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Maximum Principle for Stochastic Differential Games with Partial Information

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Abstract In this paper, we first deal with the problem of optimal control for zero-sum stochastic differential games. We give a necessary and sufficient maximum principle for that problem with partial information. Then, we use the result to solve a problem in finance. Finally, we extend our approach to general stochastic games (nonzero-sum), and obtain an equilibrium point of such game.

Keywords Jump diffusions · Stochastic control · Stochastic differential games · Sufficient maximum principle · Necessary maximum principle

1 Introduction

Game theory had been an active area of research and a useful tool in many applications, particularly in biology and economic. In the recent paper by Mataramvura and Øksendal [1], the stochastic differential game was solved with the restriction to consider only Markov controls. Then, the equilibrium point or other type of solution is constructed using the Hamilton-Jacobi-Bellman (HJB) equations. In this paper, we require that the control process is adapted to a given subfiltration of the filtration generated by the underlying Lévy processes. So, we cannot use dynamic programming and HJB equations to solve the problems. Here, we establish a maximum principle for

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such stochastic control problems. There is already a lot of literature on the maximum principle. See e.g. [2–5] and the references therein.

Our paper is organized as follows: In Sect. 2, we give a sufficient maximum principle for zero-sum stochastic differential games (Theorem 2.1). A necessary type of this problem is given in the Sect. 3. In Sect. 4, we put a problem in finance into the framework of a stochastic differential game with partial information and use Theorem 2.1 to solve it. With complete information, this problem is solved in [6] by using the HJB equations. In Sect. 4, we generalize our approach to the general case, not necessarily of zero-sum type, and also give an equilibrium point for nonzero-sum games.

2 Sufficient Maximum Principle for Zero-Sum Games

Suppose that the dynamics of a stochastic system is described by a stochastic differential equation on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ of the form

$$\begin{aligned} dX(t) &= b(t, X(t), u_0(t))dt + \sigma(t, X(t), u_0(t))dB(t) \\ &\quad + \int_{\mathbb{R}^n} \gamma(t, X(t^-), u_1(t^-, z), z)\widetilde{N}(dt, dz), \quad t \in [0, T], \end{aligned} \tag{1a}$$

$$X(0) = x \in \mathbb{R}^n. \tag{1b}$$

Here, $b : [0, T] \times \mathbb{R}^n \times K \to \mathbb{R}^n$; $\sigma : [0, T] \times \mathbb{R}^n \times K \to \mathbb{R}^{n\times n}$ and $\gamma : [0, T] \times \mathbb{R}^n \times K \times \mathbb{R}_0 \to \mathbb{R}^{n\times n}$ are given continuous functions, $B(t)$ is an n-dimensional Brownian motion, $\widetilde{N}(\cdot, \cdot)$ are n independent by compensated Poisson random measures and K is a given closed subset of $\mathbb{R}^n$. The processes $u_0(t) = u_0(t, \omega)$ and $u_1(t) = u_1(t, z, \omega)$, $\omega \in \Omega$, are our *control processes*. We assume that $u_0(t), u_1(t, z)$ have values in a given set K for a.a. t, z and that $u_0(t), u_1(t, z)$ are càdlàg and adapted to a given filtration $\{\mathcal{E}_t\}_{t\geq 0}$, where

$$\mathcal{E}_t \subseteq \mathcal{F}_t, \quad t \geq 0.$$

For example, we could have

$$\mathcal{E}_t = \mathcal{F}_{(t-\delta)^+}, \quad t \geq 0,$$

where $(t - \delta)^+ = \max(0, t - \delta)$. This models a situation where the controller only has delayed information available about the state of the system.

Let $f : [0, T] \times \mathbb{R}^n \times K \to \mathbb{R}$ be a continuous function, namely the *profit rate*, and let $g : \mathbb{R}^n \to \mathbb{R}$ be a concave function, namely the *bequest function*. We call $u = (u_0, u_1)$ an *admissible control* if (1) has a unique strong solution and

$$E^x\left[\int_0^T |f(t, X(t), u_0(t))|dt + |g(X(T))|\right] < \infty. \tag{2}$$

If u is an admissible control, we define the *performance criterion* $J(u)$ by

$$J(u) = E^x\left[\int_0^T f(t, X(t), u_0(t))\, dt + g(X(T))\right]. \tag{3}$$

Now, suppose that the controls $u_0(t)$ and $u_1(t, z)$ have the form

$$u_0(t) = (\theta_0(t), \pi_0(t)), \quad t \geq 0, \tag{4}$$

$$u_1(t,z) = (\theta_1(t,z), \pi_1(t,z)), \quad (t,z) \in [0,\infty) \times \mathbb{R}^n. \tag{5}$$

We let Θ and Π be given families of admissible controls $\theta = (\theta_0, \theta_1)$ and $\pi = (\pi_0, \pi_1)$, respectively. The *partial information zero-sum stochastic differential game problem* is to find $(\theta^*, \pi^*) \in \Theta \times \Pi$ such that

$$\Phi_{\mathcal{E}}(x) = J(\theta^*, \pi^*) = \sup_{\pi\in\Pi} \Big(\inf_{\theta\in\Theta} J(\theta,\pi) \Big). \tag{6}$$

Such a control (θ^*, π^*) is called an *optimal control* (if it exists).

The intuitive idea is that there are two players, I and II. Player I controls $\theta := (\theta_0, \theta_1)$ and player II controls $\pi := (\pi_0, \pi_1)$. The actions of the players are antagonistic, which means that between I and II there is a payoff $J(\theta,\pi)$ which is a cost for I and a reward for II.

Let K_1, K_2 be two sets such that $\theta(t,z) \in K_1$ and $\pi(t,z) \in K_2$ for a.a. t, z. As in [4], we now define the *Hamiltonian* $H : [0,T] \times \mathbb{R}^n \times K_1 \times K_2 \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \times \mathcal{R} \to \mathbb{R}$ by

$$\begin{aligned} H(t,x,\theta,\pi,p,q,r) &= f(t,x,\theta,\pi) + b^T(t,x,\theta,\pi)p + tr(\sigma^T((t,x,\theta,\pi)q) \\ &\quad + \sum_{i,j=1}^{n} \int_{\mathbb{R}_0} \gamma_{ij}(t,x,\theta,\pi,z) r_{ij}(t,z) \nu_j(dz_j), \end{aligned} \tag{7}$$

where $\mathcal{R}$ is the set of functions $r : [0,T] \times \mathbb{R}_0 \to \mathbb{R}^{n\times n}$ such that the integral in (7) converges. From now on, we assume that H is continuously differentiable with respect to x.

The *adjoint equation* in the unknown adapted processes $p(t) \in \mathbb{R}^n, q(t) \in \mathbb{R}^{n\times n}$ and $r(t,z) \in \mathbb{R}^{n\times n}$ is the backward stochastic differential equation (BSDE)

$$\begin{aligned} dp(t) &= -\nabla_x H(t, X(t), \theta(t), \pi(t), p(t), q(t), r(t,.))\,dt \\ &\quad + q(t)\,dB(t) + \int_{\mathbb{R}^n} r(t^-, z)\widetilde{N}(dt, dz), \quad t < T, \end{aligned} \tag{8a}$$

$$p(T) = \nabla g(X(T)), \tag{8b}$$

where $\nabla_y \varphi(.) = (\frac{\partial\varphi}{\partial y_1}, \dots, \frac{\partial\varphi}{\partial y_n})^T$ is the gradient of $\varphi : \mathbb{R}^n \to \mathbb{R}$ with respect to $y = (y_1, \dots, y_n)$.

