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Stability of Indices in the KKT Conditions and Metric Regularity in Convex Semi-Infinite Optimization

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Abstract This paper deals with a parametric family of convex semi-infinite optimization problems for which linear perturbations of the objective function and continuous perturbations of the right-hand side of the constraint system are allowed. In this context, Cánovas et al. (SIAM J. Optim. 18:717–732, 2007) introduced a sufficient condition (called ENC in the present paper) for the strong Lipschitz stability of the optimal set mapping. Now, we show that ENC also entails high stability for the minimal subsets of indices involved in the KKT conditions, yielding a nice behavior not only for the optimal set mapping, but also for its inverse. Roughly speaking, points near optimal solutions are optimal for proximal parameters. In particular, this fact leads us to a remarkable simplification of a certain expression for the (metric) regularity modulus given in Cánovas et al. (J. Glob. Optim. 41:1–13, 2008) (and based on Ioffe (Usp. Mat. Nauk 55(3):103–162, 2000; Control Cybern. 32:543–554, 2003)), which provides a key step in further research oriented to find more computable expressions of this regularity modulus.

Keywords Convex semi-infinite programming · KKT conditions · Modulus of metric regularity

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1 Introduction

This paper is focused on the *modulus of metric regularity* of a multifunction associated with a parametrized family of convex semi-infinite optimization problems. Metric regularity constitutes a key concept in variational analysis (see, e.g., [5, 6], and [7]). The reader is also addressed to [8] and [3] for details about the regularity modulus of generic multifunctions.

The present work deals with the *optimal set mapping* (also called argmin mapping) and its inverse. The main goal of the paper consists of deriving formulae of the regularity modulus of this inverse mapping (see Theorem 5.1), which in our context coincides with the *Lipschitz modulus* of the argmin mapping. Along this paper, we call ENC—*extended Nürnberger condition*—to the property introduced originally in [1, Condition (10)] and whose first implications are gathered in Sect. 2.2.

We consider the canonically perturbed convex programming problem, in $\mathbb{R}^n$,

$$\mathrm{P}(c,b) \quad \mathrm{Inf} \quad f(x) + \langle c, x\rangle \tag{1a}$$

$$\text{s.t.} \quad g_t(x) \le b_t,\ t \in T, \tag{1b}$$

where $x \in \mathbb{R}^n$ is the vector of variables, $c \in \mathbb{R}^n$, $\langle .,.\rangle$ represents the usual inner product in $\mathbb{R}^n$, the index set T is a compact metric space, $f : \mathbb{R}^n \to \mathbb{R}$ and $g_t : \mathbb{R}^n \to \mathbb{R}$, $t \in T$, are convex functions, $(t, x) \mapsto g_t(x)$ is assumed to be continuous on $T \times \mathbb{R}^n$ (according to [9, Theorem 10.7], it is enough to require the continuity of each $t \mapsto g_t(x)$), and $b \in C(T,\mathbb{R})$ (i.e., $t \mapsto b_t$ is also continuous). In this setting, the pair $(c, b) \in \mathbb{R}^n \times C(T,\mathbb{R})$ is regarded as the parameter to be perturbed. The topology in the parameter space $\mathbb{R}^n \times C(T,\mathbb{R})$ is described by the norm

$$\|(c,b)\| := \max\{\|c\|, \|b\|_\infty\}, \tag{2}$$

where $\|\cdot\|$ is any given norm in $\mathbb{R}^n$ and $\|b\|_\infty := \max_{t\in T} |b_t|$. The dual norm $\|\cdot\|_*$ is given by $\|u\|_* := \max\{\langle u, x\rangle \mid \|x\| \le 1\}$, and d_* denotes the corresponding distance.

The *optimal set mapping* $\mathcal{F}^* : \mathbb{R}^n \times C(T,\mathbb{R}) \rightrightarrows \mathbb{R}^n$ assigns to each parameter $(c, b) \in \mathbb{R}^n \times C(T,\mathbb{R})$ the *optimal set*—set of optimal solutions—of $\mathrm{P}(c, b)$; i.e.,

$$\mathcal{F}^*(c,b) := \arg\min\{f(x) + \langle c, x\rangle \mid g_t(x) \le b_t,\ t \in T\}.$$

We set $\mathcal{G}^* := (\mathcal{F}^*)^{-1}$; i.e., $(c, b) \in (\mathcal{F}^*)^{-1}(x) \Leftrightarrow x \in \mathcal{F}^*(c, b)$.

Here, we recall some Lipschitz/regularity concepts and related results: $\mathcal{F}^*$ is *pseudo-Lipschitz* (satisfies the *Aubin property*) at $((\overline{c}, \overline{b}), \overline{x}) \in \mathrm{gph}(\mathcal{F}^*)$ (the graph of $\mathcal{F}^*$), or equivalently, $\mathcal{G}^*$ is *metrically regular* at $(\overline{x}, (\overline{c}, \overline{b})) \in \mathrm{gph}(\mathcal{G}^*)$ (see for instance [5]), if there exist a constant $\kappa \ge 0$ and some associated neighborhood U of $\overline{x}$ and V of $(\overline{c}, \overline{b})$ such that

$$d(x, \mathcal{F}^*(c,b)) \le \kappa d((c,b), \mathcal{G}^*(x)), \tag{3}$$

for all $x \in U$ and all $(c, b) \in V$, with the convention $d(x, \emptyset) = +\infty$. Observe that we might have considered without loss of generality (w.l.o.g.) $\overline{c} = 0_n$ and $\overline{b} = 0_T$ (zero function) just by replacing $f(\cdot)$ by $f(\cdot) + \langle\overline{c}, \cdot\rangle$ and each $g_t(\cdot)$ by $g_t(\cdot) - \overline{b}_t$.

In our context of problems (1) (even in more general contexts, see for instance [5]), the pseudo-Lipschitz property of $\mathcal{F}^*$ at $((\overline{c}, \overline{b}), \overline{x})$ is equivalent to the *strong Lipschitz stability* of $\mathcal{F}^*$ at this point (see Lemma 5 in [1]), which can be read as single-valuedness and Lipschitz continuity of $\mathcal{F}^*$ near $(\overline{c}, \overline{b})$ (since $\mathcal{F}^*(c, b)$ is convex). Under this property, the so-called *modulus of metric regularity* (*regularity modulus*, for short) of $\mathcal{G}^*$ at $(\overline{x}, (\overline{c}, \overline{b}))$, denoted by $\operatorname{reg} \mathcal{G}^*(\overline{x} \mid (\overline{c}, \overline{b}))$, coincides with the *Lipschitz modulus* of $\mathcal{F}^*$ at $(\overline{c}, \overline{b})$; i.e.,

$$\operatorname{reg} \mathcal{G}^*(\overline{x} \mid (\overline{c}, \overline{b})) = \limsup_{\substack{(c^1,b^1),(c^2,b^2)\to(\overline{c},\overline{b}) \\ (c^1,b^1)\neq(c^2,b^2)}} \frac{\|\mathcal{F}^*(c^1, b^1) - \mathcal{F}^*(c^2, b^2)\|}{\|(c^1, b^1) - (c^2, b^2)\|}, \tag{4}$$

where we are using the same notation for the set $\mathcal{F}^*(c, b)$ and its unique element, for (c, b) near $(\overline{c}, \overline{b})$.

