$ is the zero element in the normed space $\mathcal{N}^n$.

(i) Let $\underline{s} \in \Pi(\mathcal{C}^1) \cap (-\Pi(\mathcal{C}^1))$. Then $\underline{s}, -\underline{s} \in \Pi(\mathcal{C}^1)$. Therefore, there exist $\underline{\tilde{u}}, \underline{\tilde{v}} \in \mathcal{C}^1$ such that $\Pi(\underline{\tilde{u}}) = \underline{s}$ and $\Pi(\underline{\tilde{v}}) = -\underline{s}$, i.e., $\pi(\tilde{u}^j) = s^j$ and $\pi(\tilde{v}^j) = -s^j$ for all $j = 1, \ldots, n$. By adding them together, we have $\pi(\tilde{u}^j \oplus \tilde{v}^j) = \pi(\tilde{u}^j) + \pi(\tilde{v}^j) = \pi(\tilde{0})$ (note that $\pi(\tilde{0})$ is the zero element of the normed space $\mathcal{N}$). Since π is one-to-one, we see that $\tilde{u}^j \oplus \tilde{v}^j = \tilde{0}$. Then we have $0 = \eta(\tilde{0}) = \eta(\tilde{u}^j \oplus \tilde{v}^j) = \eta(\tilde{u}^j) + \eta(\tilde{v}^j)$. We also have $\eta(\tilde{u}^j) \geq 0$ and $\eta(\tilde{v}^j) \geq 0$, since $\underline{\tilde{u}}, \underline{\tilde{v}} \in \mathcal{C}^1$. Therefore we obtain $\eta(\tilde{u}^j) = 0 = \eta(\tilde{v}^j)$. It shows that $\tilde{u}^j = \tilde{0} = \tilde{v}^j$ for all $j = 1, \ldots, n$, since η is a canonical linear defuzzification function on $\mathcal{F}_c(\mathbb{R})$. We conclude that $\underline{s} = (\pi(\tilde{0}), \ldots, \pi(\tilde{0}))$.

(ii) For the case of $\Pi(\mathcal{C}^2)$, from the proof of (i), we can also obtain $\tilde{u}^j \oplus \tilde{v}^j = \tilde{0}$ for all $j = 1, \ldots, n$. By Proposition 2.1, we have $0 = (\tilde{u}^j)^L_\alpha + (\tilde{v}^j)^L_\alpha = (\tilde{u}^j)^U_\alpha + (\tilde{v}^j)^U_\alpha$. Since $\underline{\tilde{u}}, \underline{\tilde{v}} \in \mathcal{C}^2$, we also have $(\tilde{u}^j)^L_\alpha \geq 0$, $(\tilde{v}^j)^L_\alpha \geq 0$, $(\tilde{u}^j)^U_\alpha \geq 0$ and $(\tilde{v}^j)^U_\alpha \geq 0$ by Remark 2.2. Therefore we obtain $0 = (\tilde{u}^j)^L_\alpha = (\tilde{v}^j)^L_\alpha = (\tilde{u}^j)^U_\alpha = (\tilde{v}^j)^U_\alpha$ for all $\alpha \in [0, 1]$ and all $j = 1, \ldots, n$. This completes the proof. □

Let Π be the function given in (6). In the sequel, we are going to show that $\text{int}(\Pi(\mathcal{C}^1)) \neq \emptyset$ and $\text{int}(\Pi(\mathcal{C}^2)) \neq \emptyset$ in order to obtain some interesting results. First of all, we need some useful lemmas given below.

Let $\tilde{a}$ be a fuzzy subset of $\mathbb{R}$ with membership function $\xi_{\tilde{a}}$ and $\tilde{a}_\alpha$ be the α-level sets of $\tilde{a}$ for $\alpha \in [0, 1]$. Zadeh [16–18] proved that the membership function $\xi_{\tilde{a}}$ can be expressed as

$$\xi_{\tilde{a}}(r) = \sup_{\alpha \in [0,1]} \alpha \cdot 1_{\tilde{a}_\alpha}(r),$$

where $1_{\tilde{a}_\alpha}$ is the indicator function of set $\tilde{a}_\alpha$. Conversely, we also have the following result.

Lemma 3.1 (Negoita and Ralescu [21]) *Let* A *be a set and let* $\{A_\alpha : \alpha \in [0, 1]\}$ *be a family of subsets of* A *such that the following conditions are satisfied*:

(a) $A_0 = A$;
(b) $A_\beta \subseteq A_\alpha$ *for* $\alpha < \beta$;
(c) $A_\alpha = \bigcap_{n=1}^{\infty} A_{\alpha_n}$ *for* $\alpha_n \uparrow \alpha$.