We can now state the following verification theorem for optimality.

Theorem 2.1 *Let* $(\hat\theta, \hat\pi) \in \Theta \times \Pi$ *with corresponding state process* $\hat X(t) = X^{(\hat\theta,\hat\pi)}(t)$. *Let* $X^\pi(t)$ *and* $X^\theta(t)$ *to be* $X^{(\hat\theta,\pi)}(t)$ *and* $X^{(\theta,\hat\pi)}(t)$, *respectively. Suppose that there exists a solution* $(\hat p(t), \hat q(t), \hat r(t,z))$ *of the corresponding adjoint equation* (8) *such that, for all* $\theta \in \Theta$ *and* $\pi \in \Pi$, *we have*

$$E\bigg[\int_0^T (\hat X(t) - X^{(\pi)}(t))^T \bigg\{\hat q \hat q^T(t) + \int_{\mathbb{R}^n} \hat r \hat r^T(t,z)\nu(dz)\bigg\}(\hat X(t) - X^{(\pi)}(t))dt\bigg] < \infty, \tag{9}$$

$$E\bigg[\int_0^T (\hat X(t) - X^{(\theta)}(t))^T \bigg\{\hat q \hat q^T(t) + \int_{\mathbb{R}^n} \hat r \hat r^T(t,z)\nu(dz)\bigg\}(\hat X(t) - X^{(\theta)}(t))dt\bigg] < \infty, \tag{10}$$

and

$$E\Big[\int_0^T \hat{p}(t)^T\Big\{\sigma\sigma^T(t,X(t),\theta(t),\hat{\pi}(t)) + \int_{\mathbb{R}^n}\gamma\gamma^T(t,X^{(\theta)}(t),\theta(t),\hat{\pi}(t),z)\nu(dz)\Big\}p(t)dt\Big] < \infty, \tag{11}$$

$$E\Big[\int_0^T \hat{p}(t)^T\Big\{\sigma\sigma^T(t,X(t),\hat{\theta}(t),\pi(t)) + \int_{\mathbb{R}^n}\gamma\gamma^T(t,X^{(\pi)}(t),\theta(t),\hat{\pi}(t),z)\nu(dz)\Big\}p(t)dt\Big] < \infty, \tag{12}$$

ensuring that the integrals with respect to B and the compensated small jump parts indeed have zero mean. Moreover, suppose that, for all $t\in[0,T]$, the following partial information maximum principle *holds*:

$$\begin{aligned}&\inf_{\theta\in K_1} E[H(t,X(t),\theta,\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))\mid \mathcal{E}_t]\\ &= E[H(t,X(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))\mid \mathcal{E}_t]\\ &= \sup_{\pi\in K_2} E[H(t,X(t),\hat{\theta}(t),\pi,\hat{p}(t),\hat{q}(t),\hat{r}(t,.))\mid \mathcal{E}_t].\end{aligned} \tag{13}$$

(i) *Suppose that, for all $t\in[0,T]$, $g(x)$ is concave and*

$$(x,\pi)\mapsto H(t,x,\hat{\theta}(t),\pi,\hat{p}(t),\hat{q}(t),\hat{r}(t,.))$$

is concave. Then,

$$J(\hat{\theta},\hat{\pi})\ge J(\hat{\theta},\pi),\quad \text{for all } \pi\in\Pi,$$

and

$$J(\hat{\theta},\hat{\pi}) = \sup_{\pi\in\Pi} J(\hat{\theta},\pi).$$

(ii) *Suppose that, for all $t\in[0,T]$, $g(x)$ is convex and*

$$(x,\theta)\mapsto H(t,x,\theta,\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))$$

is convex. Then,

$$J(\hat{\theta},\hat{\pi})\le J(\theta,\hat{\pi}),\quad \text{for all } \theta\in\Theta,$$

and

$$J(\hat{\theta},\hat{\pi}) = \inf_{\theta\in\Theta} J(\theta,\hat{\pi}).$$

(iii) *If both cases* (i) *and* (ii) *hold* (*which implies*, *in particular*, *that g is an affine function*), *then $(\theta^*,\pi^*) := (\hat{\theta},\hat{\pi})$ is an optimal control and*

$$\Phi_{\mathcal{E}}(x) = \sup_{\pi\in\Pi}\Big(\inf_{\theta\in\Theta} J(\theta,\pi)\Big) = \inf_{\theta\in\Theta}\Big(\sup_{\pi\in\Pi} J(\theta,\pi)\Big). \tag{14}$$

Proof (i) Suppose that (i) holds. Choose $(\theta, \pi) \in \Theta \times \Pi$. Let us consider

$$J(\hat{\theta}, \hat{\pi}) - J(\hat{\theta}, \pi) = I_1 + I_2,$$

where

$$I_1 = E\Big[\int_0^T \big\{f(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t)) - f(t, X^{(\pi)}(t), \hat{\theta}(t), \pi(t))\big\}dt\Big] \quad (15)$$

and

$$I_2 = E\big[g(\hat{X}(T)) - g(X^{(\pi)}(T))\big]. \quad (16)$$

g is concave in x and, from integration by parts formula for jump processes, we get the following, where the L^2 conditions (9) and (12) ensure that the stochastic integrals with respect to the local martingales have zero expectation,

$$\begin{aligned}
I_2 &= E\big[g(\hat{X}(T)) - g(X^{(\pi)}(T))\big] \\
&\geq E\big[(\hat{X}(T) - X^{(\pi)}(T))^T \bigtriangledown g(\hat{X}(T))\big] \\
&= E\big[(X^{(\hat{\theta},\hat{\pi})}(T) - X^{(\hat{\theta},\pi)}(T))^T \hat{p}(T)\big] \\
&= E\Big[\int_0^T (\hat{X}(t^-) - X^{(\pi)}(t^-))^T d\hat{p}(t) + \int_0^T \hat{p}^T(t)(d\hat{X}(t) - dX^{(\pi)}(t)) \\
&\quad + \int_0^T tr\big[\{\sigma(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t)) - \sigma(t, X^{(\pi)}(t), \hat{\theta}(t), \pi(t))\}^T \hat{q}(t)\big]dt \\
&\quad + \int_0^T \sum_{i,j=1}^n \int_{\mathbb{R}_0} \{\gamma_{ij}(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), z) \\
&\quad - \gamma_{ij}(t, X^{(\pi)}(t), \hat{\theta}(t), \pi(t), z)\}\hat{r}_{ij}(t, z)\nu_j(dz)dt\Big] \\
&= E\Big[\int_0^T (\hat{X}(t) - X^{(\pi)}(t))^T(- \bigtriangledown_x H(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t, .)))dt \\
&\quad + \int_0^T \hat{p}^T(t)\big\{b(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t)) - b(t, X^{(\pi)}(t), \hat{\theta}(t), \pi(t))\big\}dt \\
&\quad + \int_0^T tr\big[\{\sigma(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t)) - \sigma(t, X^{(\pi)}(t), \hat{\theta}(t), \pi(t))\}^T \hat{q}(t)\big]dt \\
&\quad + \int_0^T \sum_{i,j=1}^n \int_{\mathbb{R}_0} \{\gamma_{ij}(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), z) \\
&\quad - \gamma_{ij}(t, X^{(\pi)}(t), \hat{\theta}(t), \pi(t), z)\}\hat{r}_{ij}(t, z)\nu_j(dz)dt\Big].
\end{aligned} \quad (17)$$