The structure of the paper is as follows: Sect. 2 collects the preliminary concepts and results needed later. Section 3 shows that, for our purposes, under ENC, $\overline{c}$ may remain unperturbed. In Sect. 4, we formalize the idea that ENC entails a nice behavior of the minimal subsets of indices in the KKT conditions, which constitutes a key result in the sequel. Section 5 is devoted to simplify the expression for $\operatorname{reg} \mathcal{G}^*(\overline{x} \mid (\overline{c}, \overline{b}))$ provided in [2, Theorem 3].

2 Preliminaries

In this section, we provide further notation and some preliminary results.

2.1 Notations and Basic Concepts

Given $\emptyset \neq X \subset \mathbb{R}^k$, $k \in \mathbb{N}$, we denote by $\operatorname{co}(X)$ and $\operatorname{cone}(X)$ the *convex hull* and the *conical convex hull* of X, respectively. From the topological side, $\operatorname{int}(X)$ and $\operatorname{bd}(X)$ represent the *interior* and the *boundary* of X, respectively. If y is a point in any metric space, $B_\delta(y)$ and $\overline{B}_\delta(y)$ denote, respectively, the open and the closed ball centered at y with radius δ.

For all $b \in C(T, \mathbb{R})$, we consider the associated constraint system

$$\sigma(b) := \{g_t(x) \leq b_t,\ t \in T\}$$

and let $\mathcal{F}(b)$ be the corresponding set of feasible solutions. We consider also the set of *active indices* at $x \in \mathcal{F}(b)$,

$$T_b(x) := \{t \in T \mid g_t(x) = b_t\}.$$

Our system $\sigma(b)$ satisfies the *Slater constraint qualification* (SCQ) if $T_b(x^0)$ is empty for some $x^0 \in \mathcal{F}(b)$, in which case x^0 is referred to as a *Slater point* of $\sigma(b)$.

Next, we recall the well-known *Karush-Kuhn-Tucker* optimality conditions. For a convex function $h : \mathbb{R}^n \to \mathbb{R}$, $\partial h(x)$ denotes its ordinary subdifferential at x.

Lemma 2.1 [10, Chap. 7] *Let* $(c,b) \in \mathbb{R}^n \times C(T,\mathbb{R})$ *and* $x \in \mathcal{F}(b)$. *If*

$$(\partial f(x) + c) \cap \operatorname{cone}\Bigg(\bigcup_{t\in T_b(x)} (-\partial g_t(x))\Bigg) \neq \emptyset, \tag{5}$$

then $x \in \mathcal{F}^*(c,b)$. *The converse holds when* $\sigma(b)$ *satisfies SCQ.*

In this paper, we are concerned with those minimal subsets $D \subset T_b(x)$ such that (5) holds by replacing the whole $T_b(x)$ by D. According to the Carathéodory theorem, these minimal subsets have at most n elements. Along the paper, we appeal to the set of δ-*active indices* ($\delta \geq 0$) at $x \in \mathcal{F}(b)$,

$$T_b^\delta(x) := \{t \in T \mid g_t(x) \geq b_t - \delta\}.$$

2.2 Extended Nürnberger Condition and First Consequences

In [1], a sufficient condition for the metric regularity of $\mathcal{G}^*$ is introduced (see condition (10) therein). That paper points out that this sufficient condition is rather strong, but it has the virtue of being formulated exclusively in terms of the nominal problem's data, not involving problems in a neighborhood. When confined to the linear case (f and g_t's being linear functions), this condition turns out to be equivalent to the one introduced by Nürnberger in [11] (see, also, [12]) for characterizing the strong uniqueness of optimal solutions in a neighborhood of the nominal parameter. From now on, this condition, specified below, is referred to as the *extended Nürnberger condition* (extended in the sense that it is now stated for the convex case, and coincides with Nürnberger's condition for linear programs). Here, $|\cdot|$ means cardinality.

Definition 2.1 The *Extended Nürnberger Condition (ENC)* is said to be satisfied at $(\overline{x}, (\overline{c}, \overline{b})) \in \operatorname{gph}(\mathcal{G}^*)$ when

$$\sigma(\overline{b}) \text{ satisfies } SCQ \text{ and there is no } D \subset T_{\overline{b}}(\overline{x})$$
$$\text{with } |D| < n \text{ such that } (\partial f(\overline{x}) + \overline{c}) \cap \operatorname{cone}\Bigg(\bigcup_{t\in D}(-\partial g_t(\overline{x}))\Bigg) \neq \emptyset.$$

Note that ENC constitutes a specification of the KKT optimality conditions, since it additionally requires the presence of at least n active indices (i.e., exactly n active indices, taking the Carathéodory theorem into account).

Theorem 2.1 *For the convex program* (1), *let* $(\overline{x}, (\overline{c}, \overline{b})) \in \operatorname{gph}(\mathcal{G}^*)$. *If ENC is satisfied at* $(\overline{x}, (\overline{c}, \overline{b}))$, *then the following conditions hold*:

(i) [1, Proposition 9(i)] *There exists a neighborhood* U *of* $(\overline{x}, (\overline{c}, \overline{b}))$ *such that ENC is satisfied at any* $(x, (c,b)) \in U \cap \operatorname{gph}(\mathcal{G}^*)$.

(ii) [1, Proposition 9(ii)] *There exist* $u \in \partial f(\overline{x})$, $u_i \in -\partial g_{t_i}(\overline{x})$, $t_i \in T_{\overline{b}}(\overline{x})$, *and* $\lambda_i > 0$ *for* $i \in \{1,\ldots,n\}$, *such that* $\{u_1,\ldots,u_n\}$ *is a basis of* $\mathbb{R}^n$ *and*

$$u + \overline{c} = \sum_{i=1}^n \lambda_i u_i.$$

(iii) [1, Lemma 5 and Theorem 10] $\mathcal{G}^*$ *is metrically regular at* $(\overline{x}, (\overline{c}, \overline{b}))$; *or, equivalently,* $\mathcal{F}^*$ *is single-valued and Lipschitz continuous in a neighborhood of* $(\overline{c}, \overline{b})$.

Remark 2.1 [1, Theorem 16] In the linear case, ENC at $(\overline{x}, (\overline{c}, \overline{b})) \in \operatorname{gph}(\mathcal{G}^*)$ turns out to be equivalent to the metric regularity of $\mathcal{G}^*$ at $(\overline{x}, (\overline{c}, \overline{b}))$.

2.3 Variational Tools

Consider a function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and a point $z \in \mathbb{R}^n$ with $\varphi(z)$ finite. The *strong slope* [13] of φ at z is given by

$$|\nabla\varphi|(z) := \limsup_{\substack{y \to z \\ y \neq z}} \frac{(\varphi(z) - \varphi(y))^+}{\|z - y\|},$$

where $\alpha^+ := \max\{\alpha, 0\}$. A vector $v \in \mathbb{R}^n$ is called a *regular subgradient* (also called the Fréchet subgradient) of φ at z, written $v \in \widehat{\partial}\varphi(z)$, if

$$\varphi(y) \geq \varphi(z) + \langle v, y - z \rangle + o(\|y - z\|),$$

where $\lim_{\tau \searrow 0} \frac{o(\tau)}{\tau} = 0$ [7, Definition 8.3(a)]. The set $\widehat{\partial}\varphi(z)$ is closed and convex [7, Theorem 8.6] and coincides with the ordinary subdifferential set in convex analysis if φ is convex [7, Proposition 8.12].