Then, the function $\xi : A \to [0,1]$ *defined by*

$$\xi(r) = \sup_{\alpha \in [0,1]} \alpha \cdot 1_{A_\alpha}(r)$$

has the property $A_\alpha = \{r \in A : \xi(r) \geq \alpha\}$.

Let $\tilde{a}$ be a fuzzy number. Then, we see that, for $\alpha < \beta$, $\tilde{a}_\alpha^L \leq \tilde{a}_\alpha^U$, $\tilde{a}_\alpha^L \leq \tilde{a}_\beta^L$ and $\tilde{a}_\alpha^U \geq \tilde{a}_\beta^U$. Now, we propose the following definition.

Definition 3.2 Let $\tilde{a}$ be a canonical fuzzy number. Then, $\tilde{a}$ is called a *standard fuzzy number* if $\tilde{a}_\alpha^L < \tilde{a}_\alpha^U$, $\tilde{a}_\alpha^L < \tilde{a}_\beta^L$ and $\tilde{a}_\alpha^U < \tilde{a}_\beta^U$ for $\alpha < \beta$.

We denote by $\mathcal{F}_s(\mathbb{R})$ the set of all standard fuzzy numbers.

Lemma 3.2 *Let* $\tilde{a} \in \mathcal{F}_s(\mathbb{R})$ *and let* $\tilde{c}, \tilde{d} \in \mathcal{F}_c(\mathbb{R})$. *There exist a* $\zeta > 0$ *and a standard fuzzy-valued function* $\tilde{b} : [0,\zeta] \to \mathcal{F}_s(\mathbb{R})$ *defined on* $[0,\zeta]$ *such that* $\tilde{a} \oplus \lambda\tilde{c} = \tilde{b}(\lambda) \oplus \lambda\tilde{d}$ *for all* $\lambda \in [0,\zeta]$.

Proof We can use Lemma 3.1 to construct the standard fuzzy-valued function $\tilde{b}$. □

Lemma 3.3 *Let* π *be the embedding function given in* (4) *and let* η *be a linear defuzzification function on* $\mathcal{F}_c(\mathbb{R})$. *If* $\tilde{a} \in \mathcal{F}_s(\mathbb{R})$ *with* $\eta(\tilde{a}) > 0$, *then, for each* $s \in \mathcal{N}$, *there exists some* $\zeta > 0$ *such that* $\pi(\tilde{a}) + \lambda s \in \pi(\mathcal{C}_\pi^1)$ *for all* $\lambda \in [0,\zeta]$.

Proof The result can be obtained by Lemma 3.2. □

Lemma 3.4 *Let* π *be the embedding function given in* (4). *If* $\tilde{a} \in \mathcal{F}_s(\mathbb{R})$ *and* $\tilde{a}$ *is positive, then, for each* $s \in \mathcal{N}$, *there exists some* $\zeta > 0$ *such that* $\pi(\tilde{a}) + \lambda s \in \pi(\mathcal{C}_\pi^2)$ *for all* $\lambda \in [0,\zeta]$.

Proof The result can be obtained by Lemma 3.2. □

In order to discuss the nonemptiness of $\mathrm{int}(\Pi(\mathcal{C}^1))$ and $\mathrm{int}(\Pi(\mathcal{C}^2))$, we introduce the triangular fuzzy numbers. The membership function of a triangular fuzzy number $\tilde{a} = (a^L, a, a^U)$ is defined by

$$\xi_{\tilde{a}}(r) = \begin{cases} (r - a^L)/(a - a^L), & \text{if } a^L \leq r \leq a, \\ (a^U - r)/(a^U - a), & \text{if } a < r \leq a^U, \\ 0, & \text{otherwise.} \end{cases}$$

The graph of $\tilde{a}$ is a triangle with base $[a^L, a^U]$ and peak a. The α-level set of $\tilde{a}$ is then

$$\tilde{a}_\alpha = [(1-\alpha)a^L + \alpha a, (1-\alpha)a^U + \alpha a];$$

that is,

$$\tilde{a}^L_\alpha = (1-\alpha)a^L + \alpha a \quad \text{and} \quad \tilde{a}^U_\alpha = (1-\alpha)a^U + \alpha a. \tag{7}$$

We also see that $-\tilde{a} = (-a^U, -a, -a^L)$. Furthermore, the triangular fuzzy number $\tilde{a}$ is also a standard fuzzy number, and if $a^L > 0$, then $\tilde{a}$ is positive.

Proposition 3.2 *Let Π be the function given in (6). If η is a linear defuzzification function on $\mathcal{F}_c(\mathbb{R})$ such that $\eta(\tilde{u}) > 0$ for some $\tilde{u} \in \mathcal{F}_s(\mathbb{R})$, then* $\text{int}(\Pi(\mathcal{C}^1)) \neq \emptyset$.