By the definition (7) of H, we have

$$
\begin{aligned}
I_1 &= E\bigg[\int_0^T \{f(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t)) - f(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t))\}dt\bigg] \\
&= E\bigg[\int_0^T \{H(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.)) \\
&\quad - H(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))\}dt\bigg] \\
&\quad - E\bigg[\int_0^T \hat{p}^T(t)\big\{b(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t)) - b(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t))\big\}dt\bigg] \\
&\quad - E\bigg[\int_0^T tr\big[\{\sigma(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t)) - \sigma(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t))\}^T\hat{q}(t)\big]dt\bigg] \\
&\quad - E\bigg[\int_0^T \sum_{i,j=1}^n \int_{\mathbb{R}_0} \{\gamma_{ij}(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),z) \\
&\quad - \gamma_{ij}(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t),z)\}\hat{r}_{ij}(t,z)\nu_j(dz)dt\bigg].
\end{aligned}
\tag{18}
$$

By the concavity of H in x and π, we have

$$
\begin{aligned}
&H(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.)) \\
&\qquad - H(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.)) \\
&\quad \geq \nabla_x H(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))^T(\hat{X}(t) - X^{(\pi)}(t)) \\
&\qquad + \nabla_\pi H(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))^T(\hat{\pi}(t) - \pi(t)).
\end{aligned}
\tag{19}
$$

Since $\pi \to E[H(t,X^\pi(t),\hat{\theta}(t),\pi,\hat{p}(t),\hat{q}(t),\hat{r}(t,.)) \mid \mathcal{E}_t]$ is maximum for $\pi = \hat{\pi}(t)$ and since $\pi(t),\hat{\pi}(t)$ are $\mathcal{E}_t$-measurable, we get

$$
\begin{aligned}
&E\big[\nabla_\pi H(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))^T(\hat{\pi}(t) - \pi(t)) \mid \mathcal{E}_t\big] \\
&\quad = (\hat{\pi}(t) - \pi(t))\,\nabla_\pi E\big[H(t,X^\pi(t),\hat{\theta}(t),\pi(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.)) \mid \mathcal{E}_t\big]^T_{\pi=\hat{\pi}(t)} \\
&\quad \geq 0.
\end{aligned}
$$

Combining this with (19), we obtain

$$
\begin{aligned}
&E\bigg[\int_0^T \{H(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.)) \\
&\qquad - H(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))\}dt\bigg] \\
&\quad \geq E\bigg[\int_0^T \nabla_x H(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))^T(\hat{X}(t) - X^{(\pi)}(t))\bigg].
\end{aligned}
\tag{20}
$$

Hence,

$$
\begin{aligned}
I_1 \geq E\Big[\int_0^T \nabla_x H(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,.))^T(\hat{X}(t)-X^{(\pi)}(t))\Big] \\
- E\Big[\int_0^T \hat{p}^T(t)\big\{b(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t)) - b(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t))\big\}dt\Big] \\
- E\Big[\int_0^T tr\big[\{\sigma(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t)) - \sigma(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t))\}^T\hat{q}(t)\big]dt\Big] \\
- E\Big[\int_0^T \sum_{i,j=1}^n \int_{\mathbb{R}_0} \{\gamma_{ij}(t,\hat{X}(t),\hat{\theta}(t),\hat{\pi}(t),z) \\
- \gamma_{ij}(t,X^{(\pi)}(t),\hat{\theta}(t),\pi(t),z)\}\hat{r}_{ij}(t,z)\nu_j(dz)dt\Big]. \qquad (21)
\end{aligned}
$$

Adding (17), (21) above, we get

$$J(\hat{\theta},\hat{\pi}) - J(\hat{\theta},\pi) = I_1 + I_2 \geq 0. \qquad (22)$$

We conclude therefore that $J(\hat{\theta},\hat{\pi}) \geq J(\hat{\theta},\pi)$ for all $\pi \in \Pi$.

(ii) Proceeding in the same way as in (i), we can show that $J(\hat{\theta},\hat{\pi}) \leq J(\theta,\hat{\pi})$ for all $\theta \in \Theta$ if (ii) holds.

(iii) If both (i) and (ii) hold, then

$$J(\hat{\theta},\pi) \leq J(\hat{\theta},\hat{\pi}) \leq J(\theta,\hat{\pi}),$$

for any $(\theta,\pi) \in \Theta \times \Pi$. Thereby,

$$J(\hat{\theta},\hat{\pi}) \leq \inf_{\theta\in\Theta} J(\theta,\hat{\pi}) \leq \sup_{\pi\in\Pi}\Big(\inf_{\theta\in\Theta} J(\theta,\pi)\Big).$$

On the other hand,

$$J(\hat{\theta},\hat{\pi}) \geq \sup_{\pi\in\Pi} J(\hat{\theta},\pi) \geq \inf_{\theta\in\Theta}\Big(\sup_{\pi\in\Pi} J(\theta,\pi)\Big).$$

Now, due to the inequality

$$\inf_{\theta\in\Theta}\Big(\sup_{\pi\in\Pi} J(\theta,\pi)\Big) \geq \sup_{\pi\in\Pi}\Big(\inf_{\theta\in\Theta} J(\theta,\pi)\Big),$$

we have

$$\Phi_{\mathcal{E}}(x) = \sup_{\pi\in\Pi}\Big(\inf_{\theta\in\Theta} J(\theta,\pi)\Big) = \inf_{\theta\in\Theta}\Big(\sup_{\pi\in\Pi} J(\theta,\pi)\Big). \qquad \square$$

If the control process (θ,π) is admissible adapted to the filtration $\mathcal{F}_t$ we have the following corollary.

Corollary 2.1 *Suppose that $\mathcal{E}_t = \mathcal{F}_t$ for all t and that* (9)–(12) *hold. Moreover, suppose that, for all t, the following maximum principle holds*:

$$\begin{aligned}\inf_{\theta\in K_1} & H(t, X(t), \theta, \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) \\ &= H(t, X(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) \\ &= \sup_{\pi\in K_2} H(t, X(t), \hat{\theta}(t), \pi, \hat{p}(t), \hat{q}(t), \hat{r}(t,.)). \end{aligned} \tag{23}$$

Then, for all $t \in [0, T]$, we have:

(i) *If $g(x)$ is concave and $(x, \pi) \to H(t, x, \hat{\theta}(t), \pi, \hat{p}(t), \hat{q}(t), \hat{r}(t,.))$ is concave, then*

$$J(\hat{\theta}, \hat{\pi}) \geq J(\hat{\theta}, \pi), \quad \textit{for all } \pi \in \Pi,$$

and

$$J(\hat{\theta}, \hat{\pi}) = \sup_{\pi\in\Pi} J(\hat{\theta}, \pi).$$

(ii) *If $g(x)$ is convex and $(x, \theta) \to H(t, x, \theta, \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.))$ is convex, then*

$$J(\hat{\theta}, \hat{\pi}) \leq J(\theta, \hat{\pi}), \quad \textit{for all } \theta \in \Theta,$$

and

$$J(\hat{\theta}, \hat{\pi}) = \inf_{\theta\in\Theta} J(\theta, \hat{\pi}).$$

(iii) *If both cases* (i) *and* (ii) *hold, then $(\theta^*, \pi^*) := (\hat{\theta}, \hat{\pi})$ is an optimal control based on the information flow $\mathcal{F}_t$ and*

$$\Phi_{\mathcal{E}}(x) = \sup_{\pi\in\Pi}\Big(\inf_{\theta\in\Theta} J(\theta, \pi)\Big) = \inf_{\theta\in\Theta}\Big(\sup_{\pi\in\Pi} J(\theta, \pi)\Big). \tag{24}$$

3 Necessary Maximum Principle for Zero-Sum Games

In addition to the assumptions in Sect. 2, we now assume the following.