The next result comes from [4, Theorem 2.2] and [3, Proposition 3 in p. 546].

Theorem 2.2 *Let Y be a Banach space and let $F : \mathbb{R}^n \rightrightarrows Y$ be a set-valued mapping with a nonempty closed graph. Let $(\overline{x}, \overline{y}) \in \operatorname{gph}(F)$ and assume that the functions*

$$\psi_y := d(y, F(\cdot))$$

are lower semicontinuous for all y in a neighborhood of $\overline{y}$. Then,

$$\begin{aligned} \operatorname{reg} F(\overline{x} \mid \overline{y}) &= \limsup_{\substack{(x,y) \to (\overline{x}, \overline{y}) \\ y \notin F(x)}} \left(d_*(0_n, \widehat{\partial}\psi_y(x))\right)^{-1} \\ &= \limsup_{\substack{(x,y) \to (\overline{x}, \overline{y}) \\ y \notin F(x)}} (|\nabla\psi_y|(x))^{-1}. \end{aligned}$$

We shall apply this result for writing $\operatorname{reg} \mathcal{G}^*(\overline{x} \mid (\overline{c}, \overline{b}))$ in terms of the minimal norm $d_*(0_n, \widehat{\partial} f_b(z))$, for (z, b) near $(\overline{x}, \overline{b})$, where

$$f_b(z) := d(b, \widetilde{\mathcal{G}}(z)) \tag{6}$$

and $\widetilde{\mathcal{G}}$ represents the inverse of $\widetilde{\mathcal{F}} := \mathcal{F}^*(\overline{c}, \cdot) : C(T, \mathbb{R}) \rightrightarrows \mathbb{R}^n$; i.e.,

$$b \in \widetilde{\mathcal{G}}(x) \Leftrightarrow x \in \widetilde{\mathcal{F}}(b) := \mathcal{F}^*(\overline{c}, b).$$

3 Negligibility of Perturbations of the Objective Function

This section is devoted to clarify the role played by the vector c in (1) in the calculus of $\operatorname{reg}\mathcal{G}^*(\overline{x} \mid (\overline{c}, \overline{b}))$, provided that ENC (see Definition 2.1) is fulfilled at $(\overline{x}, (\overline{c}, \overline{b})) \in \operatorname{gph}(\mathcal{G}^*)$. We actually show that the parameter c can be considered fixed in our analysis. The following lemma extends [14, Lemma 2] from the linear case to the convex setting.

Lemma 3.1 *Assume that ENC is satisfied at $(\overline{x}, (\overline{c}, \overline{b})) \in \operatorname{gph}(\mathcal{G}^*)$. If a sequence $\{(x^r, (c^r, b^r))\}_{r\in\mathbb{N}} \subset \operatorname{gph}(\mathcal{G}^*)$ converges to $(\overline{x}, (\overline{c}, \overline{b}))$, then $(x^r, (\overline{c}, b^r)) \in \operatorname{gph}(\mathcal{G}^*)$, for r large enough.*

Proof Let $\{(x^r, (c^r, b^r))\}_{r\in\mathbb{N}} \subset \operatorname{gph}(\mathcal{G}^*)$ converge to $(\overline{x}, (\overline{c}, \overline{b}))$. Since $\sigma(\overline{b})$ satisfies SCQ, we may assume w.l.o.g. that $\sigma(b^r)$ also does for all r. So, according to Lemma 2.1 and taking the Carathéodory theorem into account, we write, for each $r \in \mathbb{N}$,

$$u^r + c^r = \sum_{i=1}^{n} \lambda_i^r u_i^r, \tag{7}$$

where, for all $r \in \mathbb{N}$ and all $i \in \{1, \dots, n\}$,

$$u^r \in \partial f(x^r), \quad u_i^r \in -\partial g_{t_i^r}(x^r), \quad \text{for some } t_i^r \in T_{b^r}(x^r) \text{ and } \lambda_i^r \geq 0.$$

Next, we apply a filtering procedure. Since T is a compact metric space, $\{t_1^r\}$ has a subsequence converging to a certain $t_1 \in T_{\overline{b}}(\overline{x})$, taking the continuity of $(t, x) \mapsto g_t(x)$ into account. For simplicity, we denote the associated subsequence of r's as the whole sequence. In the same way, we obtain (after filtering $n-1$ times) $t_i^r \to t_i$, for certain $t_i \in T_{\overline{b}}(\overline{x})$, $i = 2, \dots, n$. Then, [9, Theorem 24.5] ensures the boundedness of the sequences of subgradients $\{u_i^r\}_{r\in\mathbb{N}}$, for all $i = 1, \dots, n$. The same theorem yields the boundedness of $\{u^r\}$. Thus, we may assume w.l.o.g. that $u^r \to u$, $u_i^r \to u_i$, for certain

$$u \in \partial f(\overline{x}), \quad u_i \in -\partial g_{t_i}(\overline{x}), \quad i = 1, \dots, n,$$

where we have applied [9, Theorem 24.4]. Moreover, the SCQ entails the boundedness of the sequence $\{\sum_{i=1}^n \lambda_i^r\}_{r\in\mathbb{N}}$ (Gauvin-type property). Otherwise, we would have, for a suitable subsequence, $\{\sum_{i=1}^n \lambda_i^{r_k}\}_{k\in\mathbb{N}} \to +\infty$; after dividing both sides of (7) by $\sum_{i=1}^n \lambda_i^{r_k}$ and letting $k \to +\infty$, we obtain

$$0_n \in \operatorname{conv}\{u_1, \dots, u_n\} \in \operatorname{conv}\Bigg(\bigcup_{t\in T_{\overline{b}}(\overline{x})} \partial g_t(\overline{x})\Bigg),$$

which represents a contradiction with the SCQ (see [1, Lemma 3]). Therefore, w.l.o.g., $\lambda_i^r \to \lambda_i$ for some $\lambda_i \geq 0$, $i = 1, \dots, n$, and so

$$u + \overline{c} = \sum_{i=1}^{n} \lambda_i u_i.$$

Now, ENC at $(\overline{x}, (\overline{c}, \overline{b}))$ entails that $\{u_1, \ldots, u_n\}$ is a basis of $\mathbb{R}^n$ (otherwise, the Carathéodory theorem would allow us to remove some term in $\sum_{i=1}^n \lambda_i u_i$, which would contradict ENC) and $\lambda_i > 0$ for all $i = 1, \ldots, n$. Therefore,

$$u + \overline{c} \in \operatorname{int}(\operatorname{cone}(\{u_1, \ldots, u_n\}))$$

(see for instance Theorem A.7 in [10]). Thus, for r large enough, we have

$$u^r + \overline{c} \in \operatorname{cone}(\{u_1^r, \ldots, u_n^r\})$$

(see, for instance, [10, Exercise 6.12]), and so, appealing again to Lemma 2.1 and recalling that $t_i^r \in T_{b^r}(x^r)$ for all i, we conclude that $x^r \in \mathcal{F}^*(\overline{c}, b^r)$. □

By exploiting similar ideas to [14, Proposition 4] (for the linear case), we obtain the following proposition, identifying $\mathcal{F}^*(\overline{c}, b)$ with its unique element for b near $\overline{b}$.