Proof Let $\tilde{u}^j \in \mathcal{F}_s(\mathbb{R})$ with $\eta(\tilde{u}^j) > 0$, where $\tilde{u}^j$, $j = 1, \ldots, n$, are not necessarily all distinct, which is possible from the hypothesis, since we can take $\tilde{u}^j = \tilde{u}$ for all $j = 1, \ldots, n$. Let $\underline{s} = (s_1, \ldots, s_n) \in \mathcal{N}^n$. From Lemma 3.3, there exists a $\zeta^j > 0$ such that $\pi(u^j) + \lambda s^j \in \pi(\mathcal{C}^1_\pi)$ for all $\lambda \in [0, \zeta^j]$. We define $\zeta = \min\{\zeta^1, \ldots, \zeta^n\}$. Then $\pi(u^j) + \lambda s^j \in \pi(\mathcal{C}^1_\pi)$ for all $\lambda \in [0, \zeta]$ and all $j = 1, \ldots, n$. From Remark 2.5, we see that $\Pi(\underline{\tilde{u}}) + \lambda \underline{s} \in \Pi(\mathcal{C}^1)$ for all $\lambda \in [0, \zeta]$, i.e., $\Pi(\underline{\tilde{u}}) \in \text{int}(\Pi(\mathcal{C}^1))$, where $\underline{\tilde{u}} = (\tilde{u}^1, \ldots, \tilde{u}^n)$. This completes the proof. □

Proposition 3.3 *Let Π be the function given in (6). Then* $\text{int}(\Pi(\mathcal{C}^2)) \neq \emptyset$.

Proof We shall apply Lemma 3.4 and the similar arguments as in the proof of Proposition 3.2. Since the triangular fuzzy numbers are also standard fuzzy numbers, it suffices to just consider the triangular fuzzy numbers in the proof. It will always be possible to take the positive triangular fuzzy numbers $\tilde{u}^j = (u^{jL}, u^j, u^{jU})$, i.e., $u^{jL} > 0$, $j = 1, \ldots, n$. Let $\underline{s} = (s_1, \ldots, s_n) \in \mathcal{N}^n$. Then the result follows immediately from Lemma 3.4 and the arguments of Proposition 3.2. □

Now, we let

$$X^1 = \{x \in X : (\pi \circ \tilde{g}_i)(x) \preceq^1_\pi \pi(\tilde{0}), i = 1, \ldots, m\}, \tag{8a}$$

$$\mathcal{S}^1 = \{(\Pi \circ \underline{\tilde{f}})(x) : x \in X^1\}, \tag{8b}$$

and

$$X^2 = \{x \in X : (\pi \circ \tilde{g}_i)(x) \preceq^2_\pi \pi(\tilde{0}), i = 1, \ldots, m\}, \tag{9a}$$

$$\mathcal{S}^2 = \{(\Pi \circ \underline{\tilde{f}})(x) : x \in X^2\}. \tag{9b}$$

Proposition 2.9 says that problems (FMOP1) and (MOP1) have the identical feasible sets. Similarly, problems (FMOP2) and (MOP2) also have the identical feasible sets. Since π is one-to-one, we propose the following definition.

Definition 3.3 Let us consider the convex cone $\Pi(\mathcal{C}^1)$.

(i) We say that x^* is a complete $\mathcal{C}^1$-optimal solution of problem (FMOP1) if $(\Pi \circ \underline{\tilde{f}})(x^*)$ is a *strongly minimal element* of the set $\mathcal{S}^1$ defined in (8b) under the convex cone $\Pi(\mathcal{C}^1)$.

(ii) We say that x^* is a Pareto $\mathcal{C}^1$-optimal solution of problem (FMOP1) if $(\Pi \circ \underline{\tilde{f}})(x^*)$ is a *minimal element* of the set $\mathcal{S}^1$ under the convex cone $\Pi(\mathcal{C}^1)$.

(iii) We say that x^* is a weak Pareto $\mathcal{C}^1$-optimal solution of problem (FMOP1) if $(\Pi \circ \underline{\tilde{f}})(x^*)$ is a *weakly minimal element* of the set $\mathcal{S}^1$ under the convex cone $\Pi(\mathcal{C}^1)$.

We can similarly define the solution concepts based on the convex cone $\Pi(\mathcal{C}^2)$ and problem (FMOP2) by referring to $\mathcal{S}^2$ and (9b).