(A1) For all t, h such that $0 \leq t < t + h \leq T$, all bounded $\mathcal{E}_t$-measurable α, ρ, and for $s \in [0, T]$, the controls $\beta(s) := (0, \ldots, \beta_i(s), \ldots, 0)$ and $\eta(s) := (0, \ldots, \eta_i(s), \ldots, 0)$, $i = 1, \ldots, n$, with

$$\beta_i(s) := \alpha_i \chi_{[t,t+h]}(s) \quad \text{and} \quad \eta_i(s) := \rho_i \chi_{[t,t+h]}(s)$$

belong to Θ and Π, respectively.

(A2) For given $\theta, \beta \in \Theta$ and $\pi, \eta \in \Pi$, with β, η bounded, there exists $\delta > 0$ such that

$$\theta + y\beta \in \Theta \quad \text{and} \quad \pi + v\eta \in \Pi,$$

where $y, v \in (-\delta, \delta)$.

Denote $X^{\theta+y\beta}(t)=X^{(\theta+y\beta,\pi)}(t)$ and $X^{\pi+v\eta}(t)=X^{(\theta,\pi+v\eta)}(t)$. For a given $\theta,\beta\in\Theta$ and $\pi,\eta\in\Pi$ with β,η bounded, we define the processes $Y^\theta(t)$ and $Y^\pi(t)$ by

$$Y^\theta(t)=\frac{d}{dy}X^{\theta+y\beta}(t)\Big|_{y=0}=(Y_1^\theta(t),\dots,Y_n^\theta(t))^T, \tag{25}$$

$$Y^\pi(t)=\frac{d}{dv}X^{\pi+v\eta}(t)\Big|_{v=0}=(Y_1^\pi(t),\dots,Y_n^\pi(t))^T. \tag{26}$$

We have that

$$dY_i^\theta(t)=\lambda_i^\theta(t)dt+\sum_{j=1}^n\xi_{ij}^\theta(t)dB_j(t)+\sum_{j=1}^n\int_{\mathbb{R}}\zeta_{ij}^\theta(t,z)\widetilde{N}_j(dt,dz) \tag{27}$$

and

$$dY_i^\pi(t)=\lambda_i^\pi(t)dt+\sum_{j=1}^n\xi_{ij}^\pi(t)dB_j(t)+\sum_{j=1}^n\int_{\mathbb{R}}\zeta_{ij}^\pi(t,z)\widetilde{N}_j(dt,dz), \tag{28}$$

where $i=1,\dots,n$ and

$$\begin{aligned}\lambda_i^\theta(t)&=\nabla_x b_i(t,X(t),\theta(t),\pi(t))^T Y^\theta(t)\\&\quad+\nabla_\theta b_i(t,X(t),\theta(t),\pi(t))^T\beta(t), &(29a)\\ \xi_{ij}^\theta(t)&=\nabla_x\sigma_{ij}(t,X(t),\theta(t),\pi(t))^T Y^\theta(t)\\&\quad+\nabla_\theta\sigma ij(t,X(t),\theta(t),\pi(t))^T\beta(t), &(29b)\\ \zeta_{ij}^\theta(t)&=\nabla_x\gamma_{ij}(t,X(t),\theta(t),\pi(t))^T Y^\theta(t)\\&\quad+\nabla_\theta\gamma_{ij}(t,X(t),\theta(t),\pi(t))^T\beta(t), &(29c)\end{aligned}$$

and

$$\begin{aligned}\lambda_i^\pi(t)&=\nabla_x b_i(t,X(t),\theta(t),\pi(t))^T Y^\pi(t)\\&\quad+\nabla_\pi b_i(t,X(t),\theta(t),\pi(t))^T\eta(t), &(30a)\\ \xi_{ij}^\pi(t)&=\nabla_x\sigma_{ij}(t,X(t),\theta(t),\pi(t))^T Y^\pi(t)\\&\quad+\nabla_\pi\sigma ij(t,X(t),\theta(t),\pi(t))^T\eta(t), &(30b)\\ \zeta_{ij}^\pi(t)&=\nabla_x\gamma_{ij}(t,X(t),\theta(t),\pi(t))^T Y^\pi(t)\\&\quad+\nabla_\pi\gamma_{ij}(t,X(t),\theta(t),\pi(t))^T\eta(t). &(30c)\end{aligned}$$

Theorem 3.1 *Suppose that* $(\hat\theta,\hat\pi)\in\Theta\times\Pi$ *is a directional critical point for* $J(\theta,\pi)$, *in the sense that, for all bounded* $\beta\in\Theta$ *and* $\eta\in\Pi$, *there exist* $\delta>0$ *such that* $\hat\theta+y\beta\in\Theta$ *and* $\hat\pi+v\eta\in\Pi$ *for all* $y\in(-\delta,\delta)$ *and* $v\in(-\delta,\delta)$ *and*

$$h(y,v):=J(\hat\theta+y\beta,\hat\pi+v\eta),\quad y,v\in(-\delta,\delta),$$

has a critical point at $(0,0)$, *i.e.*

$$\frac{\partial h}{\partial y}(0,0)=\frac{\partial h}{\partial v}(0,0)=0. \tag{31}$$

Suppose that there exists a solution $\hat{p}(t), \hat{q}(t), \hat{r}(t,.)$ *of the associated adjoint equation*

$$d\hat{p}(t) = -\bigtriangledown_x H(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.))\,dt + \hat{q}(t)\,dB(t) + \int_{\mathbb{R}^n} \hat{r}(t^-, z)\widetilde{N}(dt, dz), \quad t < T, \tag{32a}$$

$$\hat{p}(T) = \bigtriangledown g(\hat{X}(T)). \tag{32b}$$

Moreover, suppose that, if $Y^{\hat{\theta}}, Y^{\hat{\pi}}, (\lambda_i^{\hat{\theta}}, \xi_{ij}^{\hat{\theta}}, \zeta_{ij}^{\hat{\theta}})$ *and* $(\lambda_i^{\hat{\pi}}, \xi_{ij}^{\hat{\pi}}, \zeta_{ij}^{\hat{\pi}})$ *are the corresponding coefficients* (*see* (25)–(30)), *then*

$$E\left[\int_0^T Y^{\hat{\theta}^T}(t)\left\{\hat{q}\hat{q}^T + \int_{\mathbb{R}^n} \hat{r}\hat{r}^T(t,z)\nu(dz)\right\}Y^{\hat{\theta}}(t)dt\right] < \infty, \tag{33}$$

$$E\left[\int_0^T Y^{\hat{\pi}^T}(t)\left\{\hat{q}\hat{q}^T + \int_{\mathbb{R}^n} \hat{r}\hat{r}^T(t,z)\nu(dz)\right\}Y^{\hat{\pi}}(t)dt\right] < \infty, \tag{34}$$

and

$$E\left[\int_0^T \hat{p}^T(t)\left\{\xi^{\hat{\theta}}\xi^{\hat{\theta}^T}(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t)) + \int_{\mathbb{R}} \gamma\gamma^T(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t))\nu(dz)\right\}\hat{p}(t)dt\right] < \infty, \tag{35}$$

$$E\left[\int_0^T \hat{p}^T(t)\left\{\xi^{\hat{\pi}}\xi^{\hat{\pi}^T}(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t)) + \int_{\mathbb{R}} \gamma\gamma^T(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t))\nu(dz)\right\}\hat{p}(t)dt\right] < \infty. \tag{36}$$