Proposition 3.1 *Assume that ENC is satisfied at* $(\overline{x}, (\overline{c}, \overline{b})) \in \operatorname{gph}(\mathcal{G}^*)$. *Then*,

$$\operatorname{reg} \mathcal{G}^*(\overline{x} \mid (\overline{c}, \overline{b})) = \limsup_{\substack{b, \widetilde{b} \to \overline{b} \\ b \neq \widetilde{b}}} \frac{\|\mathcal{F}^*(\overline{c}, b) - \mathcal{F}^*(\overline{c}, \widetilde{b})\|}{\|b - \widetilde{b}\|_\infty} = \operatorname{reg} \widetilde{\mathcal{G}}(\overline{x} \mid \overline{b}).$$

Proof The second equality comes from the fact that $\widetilde{\mathcal{F}}$ $(= \mathcal{F}^*(\overline{c}, \cdot))$ is single valued and Lipschitz continuous around $\overline{b}$, as a consequence of the metric regularity of $\mathcal{G}^*$ at $(\overline{x}, (\overline{c}, \overline{b}))$. According to (4), it is clear that $\operatorname{reg} \widetilde{\mathcal{G}}(\overline{x} \mid \overline{b}) \leq \operatorname{reg} \mathcal{G}^*(\overline{x} \mid (\overline{c}, \overline{b}))$. Moreover, in the nontrivial case when $\operatorname{reg} \mathcal{G}^*(\overline{x} \mid (\overline{c}, \overline{b}))$ is positive, we can write for certain $(c^r, b^r), (\widetilde{c}^r, \widetilde{b}^r) \to (\overline{c}, \overline{b})$, such that $(c^r, b^r) \neq (\widetilde{c}^r, \widetilde{b}^r)$, $r = 1, 2, \ldots,$

$$\begin{aligned} \operatorname{reg} \mathcal{G}^*(\overline{x} \mid (\overline{c}, \overline{b})) &= \limsup_{r \to \infty} \frac{\|\mathcal{F}^*(c^r, b^r) - \mathcal{F}^*(\widetilde{c}^r, \widetilde{b}^r)\|}{\|(c^r, b^r) - (\widetilde{c}^r, \widetilde{b}^r)\|} \\ &\leq \limsup_{r \to \infty} \frac{\|\mathcal{F}^*(\overline{c}, b^r) - \mathcal{F}^*(\overline{c}, \widetilde{b}^r)\|}{\|b^r - \widetilde{b}^r\|_\infty} \leq \operatorname{reg} \widetilde{\mathcal{G}}(\overline{x} \mid \overline{b}). \end{aligned}$$

In the previous expression, we have made use of (2) and the previous lemma, which yields $\mathcal{F}^*(c^r, b^r) = \mathcal{F}^*(\overline{c}, b^r)$ and $\mathcal{F}^*(\widetilde{c}^r, \widetilde{b}^r) = \mathcal{F}^*(\overline{c}, \widetilde{b}^r)$ for r large enough. Note that the assumption $\operatorname{reg} \mathcal{G}^*(\overline{x} \mid (\overline{c}, \overline{b})) > 0$ entails $b^r \neq \widetilde{b}^r$ for r large enough (because, otherwise, $\mathcal{F}^*(c^r, b^r) = \mathcal{F}^*(\overline{c}, b^r) = \mathcal{F}^*(\widetilde{c}^r, \widetilde{b}^r)$ for r large enough). □

The following example shows that the first equality in Proposition 3.1 may fail if ENC is not satisfied at $(\overline{x}, (\overline{c}, \overline{b}))$.

Example 3.1 Consider the following problem in $\mathbb{R}^2$, endowed with the Euclidean norm:

$$P(c, b) : \operatorname{Inf}\left\{\frac{1}{2}x_1^2 + c_1 x_1 + c_2 x_2 \mid |x_1| - x_2 \leq b\right\},$$

where $c := (c_1, c_2)$. For (c, b) close enough to $(\overline{c}, \overline{b}) = ((1, \frac{1}{2}), 0)$, we have

$$\mathcal{F}^*(c, b) = \{(c_2 - c_1, c_1 - c_2 - b)\}.$$

Hence, $\mathcal{F}^*$ is strongly Lipschitz stable at $((1, \frac{1}{2}), 0)$ and, so, according to Lemma 5 in [1], $\mathcal{G}^*$ is metrically regular at $(\overline{x}, (\overline{c}, \overline{b}))$ with $\overline{x} := (-\frac{1}{2}, \frac{1}{2})$. Trivially, ENC is not satisfied at $(\overline{x}, (\overline{c}, \overline{b}))$, since there is only one constraint in the model. Moreover,

$$\operatorname{reg} \widetilde{\mathcal{G}}(\overline{x} \mid \overline{b}) = \limsup_{\substack{b, \widetilde{b} \to 0 \\ b \neq \widetilde{b}}} \frac{\|(-\frac{1}{2}, \frac{1}{2} - b) - (-\frac{1}{2}, \frac{1}{2} - \tilde{b})\|}{\|b - \widetilde{b}\|_\infty} = 1,$$

whereas, considering the sequence $((1 + \frac{1}{r}, \frac{1}{2} - \frac{1}{r}), 0)$, for $r = 1, 2, \ldots,$ one has

$$\operatorname{reg} \mathcal{G}^*(\overline{x} \mid (\overline{c}, \overline{b})) \geq \lim_{r \to \infty} \frac{\|(-\frac{1}{2}, \frac{1}{2}) - (-\frac{1}{2} - \frac{2}{r}, \frac{1}{2} + \frac{2}{r})\|}{\|((1, \frac{1}{2}), 0) - ((1 + \frac{1}{r}, \frac{1}{2} - \frac{1}{r}), 0)\|} = 2.$$

4 Stability of Minimal Subsets of Indices in the KKT Conditions

This section points out the repercussions of ENC in relation to the stability of the indices involved in the KKT conditions and also to the stability of $\widetilde{\mathcal{G}}$. Roughly speaking, ENC yields high stability for both $\widetilde{\mathcal{F}}$ (which turns out to be strongly Lipschitz stable) and its inverse $\widetilde{\mathcal{G}}$. As we show in the following example, the strong Lipschitz stability of $\widetilde{\mathcal{F}}$ itself does not guarantee such a high stability as ENC does; in particular, it does not ensure that close points are optimal for close parameters (formally, $\widetilde{\mathcal{G}}$ may fail to be lower semicontinuous).

Example 4.1 Consider the following problem in $\mathbb{R}^2$, endowed with the Euclidean norm:

$$P(c, b) : \operatorname{Inf}\{c_1 x_1 + c_2 x_2 \mid |x_1| - x_2 \leq b_1,\ -x_1 \leq b_2\}.$$

Take $(\overline{c}, \overline{b}) = ((\frac{1}{2}, 1), (0, 0))$, $\overline{x} = (0, 0)$. One can easily check that $\widetilde{\mathcal{F}}(b) = \{((-b_2)^+, -b_1 + (-b_2)^+)\}$, and so $\widetilde{\mathcal{F}}$ is strongly Lipschitz stable at $\overline{b}$, although ENC is not satisfied at $(\overline{x}, (\overline{c}, \overline{b}))$, since $\overline{c} \in \operatorname{cone}(-\partial g_1(\overline{x}))$ (here $f \equiv 0$). Nevertheless, the point $(\frac{-1}{r}, \frac{1}{r})$, $r \in \mathbb{N}$, is not optimal for any b such that $(\frac{-1}{r}, \frac{1}{r}) \in \mathcal{F}(b)$ (see the expression of $\widetilde{\mathcal{F}}(b)$ above). Just as a motivation for further results, we point out that

$$\overline{c} \notin \operatorname{cone}\left(\bigcup_{i=1,2} (-\partial g_i(-1/r, 1/r)) \right), \quad r \in \mathbb{N},$$

which, roughly speaking, can be seen as a certain instability of the KKT representation of $\overline{c}$ with respect to perturbations of the point $\overline{x}$.