Let X_1^{CO}, X_1^{P}, and X_1^{WP} denote the set of complete $\mathcal{C}^1$-optimal solutions, Pareto $\mathcal{C}^1$-optimal solutions and weak Pareto $\mathcal{C}^1$-optimal solutions of problem (FMOP1), respectively. We can similarly define the sets X_2^{CO}, X_2^{P} and X_2^{WP} based on the convex cone $\Pi(\mathcal{C}^2)$ and problem (FMOP2). Then we have the following interesting results.

Proposition 3.4 *Let the problems* (FMOP1) *and* (FMOP2) *be feasible.*

(i) *If η is a linear defuzzification function on $\mathcal{F}_c(\mathbb{R})$ such that $\eta(\tilde{a}) > 0$ for some $\tilde{a} \in \mathcal{F}_s(\mathbb{R})$, then $X_1^{\mathrm{CO}} \subseteq X_1^{\mathrm{P}} \subseteq X_1^{\mathrm{WP}}$.*
(ii) *We have $X_2^{\mathrm{CO}} \subseteq X_2^{\mathrm{P}} \subseteq X_2^{\mathrm{WP}}$.*

Proof (i) Since problem (FMOP1) is feasible, the set $\mathcal{S}^1$ defined in (8b) is nonempty. From Jahn [20, p. 104, Lemma 4.10], each strongly minimal element of $\mathcal{S}^1$ is also a minimal element of $\mathcal{S}^1$. Therefore $X_1^{\mathrm{CO}} \subseteq X_1^{\mathrm{P}}$. On the other hand, from Jahn [20, p. 106, Lemma 4.14], if $\Pi(\mathcal{C}^1) \neq \mathcal{N}$ and $\mathrm{int}(\Pi(\mathcal{C}^1)) \neq \emptyset$, then each minimal element of $\mathcal{S}^1$ is also a weakly minimal element of $\mathcal{S}^1$. Since $\Pi(\mathcal{C}^1) \neq \mathcal{N}$ is obvious by definition, the result follows immediately from Proposition 3.2.

(ii) Using Proposition 3.3, we can similarly prove this result. This completes the proof. □

4 Scalarization

We define $(\mathcal{N}^n)'$ to be the set of all linear functionals from $\mathcal{N}^n$ to $\mathbb{R}$. Then the set

$$\mathcal{C}^1_{(\mathcal{N}^n)'} = \{\phi \in (\mathcal{N}^n)' : \phi(\underline{s}) \geq 0 \text{ for all } \underline{s} \in \Pi(\mathcal{C}^1)\}$$

is also a convex cone and is called a dual cone for $\Pi(\mathcal{C}^1)$. The set defined by

$$(\mathcal{C}^1)^{\circ}_{(\mathcal{N}^n)'} = \{\phi \in (\mathcal{N}^n)' : \phi(\underline{s}) > 0 \text{ for all } \underline{s} \in \Pi(\mathcal{C}^1) \setminus \{\Pi(\tilde{0}, \dots, \tilde{0})\}\}$$

is called the quasi-interior of the dual cone for $\Pi(\mathcal{C}^1)$, where $\Pi(\tilde{0}, \dots, \tilde{0})$ is the zero element in the normed space $\mathcal{N}^n$. Similarly, we have

$$\mathcal{C}^2_{(\mathcal{N}^n)'} = \{\phi \in (\mathcal{N}^n)' : \phi(\underline{s}) \geq 0 \text{ for all } \underline{s} \in \Pi(\mathcal{C}^2)\}$$

and

$$(\mathcal{C}^2)^{\circ}_{(\mathcal{N}^n)'} = \{\phi \in (\mathcal{N}^n)' : \phi(\underline{s}) > 0 \text{ for all } \underline{s} \in \Pi(\mathcal{C}^2) \setminus \{\Pi(\tilde{0}, \dots, \tilde{0})\}\}.$$

Then, we have the following interesting results.

Theorem 4.1 *Let the problems* (MOP1) *and* (MOP2) *be feasible.*

(i) *Suppose that η is a linear defuzzification function on $\mathcal{F}_c(\mathbb{R})$ such that $\eta(\tilde{a}) > 0$ for some $\tilde{a} \in \mathcal{F}_s(\mathbb{R})$. If there exists a linear functional $\phi \in \mathcal{C}^1_{(\mathcal{N}^n)'} \setminus \{0_{(\mathcal{N}^n)'}\}$, where $0_{(\mathcal{N}^n)'}$ is the zero element of $(\mathcal{N}^n)'$, and an element $x^* \in X^1$ such that*

$$\phi((\Pi \circ \underline{\tilde{f}})(x^*)) \le \phi((\Pi \circ \underline{\tilde{f}})(x)), \quad \textit{for all } x \in X^1$$

then x^* is a weak Pareto $\mathcal{C}^1$-optimal solution.