Then, for a.a. $t \in [0, T]$, *we have*

$$E[\bigtriangledown_\theta H(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) \mid \mathcal{E}_t] = E[\bigtriangledown_\pi H(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.)) \mid \mathcal{E}_t] = 0. \tag{37}$$

Proof Since h has a minimum at $y = 0$, we have

$$\begin{aligned} 0 = \frac{\partial}{\partial y}h(y,0)\Big|_{y=0} &= E\bigg[\int_0^T \bigtriangledown_x f(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t))^T \frac{d}{dy}X^{\hat{\theta}+y\beta}(t)\Big|_{y=0}dt \\ &\quad + \int_0^T \bigtriangledown_\theta f(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t))^T\beta(t)dt + \bigtriangledown g(\hat{X}(T)^T\frac{d}{dy}X^{\hat{\theta}+y\beta}(T)\Big|_{y=0}\bigg] \\ &= E\bigg[\int_0^T \bigtriangledown_x f(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t))^T Y^{\hat{\theta}}(t)dt \\ &\quad + \int_0^T \bigtriangledown_\theta f(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t))^T\beta(t)dt + \bigtriangledown g(\hat{X}(T)^T Y^{\hat{\theta}}(T)\bigg]. \end{aligned} \tag{38}$$

By the Itô formula,

$$
\begin{aligned}
E[\nabla g(\hat{X}(T)^T Y^{\hat{\theta}}(T)] &= E[\hat{p}^T(T) Y^{\hat{\theta}}(T)] \\
&= E\Bigg[\sum_{i=1}^{n} \int_0^T \Bigg\{ \hat{p}_i(t)(\nabla_x b_i(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t))^T Y^{\hat{\theta}}(t) \\
&\quad + \nabla_\theta b_i(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t))^T \beta(t)) \\
&\quad + Y_i^{\hat{\theta}}(t)(-\nabla_x H(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.))_i \\
&\quad + \sum_{j=1}^{n} \hat{q}_{ij}(t)(\nabla_x \sigma_{ij}(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t))^T Y^{\hat{\theta}}(t) \\
&\quad + \nabla_\theta \sigma_{ij}(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t))^T \beta(t)) \\
&\quad + \sum_{j=1}^{n} \int_{\mathbb{R}^n} \hat{r}_{ij}(t^-, z)(\nabla_x \gamma_{ij}(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t))^T Y^{\hat{\theta}}(t) \\
&\quad + \nabla_\theta \gamma_{ij}(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t))^T \beta(t)) \Bigg\} dt \Bigg]. \qquad (39)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\nabla_x H(t, x, \theta, \pi, p, q, r) \\
&\quad = \nabla_x f(t, x, \theta, \pi) + \sum_{i=1}^{n} \nabla_x b_i(t, x, \theta, \pi) p_i \\
&\qquad + \sum_{j,i=1}^{n} \nabla_x \sigma_{ji}(t, x, \theta, \pi) q_{ji} + \sum_{j,i=1}^{n} \int_{\mathbb{R}} \nabla_x \gamma_{ji}(t, x, \theta, \pi) r_{ji}(t, z) \nu_j(dz). \qquad (40)
\end{aligned}
$$

Substituting this into (39) and combining with (38), we get

$$
\begin{aligned}
0 &= E\Bigg[\int_0^T \sum_{i=1}^{n} \Bigg\{ \frac{\partial f}{\partial \theta_i}(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t)) \\
&\quad + \sum_{j=1}^{n} \Bigg(\hat{p}_j(t) \frac{\partial b_j}{\partial \theta_i}(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t)) + \sum_{k=1}^{n} \Bigg[\hat{q}_{kj}(t) \frac{\partial \sigma_{kj}}{\partial \theta_i}(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t)) \\
&\quad + \int_{\mathbb{R}} \hat{r}_{kj}(t, z) \frac{\partial \gamma_{kj}}{\partial \theta_i}(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t)) \nu_j(dz) \Bigg] \Bigg) \beta_i(t) \Bigg\} dt \Bigg] \\
&= E\Bigg[\int_0^T \nabla_\theta H(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,.))^T \beta(t) dt \Bigg]. \qquad (41)
\end{aligned}
$$

By Assumption (A1), (41) leads to

$$E\left[\int_t^{t+h}\frac{\partial}{\partial\theta_i}H(s,\hat{X}(s),\hat{\theta}(s),\hat{\pi}(s),\hat{p}(s),\hat{q}(s),\hat{r}(s,.))\alpha_i(s)ds\right]=0.$$

Differentiating with respect to h at $h=0$ gives

$$E\left[\frac{\partial}{\partial\theta_i}H(s,\hat{X}(s),\hat{\theta}(s),\hat{\pi}(s),\hat{p}(s),\hat{q}(s),\hat{r}(s,.))\alpha_i(s)\right]=0.$$

Since this holds for all bounded $\mathcal{E}_t$-measurable α_i, we conclude that

$$E\left[\frac{\partial}{\partial\theta_i}H(s,\hat{X}(s),\hat{\theta}(s),\hat{\pi}(s),\hat{p}(s),\hat{q}(s),\hat{r}(s,.))\mid\mathcal{E}_t\right]=0,$$

as claimed.

Proceeding in the same way by differentiating the function $h(0,v)$ with respect to v, we get

$$E\left[\frac{\partial}{\partial\pi_i}H(s,\hat{X}(s),\hat{\theta}(s),\hat{\pi}(s),\hat{p}(s),\hat{q}(s),\hat{r}(s,.))\mid\mathcal{E}_t\right]=0.$$

This completes the proof. □

4 Applications to Finance

In this section, we use our result to solve a partial information version of the problem studied in [6].

Consider the following jump diffusion market:

$$\text{(risky free asset)}\quad dS_0(t)=\rho(t)S_0(t)dt,\quad S_0(0)=1, \tag{42}$$

$$\text{(risky asset)}\quad dS_1(t)=S_1(t^-)\Big[\alpha(t)dt+\beta(t)dB(t)+\int_{\mathbb{R}}\gamma(t,z)\widetilde{N}(dt,dz)\Big],$$
$$S_1(0)>0, \tag{43}$$

where $\rho(t)$ is a deterministic function, $\alpha(t)$, $\beta(t)$ and $\gamma(t,z)$ are given $\mathcal{F}_t$-predictable functions satisfying the following integrability condition:

$$E\left[\int_0^T\left\{|\rho(s)|+|\alpha(s)|+\frac{1}{2}\beta(s)^2+\int_{\mathbb{R}}|\log(1+\gamma(s,z))-\gamma(s,z)|\nu(dz)\right\}ds\right]<\infty, \tag{44}$$

where T is fixed. We assume that

$$\gamma(t,z)\geq -1,\quad \text{for a.a. } t,z\in[0,T]\times\mathbb{R}_0, \tag{45}$$

where $\mathbb{R}_0=\mathbb{R}\backslash\{0\}$. This model represents a natural generalization of the classical Black–Scholes market model to the case where the coefficients are not necessarily constants, but allowed to be (adapted) stochastic processes. Moreover, we have added a jump component. See e.g. [7] or [8] for discussions of such markets.