ENC precludes the previous situation (see Theorem 4.3). Now, we introduce additional tools in relation to KKT conditions for our model (1). Hereafter, the vector c remains unchanged ($c = \overline{c}$). So, the notation introduced below does not rely on c.

The following definition is intended to isolate the part of KKT conditions which depends only on the point x and the nonparametric elements of the model, i.e, the functions f and g_t's.

Definition 4.1 The subset $D \subset T$ is said to be a quasi-KKT set of indices at $x \in \mathbb{R}^n$ if

$$(\partial f(x) + \overline{c}) \cap \operatorname{cone}\left(\bigcup_{t \in D}(-\partial g_t(x))\right) \neq \varnothing.$$

Note that (under SCQ) x is a KKT point for $P(\overline{c}, b)$ if and only if $x \in \mathcal{F}(b)$ and $T_b(x)$ is a quasi-KKT set of indices at x. For $x \in \mathbb{R}^n$, we introduce the set

$$\mathcal{D}(x) := \{D : |D| = n \text{ and } D \text{ is a quasi-KKT set of indices at } x\}.$$

The ENC assumption at $(\overline{x}, (\overline{c}, \overline{b})) \in \operatorname{gph}(\mathcal{G}^*)$, together with the Carathéodory theorem, entails that $\mathcal{D}(\overline{x})$ consists of the minimal quasi-KKT sets of indices at $\overline{x}$.

For $x \in \mathcal{F}(b)$, we set

$$\mathcal{T}_b^\delta(x) := \{D \in \mathcal{D}(x) : D \subset T_b^\delta(x)\}, \text{ with } \delta \geq 0.$$

For simplicity, we write $\mathcal{T}_b(x)$ instead of $\mathcal{T}_b^0(x)$. Note that, as a consequence of ENC at $(\overline{x}, (\overline{c}, \overline{b})) \in \operatorname{gph}(\mathcal{G}^*)$, and according to Theorem 2.1(i), $\mathcal{T}_b(x) \neq \varnothing$ for $(x, b) \in \operatorname{gph}(\widetilde{\mathcal{G}})$ close enough to $(\overline{x}, \overline{b})$. The following theorem shows that, in Theorem 2.1(ii), we can replace everywhere "there exist" by "for all".

Theorem 4.1 *Assume that* ENC *is satisfied at* $(\overline{x}, (\overline{c}, \overline{b})) \in \operatorname{gph}(\mathcal{G}^*)$. *Then, for all* $D := \{t_1, \ldots, t_n\} \in \mathcal{T}_{\overline{b}}(\overline{x})$ *and every* $u_i \in -\partial g_{t_i}(\overline{x})$, $i = 1, \ldots, n$, *one has*

$$\partial f(\overline{x}) + \overline{c} \subset \operatorname{int}(\operatorname{cone}(\{u_1, \ldots, u_n\})).$$

Proof Let us fix an arbitrary $D := \{t_1, \ldots, t_n\} \in \mathcal{T}_{\overline{b}}(\overline{x})$. Along the proof, we will frequently appeal to the fact that $\operatorname{cone}(\bigcup_{i=1}^n(-\partial g_{t_i}(\overline{x})))$ is a pointed closed convex cone, since $0_n \notin \operatorname{co}(\bigcup_{i=1}^n \partial g_{t_i}(\overline{x}))$, which is derived from the fact that $\sigma(\overline{b})$ satisfies the Slater condition (see [1, Lemma 3(v)]). The proof is built in three steps.

Step 1 Let us prove the existence of $\overline{u}_i \in -\partial g_{t_i}(\overline{x})$, $i = 1, \ldots, n$, such that

$$\partial f(\overline{x}) + \overline{c} \subset \operatorname{int}(\operatorname{cone}(\{\overline{u}_1, \ldots, \overline{u}_n\})). \tag{8}$$

The fulfillment of ENC at $(\overline{x}, (\overline{c}, \overline{b}))$ yields, according to Theorem 2.1(ii), the existence of $\overline{u} \in \partial f(\overline{x})$, $\overline{u}_i \in -\partial g_{t_i}(\overline{x})$ and $\overline{\mu}_i > 0$, $i = 1, \ldots, n$, such that $\{\overline{u}_1, \ldots, \overline{u}_n\}$ is a basis of $\mathbb{R}^n$ and $\overline{u} + \overline{c} = \sum_{i=1}^n \overline{\mu}_i \overline{u}_i$, which leads us to $\overline{u} + \overline{c} \in \operatorname{int}(\operatorname{cone}(\{\overline{u}_1, \ldots, \overline{u}_n\}))$ (see, for instance, [10, Theorem A.7]). Then, one can easily check that (8) holds since otherwise, because of the convexity of $\partial f(\overline{x}) + \overline{c}$, there would exist $\widetilde{u} \in \partial f(\overline{x})$ such that $\widetilde{u} + \overline{c} \in \operatorname{bd}(\operatorname{cone}(\{\overline{u}_1, \ldots, \overline{u}_n\}))$, leading us to a representation of $\widetilde{u} + \overline{c}$ which contradicts ENC (applying again [10, Theorem A.7]).

Step 2 Consider, according to the previous step, an arbitrarily fixed $u \in \partial f(\overline{x})$ and take the same vectors $\overline{u}_i \in -\partial g_{t_i}(\overline{x})$, $i = 1, \dots, n$, as before, and certain $\mu_1, \dots, \mu_n > 0$ (positive because of ENC) such that

$$\sum_{i=1}^{n} \mu_i \overline{u}_i - (u + \overline{c}) = 0_n. \tag{9}$$

In this step, we prove that, for every $u_1 \in -\partial g_{t_1}(\overline{x})$, the following condition holds:

$$u + \overline{c} \in \operatorname{cone}(\{u_1, \overline{u}_2, \dots, \overline{u}_n\}). \tag{10}$$

Reasoning by contradiction, we assume that, for a certain $\widetilde{u}_1 \in -\partial g_{t_1}(\overline{x})$, (10) does not hold. Then from the separation theorem, and since $\operatorname{cone}(\{\widetilde{u}_1, \overline{u}_2, \dots, \overline{u}_n\})$ is pointed, closed and convex, there exists $w \in \mathbb{R}^n \setminus \{0\}$ with

$$\langle u + \overline{c}, w\rangle < 0, \qquad \langle \widetilde{u}_1, w\rangle > 0, \quad \text{and} \quad \langle \overline{u}_i, w\rangle > 0, \quad \text{for } i = 2, \dots, n.$$

(Obviously, $\widetilde{u}_1 \neq 0_n$ because $0_n \notin \operatorname{co}(\bigcup_{i=1}^n \partial g_{t_i}(\overline{x}))$.) For an appropriate choice of positive scalars λ_i, $i = 1, 2, \dots, n$ (which are in fact determined by the equations below), we can guarantee that

$$\langle u + \overline{c} + \lambda_1 \widetilde{u}_1, w\rangle = \langle u + \overline{c} + \lambda_i \overline{u}_i, w\rangle = 0, \quad \text{for } i = 2, \dots, n,$$

so that the set $\{u + \overline{c} + \lambda_1 \widetilde{u}_1; u + \overline{c} + \lambda_i \overline{u}_i, i = 2, \dots, n\}$ is linearly dependent. Let us then consider n scalars $\alpha_1, \dots, \alpha_n$, not all zero, satisfying

$$\alpha_1 \lambda_1 \widetilde{u}_1 + \sum_{i=2}^{n} \alpha_i \lambda_i \overline{u}_i + \sum_{i=1}^{n} \alpha_i (u + \overline{c}) = 0_n. \tag{11}$$