(ii) *If there exists a linear functional $\phi \in \mathcal{C}^2_{(\mathcal{N}^n)'} \setminus \{0_{(\mathcal{N}^n)'}\}$ and an element $x^* \in X^2$ such that*

$$\phi((\Pi \circ \underline{\tilde{f}})(x^*)) \le \phi((\Pi \circ \underline{\tilde{f}})(x)), \quad \textit{for all } x \in X^2,$$

then x^* is a weak Pareto $\mathcal{C}^2$-optimal solution.

Proof From Jahn [20, p. 136, Theorem 5.28], if $\text{int}(\Pi(\mathcal{C}^1)) \neq \emptyset$ and there exists a linear functional $\phi \in \mathcal{C}^1_{(\mathcal{N}^n)'} \setminus \{0_{(\mathcal{N}^n)'}\}$ and an element $y^* \in \mathcal{S}^1$ with $\phi(y^*) \le \phi(y)$ for all $y \in \mathcal{S}^1$, then y^* is a weakly minimal element of $\mathcal{S}^1$. The results follow immediately from Propositions 3.2 and 3.3. □

Theorem 4.2 *Suppose that problem* (MOP1) *is feasible and η is a canonical linear defuzzification function on $\mathcal{F}_c(\mathbb{R})$.*

(i) *If there exists a linear functional $\phi \in \mathcal{C}^1_{(\mathcal{N}^n)'}$ and an element $x^* \in X^1$ such that*

$$\phi((\Pi \circ \underline{\tilde{f}})(x^*)) < \phi((\Pi \circ \underline{\tilde{f}})(x)), \quad \textit{for all } x \in X^1 \setminus \{x^*\},$$

then x^* is a Pareto $\mathcal{C}^1$-optimal solution.

(ii) *If there exists a linear functional $\phi \in (\mathcal{C}^1)^\circ_{(\mathcal{N}^n)'}$ and an element $x^* \in X^1$ such that*

$$\phi((\Pi \circ \underline{\tilde{f}})(x^*)) \le \phi((\Pi \circ \underline{\tilde{f}})(x)), \quad \textit{for all } x \in X^1,$$

then x^* is a Pareto $\mathcal{C}^1$-optimal solution.

Proof From Jahn [20, p. 128, Theorem 5.18], if $\Pi(\mathcal{C}^1)$ is a pointed convex cone, then we have:

(i) if there exists a linear functional $\phi \in \mathcal{C}^1_{(\mathcal{N}^n)'}$ and an element $y^* \in \mathcal{S}^1$ with $\phi(y^*) < \phi(y)$ for all $y \in \mathcal{S}^1 \setminus \{y^*\}$, then y^* is a minimal element of $\mathcal{S}^1$;

(ii) if there exists a linear functional $\phi \in (\mathcal{C}^1)^\circ_{(\mathcal{N}^n)'}$ and an element $y^* \in \mathcal{S}^1$ with $\phi(y^*) \le \phi(y)$ for all $y \in \mathcal{S}^1$, then y^* is a minimal element of $\mathcal{S}^1$.

The results follow immediately from Proposition 3.1 (i). □

Theorem 4.3 *Suppose that problem* (MOP2) *is feasible.*

(i) *If there exists a linear functional $\phi \in \mathcal{C}^2_{(\mathcal{N}^n)'}$ and an element $x^* \in X^2$ such that*

$$\phi((\Pi \circ \underline{\tilde{f}})(x^*)) < \phi((\Pi \circ \underline{\tilde{f}})(x)), \quad \textit{for all } x \in X^2 \setminus \{x^*\},$$

then x^* is a Pareto $\mathcal{C}^2$-optimal solution.

(ii) *If there exists a linear functional $\phi \in (\mathcal{C}^2)^{\circ}_{(\mathcal{N}^n)'}$ and an element $x^* \in X^2$ such that*

$$\phi((\Pi \circ \underline{\tilde{f}})(x^*)) \le \phi((\Pi \circ \underline{\tilde{f}})(x)), \quad \text{for all } x \in X^2,$$

then x^* is a Pareto $\mathcal{C}^2$-optimal solution.

Proof The results follow from Proposition 3.1 (ii) and the similar arguments in the proof of Theorem 4.2. □

Remark 4.1 Theorems 4.2 and 4.3 still hold true even when the objective functions $\tilde{f}_j$, $j = 1, \ldots, n$, and the constraint functions $\tilde{g}_i$, $i = 1, \ldots, m$, assume values on $\mathcal{F}(\mathbb{R})$ instead of $\mathcal{F}_c(\mathbb{R})$, since the nonemptiness of $\mathrm{int}(\Pi(\mathcal{C}^1))$ and $\mathrm{int}(\Pi(\mathcal{C}^2))$ are not needed in those two theorems.