Let $\mathcal{E}_t \subseteq \mathcal{F}_t$ be a given subfiltration. Let $\pi(t)$ be a portfolio, which is an $\mathcal{E}_t$-measurable random variable represented by the *fraction* of the wealth invested in the risky asset at time t. The dynamics of the corresponding wealth process $V^{(\pi)}(t)$ is

$$\begin{aligned} dV^{(\pi)}(t) &= V^{(\pi)}(t^-)[\{\rho(t) + (\alpha(t) - \rho(t))\pi(t)\}dt \\ &\qquad + \pi(t)\beta(t)dB(t) + \pi(t^-)\int_{\mathbb{R}} \gamma(t,z)\widetilde{N}(dt,dz)], \end{aligned} \tag{46a}$$

$$V^{(\pi)}(0) = v > 0. \tag{46b}$$

A portfolio π is called *admissible* if it is a measurable càdlàg stochastic process adapted to the filtration $\mathcal{E}_t$ and satisfies

$$\pi(t^-)\gamma(t,z) > -1, \quad \text{a.s.,}$$

and

$$\begin{aligned} \int_0^T \Big\{ &|\rho(t) + (\alpha(t) - \rho(t))\pi(t)| + \pi^2(t)\beta^2(t) \\ &+ \pi^2(t)\int_{\mathbb{R}} \gamma^2(t,z)\nu(dz)\Big\} dt < \infty, \quad \text{a.s.} \end{aligned} \tag{47}$$

The requirement that π should be adapted to the filtration $\mathcal{E}_t$ is a mathematical way of requiring that the choice of the portfolio value $\pi(t)$ at time t is allowed to depend on the information (σ-algebra) $\mathcal{E}_t$ only. The wealth process corresponding to an admissible portfolio π is the solution of (46),

$$\begin{aligned} V^{(\pi)}(t) = v\exp\Big[\int_0^t \Big\{ &\rho(t) + (\alpha(t) - \rho(t))\pi(t) - \frac{1}{2}\pi^2(t)\beta^2(t) \\ &+ \int_{\mathbb{R}} (\ln(1 + \pi(s)\gamma(s,z)) - \pi(z)\gamma(s,z))\nu(dz)\Big\} ds \\ &+ \int_0^t \pi(s)\beta(s)dB(s) + \int_0^t\int_{\mathbb{R}} \ln(1 + \pi(s)\gamma(s,z))\widetilde{N}(ds,dz)\Big]. \end{aligned} \tag{48}$$

The family of admissible portfolios is denoted by Π.

Now, we introduce a family $\mathcal{Q}$ of measures Q_θ parameterized by processes $\theta = (\theta_0(t), \theta_1(t,z))$ such that

$$dQ_\theta(\omega) = Z_\theta(T)dP(\omega), \quad \text{on } \mathcal{F}_T, \tag{49}$$

where

$$dZ_\theta(t) = Z_\theta(t^-)[-\theta_0(t)dB(t) - \int_{\mathbb{R}} \theta_1(t,z)\widetilde{N}(dt,dz)], \tag{50a}$$

$$Z_\theta(0) = 1. \tag{50b}$$

We assume that $\theta_1(t,z) \le 1$ for a.a. t, z and

$$\int_0^T \Big\{\theta_0^2(s) + \int_{\mathbb{R}} \theta_1^2(s,z)\Big\} ds < \infty \quad \text{a.s.} \tag{51}$$

Then, by the Itô formula, the solution of (50) is given by

$$\begin{aligned} Z_\theta(t) = \exp\Big[&-\int_0^t \theta_0(s)dB(s) - \frac{1}{2}\int_0^t \theta_0^2(s)ds + \int_0^t\int_{\mathbb{R}} \ln(1-\theta_1(s,z))\widetilde{N}(dt,dz) \\ &+ \int_0^t\int_{\mathbb{R}} \{\ln(1-\theta_1(s,z)) + \theta_1(s,z)\}\nu(dz)ds\Big]. \end{aligned} \tag{52}$$

If $\theta = (\theta_0(t), \theta_1(t,z))$ satisfies

$$E[Z_\theta(T)] = 1, \tag{53}$$

then Q_θ is a probability measure. If, in addition,

$$\beta(t)\theta_0(t) + \int_{\mathbb{R}} \gamma(t,z)\theta_1(t,z)\nu(dz) = \alpha(t) - r(t), \quad t \in [0,T], \tag{54}$$

then $dQ_\theta(\omega) = Z_\theta(T)dP(\omega)$ is an *equivalent local martingale measure*. See e.g. [8], Chap. 1. But here we do not assume that (54) holds.

All $\theta = (\theta_0, \theta_1)$ adapted to the subfiltration $\mathcal{E}_t$ and satisfying (51)–(53) are called *admissible controls of the market*. The family of admissible controls θ is denoted by Θ.

The problem is to find $(\theta, \pi) \in \Theta \times \Pi$ such that

$$\inf_{\theta\in\Theta}\Big(\sup_{\pi\in\Pi} E_{Q_\theta}[U(X^\pi(T))]\Big) = E_{Q_{\theta^*}}[U(X^{\pi^*}(T))], \tag{55}$$

where $U : [0,\infty) \to [-\infty,\infty)$ is a given utility function, which is increasing, concave and twice continuously differentiable on $(0,\infty)$.

We can consider this problem as a stochastic differential game between the *agent* and the *market*. The agent wants to maximize her expected discounted utility over all portfolios π and the market wants to minimize the maximal expected utility of the representative agent over all scenarios, represented by all probability measures $Q_\theta; \theta \in \Theta$.

To put this problem into the Markovian context discussed in the previous sections, we combine the Radon-Nikodym process $Z_\theta(t)$ and the wealth process $V^{(\pi)}(t)$ into a 2-dimensional state process $X(t)$, as follows: We put

$$\begin{aligned} dX(t) &= \begin{bmatrix} dX_1(t) \\ dX_2(t) \end{bmatrix} = \begin{bmatrix} dZ_\theta(t) \\ dV^{(\pi)}(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ V^{(\pi)}(t^-)\{\rho(t) + (\alpha(t)-\rho(t))\pi\} \end{bmatrix} dt \\ &\quad + \begin{bmatrix} -Z_\theta(t^-)\theta_0(t) \\ V^{(\pi)}(t^-)\beta(t)\pi(t) \end{bmatrix} dB(t) + \begin{bmatrix} -Z_\theta(t^-)\int_{\mathbb{R}}\theta_1(t,z) \\ V^{(\pi)}(t^-)\pi(t)\int_{\mathbb{R}}\gamma(t,z) \end{bmatrix} \widetilde{N}(dt,dz). \end{aligned} \tag{56}$$

To solve this problem, we first write down the Hamiltonian function

$$
\begin{aligned}
&H(t,x_1,x_2,\theta,\pi,p,q,r)\\
&\quad = x_2\{\rho(t)+(\alpha(t)-\rho(t))\pi(t)\}p_2 - x_1\theta_0 q_1\\
&\qquad + x_2\beta(t)\pi(t)q_2 + \int_{\mathbb{R}}\{-x_1\theta_1(t,z)r_1(t,z)+x_2\pi(t)\gamma(t,z)r_2(t,z)\}\nu(dz). \qquad (57)
\end{aligned}
$$