We can assume w.l.o.g. that $\alpha_1 \geq 0$ (otherwise, just multiply both sides of (11) by -1). Moreover, at least one of the scalars α_i, for $i \in \{2, \dots, n\}$, should be negative, because otherwise we would have $\sum_{i=1}^n \alpha_i > 0$, in which case (11) leads us to the contradiction $-(u + \overline{c}) \in \operatorname{cone}(\bigcup_{i=1}^n (-\partial g_{t_i}(\overline{x})))$ (recall that $\operatorname{cone}(\bigcup_{i=1}^n (-\partial g_{t_i}(\overline{x})))$ is a closed convex pointed cone containing $u + \overline{c}$).

Now, we consider

$$\gamma := \min\left\{ \frac{-\mu_i}{\alpha_i \lambda_i} \,\middle|\, \alpha_i < 0 \right\} > 0.$$

Note that $\gamma = -\mu_{i_0}/(\alpha_{i_0} \lambda_{i_0})$ for some $i_0 \in \{2, \dots, n\}$. Then, multiplying in (11) by γ and adding (9), we obtain

$$\gamma \alpha_1 \lambda_1 \widetilde{u}_1 + \mu_1 \overline{u}_1 + \sum_{\substack{i=2 \\ i \neq i_0}}^{n} (\gamma \alpha_i \lambda_i + \mu_i) \overline{u}_i + \left(\gamma \sum_{i=1}^{n} \alpha_i - 1 \right) (u + \overline{c}) = 0_n, \tag{12}$$

with $\gamma \alpha_i \lambda_i + \mu_i \geq 0$ for all $i = 2, \dots, n$.

Now we distinguish three cases:

(i) If $\gamma \sum_{i=1}^{n} \alpha_i - 1 = 0$, we attain the contradiction

$$0_n \in \text{co}\Bigg(\bigcup_{i\in\{1,\dots,n\}} (-\partial g_{t_i}(\overline{x})) \Bigg).$$

(ii) If $\gamma \sum_{i=1}^{n} \alpha_i - 1 > 0$, then

$$-(u+\overline{c}) \in \text{cone}\Bigg(\bigcup_{i\in\{1,\dots,n\}\setminus\{i_0\}} (-\partial g_{t_i}(\overline{x})) \Bigg),$$

which again contradicts the pointedness of $\text{cone}(\bigcup_{i=1}^{n} (-\partial g_{t_i}(\overline{x})))$.

(iii) Finally, if one has $\gamma \sum_{i=1}^{n} \alpha_i - 1 < 0$, then

$$u+\overline{c} \in \text{cone}\Bigg(\bigcup_{i\in\{1,\dots,n\}\setminus\{i_0\}} (-\partial g_{t_i}(\overline{x})) \Bigg),$$

contradicting ENC.

Step 3 From the previous step, (10) holds for arbitrarily chosen $u \in \partial f(\overline{x})$ and $u_1 \in -\partial g_1(\overline{x})$. Moreover, ENC and the Carathéodory theorem ensure that $\{u_1, \overline{u}_2, \dots, \overline{u}_n\}$ is a basis of $\mathbb{R}^n$ and the associated coefficients generating $u+\overline{c}$ are positive. Then, we can apply Step 2 to replace $\overline{u}_2$ by an arbitrary $u_2 \in -\partial g_2(\overline{x})$. By repeating this procedure, we finish the proof. □

Along the paper, we claim that ENC provides high stability for the minimal subsets of indices involved in the KKT conditions. This is formalized in the following theorem. In particular, condition (i) below yields $\mathcal{T}_{\overline{b}}(\overline{x}) \subset \mathcal{D}(x)$ for all x in a neighborhood of $\overline{x}$ (since $\mathcal{T}_{\overline{b}}(\overline{x}) \subset \mathcal{T}_{\overline{b}}^{\delta}(\overline{x})$ for all $\delta > 0$). In other words, those minimal subsets of indices in the KKT conditions at $\overline{x}$ (for $P(\overline{c}, \overline{b})$) are also quasi-KKT sets of indices at every x in a neighborhood of $\overline{x}$. This fact is crucial for establishing that points near $\overline{x}$ are optimal for some $(\overline{c}, b)$ with b near $\overline{b}$ (see Theorem 4.3).

Theorem 4.2 *Assume that* ENC *is satisfied at* $(\overline{x}, (\overline{c}, \overline{b})) \in \text{gph}(\mathcal{G}^*)$. *Then, the following conditions hold*:

(i) *There exist* $\delta_0 > 0$ *and a neighborhood* U_0 *of* $\overline{x}$ *such that*

$$\mathcal{T}_{\overline{b}}^{\delta_0}(\overline{x}) \subset \mathcal{D}(x), \quad \textit{for all } x \in U_0.$$

(ii) *For every* $\delta > 0$, *there exist neighborhoods* U_δ *of* $\overline{x}$ *and* V_δ *of* $\overline{b}$ *such that*

$$\mathcal{T}_b(x) \subset \mathcal{T}_{\overline{b}}^{\delta}(\overline{x}), \quad \textit{for all } x \in U_\delta \quad \textit{and all} \quad b \in V_\delta.$$

Proof

(i) We proceed by contradiction, supposing that there are sequences $\delta_r \searrow 0$, $x^r \to \overline{x}$, and $D^r := \{t_1^r, \dots, t_n^r\} \in \mathcal{T}_{\overline{b}}^{\delta_r}(\overline{x}) \setminus \mathcal{D}(x^r)$, $r = 1, 2, \dots,$ and so

$$(\partial f(x^r) + \overline{c}) \cap \operatorname{cone}\left(\bigcup_{i=1}^{n}(-\partial g_{t_i^r}(x^r))\right) = \varnothing, \quad \text{for } r = 1, 2, \dots. \tag{13}$$

From $D^r \in \mathcal{T}_{\overline{b}}^{\delta_r}(\overline{x}) \subset \mathcal{D}(\overline{x})$, we conclude the existence of $u_i^r \in -\partial g_{t_i^r}(\overline{x})$, $\lambda_i^r \geq 0$, with $i = 1, \dots, n$, and $u^r \in \partial f(\overline{x})$ such that

$$u^r + \overline{c} = \sum_{i=1}^{n} \lambda_i^r u_i^r. \tag{14}$$

We may assume w.l.o.g. that $t_i^r \to t_i \in T_{\overline{b}}(\overline{x})$ (recall that $\overline{b}_{t_i^r} - \delta_r \leq g_{t_i^r}(\overline{x}) \leq \overline{b}_{t_i^r}$ for all r), for $i = 1, \dots, n$, so that [9, Theorem 24.5] entails, for a suitable subsequence of r's, $u_i^r \to u_i \in -\partial g_{t_i}(\overline{x})$, $u^r \to u \in \partial f(\overline{x})$. Moreover, SCQ allows us to assume (for a new subsequence, in the line of the proof of Lemma 3.1) that $\lambda_i^r \to \lambda_i$, $i = 1, \dots, n$. Hence, setting $D := \{t_1, \dots, t_n\}$ and letting $r \to \infty$ in (14), we have

$$u + \overline{c} = \sum_{i=1}^{n} \lambda_i u_i.$$

Thus, $D \in \mathcal{T}_{\overline{b}}(\overline{x})$.