5 Practical Problems

In order to interpret the constraints of problem (FMOP2), we consider the ordering $\pi(\tilde{a}) \preceq^2_{\pi} \pi(\tilde{0})$. Since $\pi(\tilde{0})$ is the zero element of the normed space $(\mathcal{N}, \|\cdot\|)$, by definition, we have $-\pi(\tilde{a}) = \pi(\tilde{0}) - \pi(\tilde{a}) \in \pi(\mathcal{C}^2_{\pi})$. Although $-\pi(\tilde{a}) \neq \pi((-1)\tilde{a})$ in general, we see that $-\pi(\tilde{a}) = \pi(\tilde{b})$ for some nonnegative $\tilde{b}$ in $\mathcal{C}^2_{\pi}$. By adding $\pi(\tilde{a})$ on both sides, we have $\pi(\tilde{0}) = \pi(\tilde{a}) + \pi(\tilde{b}) = \pi(\tilde{a} \oplus \tilde{b})$. Since π is one-to-one, we obtain $\tilde{a} \oplus \tilde{b} = \tilde{0}$. From Proposition 2.1, we conclude that $\tilde{a}^L_{\alpha} = -\tilde{b}^L_{\alpha} \le 0$ and $\tilde{a}^U_{\alpha} = -\tilde{b}^U_{\alpha} \le 0$ for all $\alpha \in [0, 1]$. Therefore, the feasible set X^2 of problem (FMOP2) shown in (9b) can be rewritten as

$$X^2 = \{x \in X : (\tilde{g}_i(x))^L_{\alpha} \le 0 \text{ and } (\tilde{g}_i(x))^U_{\alpha} \le 0, \text{ for } \alpha \in [0, 1] \text{ and } i = 1, \ldots, m\}. \tag{10}$$

Similarly, for problem (FMOP1), we can also obtain $\tilde{a} \oplus \tilde{b} = \tilde{0}$, where $\tilde{b} \in \mathcal{C}^1_{\pi}$, i.e., $\eta(\tilde{b}) \ge 0$, which implies $\eta(\tilde{a}) + \eta(\tilde{b}) = \eta(\tilde{a} \oplus \tilde{b}) = \eta(\tilde{0}) = 0$. Therefore, we have $\eta(\tilde{a}) \le 0$. It says that the feasible set X^1 of problem (FMOP1) shown in (8b) can be rewritten as

$$X^1 = \{x \in X : \eta(\tilde{g}_i(x)) \le 0, i = 1, \ldots, m\}. \tag{11}$$

In order to apply the previous theorems, we need to specify the linear functional $\phi : \mathcal{N}^n \to \mathbb{R}$. Here, we are going to define ϕ as

$$\phi([\![\tilde{a}_1, \tilde{b}_1]\!], \ldots, [\![\tilde{a}_n, \tilde{b}_n]\!]) = \sum_{i=1}^{n} \eta(\tilde{a}_i) - \sum_{i=1}^{n} \eta(\tilde{b}_i),$$

where η is a linear defuzzification function. Of course, we need to show that it is well-defined. Suppose that $(\tilde{c}_i, \tilde{d}_i) \in [\![\tilde{a}_i, \tilde{b}_i]\!]$ for $i = 1, \ldots, n$. By definition, we have $[\![\tilde{c}_i, \tilde{d}_i]\!] = [\![\tilde{a}_i, \tilde{b}_i]\!]$ and $\tilde{a}_i \oplus \tilde{d}_i = \tilde{b}_i \oplus \tilde{c}_i$, which also implies $\eta(\tilde{a}_i) - \eta(\tilde{b}_i) = \eta(\tilde{c}_i) -$

$\eta(\tilde{d}_i)$ for all $i = 1, \ldots, m$. Therefore, we see that

$$\begin{aligned}\phi([\![\tilde{a}_1, \tilde{b}_1]\!], \ldots, [\![\tilde{a}_n, \tilde{b}_n]\!]) &= \phi([\![\tilde{c}_1, \tilde{d}_1]\!], \ldots, [\![\tilde{c}_n, \tilde{d}_n]\!]) \\ &= \sum_{i=1}^{n} \eta(\tilde{a}_i) - \sum_{i=1}^{n} \eta(\tilde{b}_i) = \sum_{i=1}^{n} \eta(\tilde{c}_i) - \sum_{i=1}^{n} \eta(\tilde{d}_i),\end{aligned}$$

which says that ϕ is well-defined. The linearity of ϕ is also not difficult to prove. We omit the details. Since $\pi(\tilde{a}) = [\![\tilde{a}, \tilde{0}]\!]$, we also see that

$$f(x) \equiv \phi((\Pi \circ \underline{\tilde{f}})(x)) = \sum_{i=1}^{n} \eta(\tilde{f}_i(x)). \tag{12}$$

Therefore the previous theorems say that, in order to solve problem (FMOP2), we need to minimize the objective function f given in (12) subject to the feasible set X^2 shown in (10), which is a semi-infinite programming problem, since X^2 consists of infinite constraints. There are many semi-infinite programming algorithms available for solving this problem by referring to Hettich and Kortanek [22].