The adjoint equations are

$$
\begin{aligned}
dp_1(t) &= (\theta_0(t)q_1(t)+\int_{\mathbb{R}}\theta_1(t,z)r_1(t,z)\nu(dz))dt + q_1(t)dB(t)\\
&\quad + \int_{\mathbb{R}} r_1(t,z)\widetilde{N}(dt,dz), \qquad (58a)\\
p_1(T) &= \nabla_{x_1}U(X_2(T)) \qquad (58b)
\end{aligned}
$$

and

$$
\begin{aligned}
dp_2(t) &= -[\{\rho(t)+(\alpha(t)-\rho(t))\pi(t)\}p_2(t)+\beta(t)\pi(t)q_2(t)\\
&\quad + \int_{\mathbb{R}}\pi(t)\gamma(t,z)r_2(t,z)\nu(dz)]dt + q_2(t)dB(t) + \int_{\mathbb{R}} r_2(t,z)\widetilde{N}(dt,dz), \qquad (59a)\\
p_2(T) &= \nabla_{x_2}U(X_2(T)). \qquad (59b)
\end{aligned}
$$

Let $(\hat{\theta},\hat{\pi})$ be a candidate optimal control and let $\hat{X}(t)=(\hat{X}_1(t),\hat{X}_2(t))$ be the corresponding optimal processes with corresponding solution $\hat{p}(t)=(\hat{p}_1(t),\hat{p}_2(t))$, $\hat{q}(t)=(\hat{q}_1(t),\hat{q}_2(t))$, $\hat{r}(t,.)=(\hat{r}_1(t,.),\hat{r}_2(t,.))$ of the adjoint equations.

We first maximize the Hamiltonian $E[H(t,x_1,x_2,\theta,\pi,p,q,r)\mid\mathcal{E}_t]$ over all $\pi\in K_2$. This gives the following condition for a maximum point $\hat{\pi}$:

$$
\begin{aligned}
&E[(\alpha(t)-\rho(t))\hat{p}_2(t)\mid\mathcal{E}_t] + E[\beta(t)\hat{q}_2(t)\mid\mathcal{E}_t]\\
&\quad + \int_{\mathbb{R}}\gamma(t,z)E[\gamma(t,z)\hat{r}_2(t,z)\mid\mathcal{E}_t]\nu(dz)=0. \qquad (60)
\end{aligned}
$$

Then, we minimize $E[H(t,x_1,x_2,\theta,\pi,p,q,r)\mid\mathcal{E}_t]$ over all $\theta\in K_1$ and get the following conditions for a minimum point $\hat{\theta}=(\theta_0,\theta_1)$:

$$
E[-\hat{X}_1(t)\hat{q}_1(t)\mid\mathcal{E}_t]=0 \qquad (61)
$$

and

$$
\int_{\mathbb{R}} E[-\hat{X}_1(t)\hat{r}_1(t,z)\mid\mathcal{E}_t]\nu(dz)=0. \qquad (62)
$$

We try a process $\hat{p}_1(t)$ of the form

$$
\hat{p}_1(t)=U(f(t)\hat{X}_2(t)), \qquad (63)
$$

with f a deterministic differentiable function. Differentiating (63) and using (46), we get

$$
\begin{aligned}
d\hat{p}_1(t) = {} & f^{'}(t)\hat{X}_2(t)U^{'}(f(t)\hat{X}_2(t))dt \\
& + \hat{X}_2(t)[(\rho(t) - (\alpha(t) - \rho(t))\hat{\pi}(t))dt \\
& + \beta(t)\hat{\pi}(t)dB(t)]f(t)U^{'}(f(t)\hat{X}_2(t)) \\
& + \frac{1}{2}f^2(t)\hat{X}_2^2(t)\beta^2(t)\pi^2(t)U^{''}(f(t)\hat{X}_2(t))dt \\
& + \int_{\mathbb{R}} \{U(\hat{X}_2(t)(f(t) + \hat{\pi}(t)\gamma(t,z)) - U(f(t)\hat{X}_2(t))) \\
& - \hat{X}_2(t)\hat{\pi}(t)\gamma(t,z)f(t)U^{'}(f(t)\hat{X}_2(t))\}\nu(dz)dt \\
& + \int_{\mathbb{R}} \{U(\hat{X}_2(t)(f(t) + \hat{\pi}(t)\gamma(t,z)) - U(f(t)\hat{X}_2(t)))\}\widetilde{N}(dt,dz) \\
= {} & \Big\{ f^{'}(t)\hat{X}_2(t)U^{'}(f(t)\hat{X}_2(t)) + \frac{1}{2}f^2(t)\hat{X}_2^2(t)\beta^2(t)\pi^2(t)U^{''}(f(t)\hat{X}_2(t)) \\
& + \hat{X}_2(t)(\rho(t) - (\alpha(t) - \rho(t))\hat{\pi}(t))f(t)U^{'}(f(t)\hat{X}_2(t)) \\
& + \int_{\mathbb{R}} \{U(\hat{X}_2(t)(f(t) + \hat{\pi}(t)\gamma(t,z)) - U(f(t)\hat{X}_2(t))) \\
& - \hat{X}_2(t)\hat{\pi}(t)\gamma(t,z)f(t)U^{'}(f(t)\hat{X}_2(t))\}\nu(dz) \Big\} dt \\
& + \hat{X}_2(t)\beta(t)\hat{\pi}(t)f(t)U^{'}(f(t)\hat{X}_2(t))dB(t) \\
& + \int_{\mathbb{R}} \{U(\hat{X}_2(t)(f(t) + \hat{\pi}(t)\gamma(t,z)) - U(f(t)\hat{X}_2(t)))\}\widetilde{N}(dt,dz).
\end{aligned}
$$

Comparing this with (58) by equating the $dt, dB(t)$, $\widetilde{N}(dt,dz)$ coefficients respectively, we get

$$\hat{q}_1(t) = \hat{X}_2(t)\beta(t)\hat{\pi}(t)U^{'}(f(t)\hat{X}_2(t)), \tag{64}$$

$$\hat{r}_1(t,z) = U(\hat{X}_2(t)(f(t) + \hat{\pi}(t)\gamma(t,z))) - U(f(t)\hat{X}_2(t)), \tag{65}$$

and

$$
\begin{aligned}
& f^{'}(t)\hat{X}_2(t)U^{'}(f(t)\hat{X}_2(t)) + \frac{1}{2}\hat{X}_2^2(t)\beta^2(t)\pi^2(t)U^{''}(f(t)\hat{X}_2(t)) \\
& \quad + \hat{X}_2(t)(\rho(t) - (\alpha(t) - \rho(t))\hat{\pi}(t))f(t)U^{'}(f(t)\hat{X}_2(t)) \\
& \quad + \int_{\mathbb{R}} \{U(\hat{X}_2(t)(f(t) + \hat{\pi}(t)\gamma(t,z)) - U(f(t)\hat{X}_2(t))) \\
& \quad - \hat{X}_2(t)\hat{\pi}(t)\gamma(t,z)f(t)U^{'}(f(t)\hat{X}_2(t))\}\nu(dz) \\
& = \hat{\theta}_0(t)\hat{q}_1(t) + \int_{\mathbb{R}} \hat{\theta}_1(t,z)\hat{r}_1(t,z)\nu(dz).
\end{aligned} \tag{66}
$$