Now, let us consider, for each r, arbitrarily chosen $v_i^r \in -\partial g_{t_i^r}(x^r)$, $i = 1, \dots, n$, and $v^r \in \partial f(x^r)$. Applying again [9, Theorem 24.5], we may assume (for suitable subsequences) $v^r \to v \in \partial f(\overline{x})$, $v_i^r \to v_i \in -\partial g_{t_i}(\overline{x})$, $i = 1, \dots, n$. Applying Theorem 4.1, we obtain $v + \overline{c} \in \operatorname{int}(\operatorname{cone}(\{v_1, \dots, v_n\}))$. Then, for r large enough, we have $v^r + \overline{c} \in \operatorname{int}(\operatorname{cone}(\{v_1^r, \dots, v_n^r\}))$ (see [10, Exercise 6.12]). This contradicts (13).

(ii) Reasoning by contradiction, assume the existence of $\delta_0 > 0$ and sequences $x^r \to \overline{x}$ and $b^r \to \overline{b}$ such that $\mathcal{T}_{b^r}(x^r) \not\subset \mathcal{T}_{\overline{b}}^{\delta_0}(\overline{x})$ for all r. Let

$$D^r := \{t_1^r, \dots, t_n^r\} \in \mathcal{T}_{b^r}(x^r) \setminus \mathcal{T}_{\overline{b}}^{\delta_0}(\overline{x}), \quad r = 1, 2, \dots. \tag{15}$$

We may assume w.l.o.g. that $t_i^r \to t_i \in T$, for $i = 1, \dots, n$. So, the fact that $D^r \subset T_{b^r}(x^r)$ for all r, implies $t_i \in T_{\overline{b}}(\overline{x})$, for all $i = 1, \dots, n$, appealing once more to the continuity of $(t, x) \mapsto g_t(x)$. So, we have $D^r \subset T_{\overline{b}}^{\delta_0}(\overline{x})$ for r large enough (assume for all r). Thus, (15) necessarily implies

$$(\partial f(\overline{x}) + \overline{c}) \cap \operatorname{cone}\left(\bigcup_{i=1}^{n}(-\partial g_{t_i^r}(\overline{x}))\right) = \varnothing, \quad \text{for all } r. \tag{16}$$

Take any $u \in \partial f(\overline{x})$ and, for each r and each i, consider any $u_i^r \in -\partial g_{t_i^r}(\overline{x})$. From (16), we have

$$u + \overline{c} \notin \operatorname{cone}(\{u_1^r, \dots, u_n^r\}), \quad r = 1, 2, \dots.$$

Moreover, [9, Theorem 24.5] entails, for a suitable subsequence of r's, $u_i^r \to u_i \in -\partial g_{t_i}(\overline{x})$. Note that, again from [10, Exercise 6.12], one has

$$u + \overline{c} \notin \operatorname{int}(\operatorname{cone}(\{u_1, \dots, u_n\})).$$

Now, we shall prove that $D := \{t_1, \dots, t_n\} \in \mathcal{T}_{\overline{b}}(\overline{x})$, which represents a contradiction with Theorem 4.1. From one side, $D \subset T_{\overline{b}}(\overline{x})$. Moreover, since $D^r \in \mathcal{T}_{b^r}(x^r)$, we can find, for each r, $v^r \in \partial f(x^r)$, $v_i^r \in -\partial g_{t_i^r}(x^r)$ for all i, and $\lambda_i^r \geq 0$, such that $v^r + \overline{c} = \sum_{i=1}^n \lambda_i^r v_i^r$. Again, from [9, Theorem 24.5], and taking SCQ into account, we may assume $v^r \to v \in \partial f(\overline{x})$, $v_i^r \to v_i \in -\partial g_{t_i}(\overline{x})$, $\lambda_i^r \to \lambda_i \geq 0$, $i = 1, \dots, n$. Hence, $v + \overline{c} = \sum_{i=1}^n \lambda_i v_i$ and then $D \in \mathcal{T}_{\overline{b}}(\overline{x})$. □

Remark 4.1 Observe that Example 4.1 also shows that ENC cannot be removed as an assumption in the previous theorem. Specifically, $\mathcal{D}(\frac{-1}{r}, \frac{1}{r}) = \emptyset$ for all $r \in \mathbb{N}$, whereas for every $\delta > 0$ one has $\mathcal{T}^{\delta}_{(0,0)}(0, 0) = \{\{1, 2\}\}$.

Theorems 4.1 and 4.2 allow us to establish the *lower* (*or inner*) *semicontinuity* of $\widetilde{\mathcal{G}}$ at $(\overline{x}, \overline{b}) \in \operatorname{gph}(\widetilde{\mathcal{G}})$ in the following sense (see for instance [6, Definition 1.63(i)]): For all neighborhood V of $\overline{b}$, there exists a neighborhood U of $\overline{x}$ such that

$$\widetilde{\mathcal{G}}(x) \cap V \neq \emptyset, \quad \text{for all } x \in U.$$

We need the following technical lemma, whose proof follows from standard arguments of continuous functions defined on compact spaces.

Lemma 4.1 *For each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\max_{t \in T} |g_t(x) - g_t(\overline{x})| < \varepsilon, \quad \textit{for all } x \in B_\delta(\overline{x}). \tag{17}$$

Theorem 4.3 *Assume that ENC is satisfied at $(\overline{x}, (\overline{c}, \overline{b})) \in \operatorname{gph}(\mathcal{G}^*)$. Then, $\widetilde{\mathcal{G}}$ is lower semicontinuous at $(\overline{x}, \overline{b})$.*

Proof Take any neighborhood V of $\overline{b}$. We show the existence of a neighborhood U of $\overline{x}$ such that $\widetilde{\mathcal{G}}(x) \cap V \neq \emptyset$ for all $x \in U$. Fix $D \in \mathcal{T}_{\overline{b}}(\overline{x})$, take $\varepsilon > 0$ such that $B_\varepsilon(\overline{b}) \subset V$, and define $U := B_\delta(\overline{x})$ where δ is such that $\|x - \overline{x}\| < \delta$ implies

$$D \in \mathcal{D}(x) \quad \text{and} \quad \max_{t \in T} |g_t(x) - g_t(\overline{x})| < \varepsilon.$$

The existence of such a δ comes from Theorem 4.2(i) and the previous lemma. In particular, since $D \subset T_{\overline{b}}(\overline{x})$, we have $\max_{t \in D} |g_t(x) - \overline{b}_t| < \varepsilon$.