Now we illustrate a numerical example to solve problem (FMOP1). Suppose that we consider the triangular fuzzy number $\tilde{a} = (a - h, a, a + h)$ for some $h \in \mathbb{R}^+$, where $a \in \mathbb{R}$ is the core value of $\tilde{a}$. If we adopt the linear defuzzification function η in Example 2.1, then, according to (7), we obtain $\eta(\tilde{a}) = a$. We consider the following biobjective problem:

$$\begin{array}{ll}\min & (\tilde{f}_1(x_1, \ldots, x_5), \tilde{f}_2(x_1, \ldots, x_5)), \\ \text{s.t.} & \tilde{2}x_1 \oplus \tilde{3}x_2 \oplus \tilde{3}x_3 \oplus \tilde{2}x_4 \oplus \tilde{2}x_5 \oplus (\widetilde{-20}) \preceq_{\pi}^{1} \tilde{0}, \\ & \tilde{3}x_1 \oplus \tilde{5}x_2 \oplus \tilde{4}x_3 \oplus \tilde{2}x_4 \oplus \tilde{4}x_5 \oplus (\widetilde{-30}) \preceq_{\pi}^{1} \tilde{0}, \\ & x_1, x_2, x_3, x_4, x_5 \geq 0,\end{array}$$

where

$$\begin{aligned}\tilde{f}_1(x_1, \ldots, x_5) &= \tilde{2}x_1 \oplus \tilde{1}x_2 \oplus (\widetilde{-2})x_3 \oplus \tilde{2}x_4 \oplus (\widetilde{-9})x_5, \\ \tilde{f}_2(x_1, \ldots, x_5) &= (\widetilde{-7})x_1 \oplus (\widetilde{-9})x_2 \oplus \tilde{9}x_3 \oplus (\widetilde{-6})x_4 \oplus \tilde{3}x_5.\end{aligned}$$

According to (11) and (12), its corresponding scalar optimization problem is given by

$$\begin{array}{ll}\min & -5x_1 - 8x_2 - 7x_3 - 4x_4 - 6x_5, \\ \text{s.t.} & 2x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 \leq 20, \\ & 3x_1 + 5x_2 + 4x_3 + 2x_4 + 4x_5 \leq 30, \\ & x_1, x_2, x_3, x_4, x_5 \geq 0.\end{array}$$

The optimal solution of this problem is $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) = (0, 5, 0, 2.5, 0)$. Therefore, from Theorem 4.1, $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) = (0, 5, 0, 2.5, 0)$ is a weak Pareto $\mathcal{C}^1$-optimal solution.

6 Conclusions

Three sets of fuzzy numbers $\mathcal{F}(\mathbb{R})$, $\mathcal{F}_c(\mathbb{R})$ and $\mathcal{F}_s(\mathbb{R})$ with the relation $\mathcal{F}_s(\mathbb{R}) \subset \mathcal{F}_c(\mathbb{R}) \subset \mathcal{F}(\mathbb{R})$ are considered in this paper. The purpose of introducing the sets $\mathcal{F}_c(\mathbb{R})$ and $\mathcal{F}_s(\mathbb{R})$ is to guarantee the nonemptiness of $\text{int}(\Pi(\mathcal{C}^1))$ and $\text{int}(\Pi(\mathcal{C}^2))$. However, whether the nonemptiness of $\text{int}(\Pi(\mathcal{C}^1))$ and $\text{int}(\Pi(\mathcal{C}^2))$ still holds true based on the set $\mathcal{F}(\mathbb{R})$ instead of the set $\mathcal{F}_c(\mathbb{R})$ remains open. Although Theorems 4.2 and 4.3 are created on the set $\mathcal{F}_c(\mathbb{R})$, the arguments in the proofs are still valid when these are created on the set $\mathcal{F}(\mathbb{R})$ instead of the set $\mathcal{F}_c(\mathbb{R})$. The main reason is that the nonemptiness of $\text{int}(\Pi(\mathcal{C}^1))$ and $\text{int}(\Pi(\mathcal{C}^2))$ are not the conditions that guarantee Theorems 4.2 and 4.3.