Now, let us prove that, for each $x \in U$, we can find a continuous function $b \in \widetilde{\mathcal{G}}(x) \cap V$ and so this set is nonempty. Fix $x \in U$ and define

$$b_t := \varphi(t) g_t(x) + (1 - \varphi(t)) \overline{b}_t, \quad \text{for } t \in T,$$

where $\varphi : T \to [0,1]$ is defined by virtue of Urysohn's lemma as a continuous function satisfying

$$\varphi(t) := \begin{cases} 1, & \text{if } t \in D \text{ or } g_t(x) \geq \overline{b}_t, \\ 0, & \text{if } g_t(x) \leq \overline{b}_t - \varepsilon. \end{cases}$$

(When $\{t \in T : g_t(x) \leq \overline{b}_t - \varepsilon\} = \emptyset$, then we take $\varphi \equiv 1$.) Now, we check that $b \in \widetilde{\mathcal{G}}(x)$ and $\|b - \overline{b}\|_\infty < \varepsilon$. To start with, note that the definition of φ implies

$$g_t(x) - b_t = (1 - \varphi(t))(g_t(x) - \overline{b}_t) \leq 0,$$

which states the feasibility of x; i.e., $x \in \mathcal{F}(b)$. Moreover, observe that $b_t = g_t(x)$, for $t \in D$, as well as $D \in \mathcal{D}(x)$, which provide the KKT optimality conditions, entailing $b \in \widetilde{\mathcal{G}}(x)$ (see Lemma 2.1). Moreover, $|b_t - \overline{b}_t| = \varphi(t)|g_t(x) - \overline{b}_t|$ and, in the nontrivial case $\varphi(t) > 0$, we have

$$-\varepsilon < g_t(x) - \overline{b}_t \leq g_t(x) - g_t(\overline{x}) < \varepsilon,$$

yielding $|b_t - \overline{b}_t| \leq |g_t(x) - \overline{b}_t| < \varepsilon$. Then, $\|b - \overline{b}\|_\infty = \max_{t \in T} |b_t - \overline{b}_t| < \varepsilon$. □

5 Regularity Modulus of $\mathcal{G}^*$

Theorem 4.3 allows us to sharpen some arguments in the methodology followed in [2] in order to obtain in Theorem 5.1 a more simplified expression of $\operatorname{reg} \mathcal{G}^*(\overline{x} \mid (\overline{c}, \overline{b}))$. For comparative purposes, after Theorem 5.1 we refer to the original expression of [2, Theorem 3]. For the sake of completeness, we introduce all the necessary ingredients related to the fulfillment of the hypotheses of Theorem 2.2, but adapted to the current case in which $\overline{c}$ remains unperturbed.

Assume the ENC at $(\overline{x}, (\overline{c}, \overline{b})) \in \operatorname{gph}(\mathcal{G}^*)$. In particular, $\sigma(\overline{b})$ satisfies SCQ, and we consider $x^0 \in \mathbb{R}^n$ and $\rho > 0$ such that $g_t(x^0) \leq \overline{b}_t - 2\rho$ for all $t \in T$. Let

$$W := \{b \in C(T, \mathbb{R}) : b_t \geq g_t(x^0) + \rho, \text{ for all } t \in T\}. \tag{18}$$

Note that W is a closed neighborhood of $\overline{b}$ containing $\overline{B}_\rho(\overline{b})$.

Now, associated with W, we introduce $\widetilde{\mathcal{G}}_W : \mathbb{R}^n \rightrightarrows C(T, \mathbb{R})$ given by

$$\widetilde{\mathcal{G}}_W(x) := \widetilde{\mathcal{G}}(x) \cap W, \tag{19}$$

and consider, associated with each $b \in W$, the distance function $f_{b,W}$ defined by

$$f_{b,W}(x) := d(b, \widetilde{\mathcal{G}}_W(x)). \tag{20}$$

Then, we have the following lemma, which constitutes the counterpart of [2, Theorem 2] in our current situation (under ENC) where perturbations of c are negligible. In fact, condition (i) in the following lemma is a consequence of the referred Theorem 2(i) in [2]. It is not the case of condition (ii) although its proof follows from a very similar argument and so it is omitted here.

Lemma 5.1 *Assume the* ENC *at* $(\overline{x}, (\overline{c}, \overline{b})) \in \mathrm{gph}(\mathcal{G}^*)$, *take* W *as in (18), and consider* $\widetilde{\mathcal{G}}_W$ *and* $f_{b,W}$ *given by* (19) *and* (20), *respectively. Then*:

(i) $\mathrm{gph}(\widetilde{\mathcal{G}}_W)$ *is closed and nonempty*;
(ii) $f_{b,W}$ *is lower semicontinuous on* $\mathbb{R}^n$ *for all* $b \in W$.

In our context, we can go a little bit further and ensure that, for b close enough to $\overline{b}$, $f_{b,W}$ is finite-valued in a neighborhood of $\overline{x}$, and that W may be removed from the definition of $f_{b,W}$.

Lemma 5.2 *Under the hypotheses of the previous lemma, there exist neighborhoods* U *of* $\overline{x}$ *and* V *of* $\overline{b}$, $V \subset W$, *such that*

$$f_{b,W}(x) = d(b, \widetilde{\mathcal{G}}(x)) < +\infty, \quad \textit{for all } x \in U \textit{ and } b \in V.$$

Proof Take $V := B_\varepsilon(\overline{b})$ such that $B_{3\varepsilon}(\overline{b}) \subset W$, and let U be a neighborhood of $\overline{x}$ such that $\widetilde{\mathcal{G}}(x) \cap V \neq \varnothing$ for all $x \in U$, according to Theorem 4.3. In this way, given $b \in V$, $x \in U$ and $\widetilde{b} \in \widetilde{\mathcal{G}}(x) \cap V$, we have $f_{b,W}(x) \le d(b, \widetilde{b}) < 2\varepsilon$ and, if $\widehat{b} \in \widetilde{\mathcal{G}}(x) \backslash W$, then

$$d(b, \widehat{b}) \ge d(\widehat{b}, \overline{b}) - d(\overline{b}, b) > 3\varepsilon - \varepsilon > d(b, \widetilde{\mathcal{G}}(x) \cap W).$$

So, $f_{b,W}(x) = d(b, \widetilde{\mathcal{G}}(x))$. □

Because of the two previous lemmas, $f_{b,W}$ verifies the hypothesis of Theorem 2.2. In fact, Lemma 5.2 enables us to express the thesis of Theorem 2.2 directly in terms of the functions f_b (see (6)) for b close enough to $\overline{b}$. Specifically we obtain the following result, which is the counterpart of [2, Theorem 3] in our context.

Theorem 5.1 *Assume that* ENC *is satisfied at* $(\overline{x}, (\overline{c}, \overline{b})) \in \mathrm{gph}(\mathcal{G}^*)$ *and let* f_b, $b \in C(T, \mathbb{R})$ *be the functions defined in* (6). *Then we have*

$$\mathrm{reg}\, \mathcal{G}^*(\overline{x} \mid (\overline{c}, \overline{b})) = \limsup_{\substack{(z,b) \to (\overline{x}, \overline{b}) \\ f_b(z) > 0}} (d_*(0_n, \widehat{\partial} f_b(z)))^{-1} = \limsup_{\substack{(z,b) \to (\overline{x}, \overline{b}) \\ f_b(z) > 0}} (|\nabla f_b|(z))^{-1}.$$

Remark 5.1 Theorem 3 in [2] provides a more involved expression for $\mathrm{reg}\, \mathcal{G}^*(\overline{x} \mid (\overline{c}, \overline{b}))$, where the limsup is taken on $(z, c, b) \to (\overline{x}, \overline{c}, \overline{b})$ with $f_{c,b}(z) > 0$, the latter being defined as $f_{c,b}(z) := d((c, b), \mathcal{G}^*(z) \cap V)$, where $V := \mathbb{R}^n \times W$.

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