As mentioned in the section of introduction, there are many existing methods available for solving the fuzzy multiobjective programming problem. These methods can be roughly classified into parametric programming approach, two-phase approach, interactive method, dynamic programming approach and differential equation approach. Since the solution concepts adopted by the researchers are different, these methods may be incomparable. For example, the solution obtained by applying the parametric programming approach may not be comparable with the solution obtained by the interactive method. The main reason is that the solution concepts adopted by these two methods are different, which also means that the parametric programming approach may not be used to obtain the solution that is based on the solution concept adopted by the interactive method. The proposed methodology in this paper is based on the solution concepts that are induced by the convex cones, which is also different from the existing methods. In other words, the proposed method in this paper is also incomparable with the existing methods.

The technique for solving fuzzy optimization problems using embedding theorem was proposed by Wu [13]. However, the solution concept in Wu [13] is different from the solution concept adopted in this paper. Therefore, these two methods are incomparable. Although the solution concept in Wu [7] was also based on the convex cone that is the same approach adopted in this paper, the difference is that the notions of convex cones are different. The reason is that the vector space adopted in Wu [7] consists of the equivalence classes obtained from the set of all fuzzy numbers, and the vector space adopted in this paper is based on the embedding theorem. This implies that these two methods are also incomparable.

References

1. Bellman, R.E., Zadeh, L.A.: Decision making in a fuzzy environment. Manage. Sci. **17**, 141–164 (1970)
2. Słowiński, R. (ed.): Fuzzy Sets in Decision Analysis, Operations Research and Statistics. Kluwer Academic, Dordrecht (1998)
3. Delgado, M., Kacprzyk, J., Verdegay, J.-L., Vila, M.A. (eds.): Fuzzy Optimization: Recent Advances. Physica-Verlag, Heidelberg (1994)
4. Zimmermann, H.-J.: Fuzzy Set Theory—And Its Applications, 3rd edn. Kluwer Academic, Dordrecht (1996)
5. Lai, Y.-J., Hwang, C.L.: Fuzzy Mathematical Programming: Methods and Applications. Lecture Notes in Economics and Mathematical Systems, vol. 394. Springer, Berlin (1992)

6. Lai, Y.-J., Hwang, C.L.: Fuzzy Multiple Objective Decision Making: Methods and Applications. Lecture Notes in Economics and Mathematical Systems, vol. 404. Springer, Berlin (1994)
7. Wu, H.-C.: A solution concept for fuzzy multiobjective programming problems based on convex cones. J. Optim. Theory Appl. **121**, 397–417 (2004)
8. Benson, H.P.: Existence of efficient solutions for vector maximization problems. J. Optim. Theory Appl. **26**, 569–580 (1978)
9. Benson, H.P.: An improved definition of proper efficiency for vector maximization with respect to cones. J. Math. Anal. Appl. **71**, 232–241 (1979)
10. Benson, H.P.: Efficiency and proper efficiency in vector maximization with respect to cones. J. Math. Anal. Appl. **93**, 273–289 (1983)
11. Borwein, J.M.: Proper efficient points for maximization with respect to cones. SIAM J. Control Optim. **15**, 57–63 (1977)
12. Yu, P.L.: Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives. J. Optim. Theory Appl. **14**, 319–377 (1974)
13. Wu, H.-C.: An (α, β)-optimal solution concept in fuzzy optimization problems. Optimization **53**, 203–221 (2004)
14. Puri, M.L., Ralescu, D.A.: Differentials of fuzzy functions. J. Math. Anal. Appl. **91**, 552–558 (1983)
15. Kaleva, O.: The calculus of fuzzy valued functions. Appl. Math. Lett. **3**, 55–59 (1990)
16. Zadeh, L.A.: The concept of linguistic variable and its application to approximate reasoning, I. Inf. Sci. **8**, 199–249 (1975)
17. Zadeh, L.A.: The concept of linguistic variable and its application to approximate reasoning, II. Inf. Sci. **8**, 301–357 (1975)
18. Zadeh, L.A.: The concept of linguistic variable and its application to approximate reasoning, III. Inf. Sci. **9**, 43–80 (1975)
19. Kreyszig, E.: Introductory Functional Analysis with Applications. Wiley, New York (1978)
20. Jahn, J.: Mathematical Vector Optimization in Partially Ordered Linear Spaces. Peter Lang, Frankfurt am Main (1986)
21. Negoita, C.V., Ralescu, D.A.: Applications of Fuzzy Sets to Systems Analysis. Wiley, New York (1975)
22. Hettich, R., Kortanek, K.O.: Semi-infinite programming: theory, methods and applications. SIAM Rev. **35**, 380–429 (1993)