Substituting (64) into (61), we get

$$-\hat{\pi}(t)E[\hat{X}_1(t)\hat{X}_2(t)\beta(t)U^{''}(f(t)\hat{X}_2(t)) \mid \mathcal{E}_t] = 0 \tag{67}$$

or

$$\hat{\pi}(t) = 0. \tag{68}$$

Now, we try a process $\hat{p}_2(t)$ of the form

$$\hat{p}_2(t) = \hat{X}_1(t)f(t)U^{'}(f(t)\hat{X}_2(t)). \tag{69}$$

Differentiating (69) and using (68), we get

$$\begin{aligned} d\hat{p}_2(t) &= f^{'}(t)\hat{X}_1(t)U^{'}(f(t)\hat{X}_2(t))dt + f(t)U^{'}(f(t)\hat{X}_2(t))d\hat{X}_1(t) \\ &\quad + f(t)\hat{X}_1(t)dU^{'}(f(t)\hat{X}_2(t)) \\ &= \hat{X}_1(t)(f^{'}(t)U^{'}(f(t)\hat{X}_2(t)) + f(t)f^{'}(t)\hat{X}_2(t)U^{''}(f(t)\hat{X}_2(t)) \\ &\quad + f^2(t)\hat{X}_2(t)\rho(t)U^{''}(f(t)\hat{X}_2(t)))dt \\ &\quad - f(t)\hat{X}_1(t)\theta_0(t)U^{'}(f(t)\hat{X}_2(t))dB(t) \\ &\quad - \int_{\mathbb{R}} f(t)\hat{X}_1(t)\theta_1(t,z)U^{'}(f(t)\hat{X}_2(t))\widetilde{N}(dt,dz). \end{aligned} \tag{70}$$

Comparing this with (59), we get

$$\hat{q}_2(t) = -f(t)\hat{X}_1(t)\theta_0(t)U^{'}(f(t)\hat{X}_2(t)), \tag{71}$$

$$\hat{r}_2(t,z) = -f(t)\hat{X}_1(t)\theta_1(t,z)U^{'}(f(t)\hat{X}_2(t)) \tag{72}$$

and

$$\begin{aligned} &f^{'}(t)U^{'}(f(t)\hat{X}_2(t)) + f(t)\hat{X}_2(t)U^{''}(f(t)\hat{X}_2(t))(f^{'}(t) + f(t)\rho(t)) \\ &\quad = -\rho(t)f(t)U^{'}(f(t)\hat{X}_2(t)). \end{aligned} \tag{73}$$

Substituting (71), (72) into (60), we get

$$\begin{aligned} &E[(\alpha(t) - \rho(t))f(t)\hat{X}_1(t)U^{'}(f(t)\hat{X}_2(t)) \mid \mathcal{E}_t] \\ &\quad - \theta_0(t)E[\beta(t)f(t)\hat{X}_1(t)U^{'}(f(t)\hat{X}_2(t)) \mid \mathcal{E}_t] \\ &\quad - \int_{\mathbb{R}} \theta_1(t,z)E[\gamma(t,z)f(t)\hat{X}_1(t)U^{'}(f(t)\hat{X}_2(t)) \mid \mathcal{E}_t] = 0. \end{aligned} \tag{74}$$

This can be written as

$$\hat{\theta}_0(t)E[\beta(t) \mid \mathcal{E}_t] - \int_{\mathbb{R}} \hat{\theta}_1(t,z)E[\gamma(t,z) \mid \mathcal{E}_t]\nu(dz) = E[\alpha(t) \mid \mathcal{E}_t] - \rho(t). \tag{75}$$

From (73), we get

$$\big(U^{'}(f(t)\hat{X}_2(t)) + \hat{X}_2(t)f(t)U^{''}(f(t)\hat{X}_2(t))\big)(f^{'}(t) + r(t)f(t)) = 0 \tag{76}$$

or

$$f^{'}(t) + r(t)f(t) = 0, \tag{77}$$

i.e.

$$f(t) = \exp\left(\int_t^T r(s)ds\right). \tag{78}$$

We have proved the following theorem.

Theorem 4.1 *The optimal portfolio $\pi \in \Pi$ for the agent is*

$$\pi(t) = \hat{\pi}(t) = 0 \tag{79}$$

and the optimal measure $Q_{\hat{\theta}}$ for the market is to choose $\hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1)$ such that

$$\hat{\theta}_0(t)E[\beta(t) \mid \mathcal{E}_t] - \int_{\mathbb{R}} \hat{\theta}_1(t,z)E[\gamma(t,z) \mid \mathcal{E}_t]\nu(dz) = E[\alpha(t) \mid \mathcal{E}_t] - \rho(t). \tag{80}$$

Remark 4.1 In the case when $\mathcal{E}_t = \mathcal{F}_t$ for all t, this was proved in [6]. In this case, the interpretation of this result is the following: The market minimizes the maximal expected utility of the agent by choosing a scenario (represented by a probability law $dQ_\theta = Z_\theta(T)dP$), which is an equivalent martingale measure for the market (see (54)). In this case, the optimal strategy for the agent is to place all the money in the risk free asset, i.e. to choose $\pi(t) = 0$ for all t. Theorem 4.1 states that an analogue result holds also in the case when both players have only partial information $\mathcal{E}_t \subseteq \mathcal{F}_t$ to their disposal, but now the coefficients $\beta(t)$, $\gamma(t,z)$ and $\alpha(t)$ must be replaced by their conditional expectations $E[\beta(t) \mid \mathcal{E}_t]$, $E[\gamma(t,z) \mid \mathcal{E}_t]$ and $E[\alpha(t) \mid \mathcal{E}_t]$.

5 Sufficient Maximum Principle for Nonzero-Sum Game

Let $X(t)$ be a stochastic process describing the state of the system. We consider now the case when two controllers *I* and *II* intervene on the dynamics of the system and their advantages are not necessarily antagonistic but each one acts so as to save her own interest. This situation is a nonzero-sum game.

Let $\mathcal{E}_t^1, \mathcal{E}_t^2$ be the filtrations satisfying

$$\mathcal{E}_t^i \subseteq \mathcal{F}_t, \quad t \geq 0, \quad i = 1, 2.$$

Let $u = (\theta, \pi)$, where $\theta = (\theta_0, \theta_1)$ and $\pi = (\pi_0, \pi_1)$ are the controls for players *I* and *II*, respectively. We assume that $\theta = (\theta_0, \theta_1)$ are adapted to $\mathcal{E}_t^1$ and $\pi = (\pi_0, \pi_1)$ are adapted to $\mathcal{E}_t^2$. Denote by Θ and Π the sets of admissible controls θ and π, respectively. Suppose that the players act on the system with strategy $(\theta, \pi) \in \Theta \times \Pi$;

then, the costs associated with I and II are, respectively, $J_1^{(\theta,\pi)}(x)$ and $J_2^{(\theta,\pi)}(x)$ of the form

$$J_i(\theta,\pi) = E^x\left[\int_0^T f_i(t, X(t), u(t))\,dt + g_i(X(T))\right], \quad i = 1, 2. \tag{81}$$

The problem is to find a control $(\theta^*, \pi^*) \in \Theta \times \Pi$ such that

$$J_1(\theta,\pi^*) \le J_1(\theta^*,\pi^*), \quad \text{for all } \theta \in \Theta, \tag{82}$$

$$J_2(\theta^*,\pi) \le J_2(\theta^*,\pi^*), \quad \text{for all } \pi \in \Pi. \tag{83}$$

The pair of controls (θ^*, π^*) is called a *Nash equilibrium point* for the game because, when player I (resp. II) acts with the strategy θ^* (resp. π^*), the best that II (resp. I) can do is to act with strategy π^* (resp. θ^*).

Let us introduce the Hamiltonian functions associated with this game, namely H_1 and H_2, from $[0,T] \times \mathbb{R}^n \times K_1 \times K_2 \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \times \mathcal{R}$ to $\mathbb{R}$, which are defined by

$$\begin{aligned} &H_i(t,x,\theta,\pi,p_i,q_i,r^i) \\ &\quad = f_i(t,x,\theta,\pi) + b^{\mathsf{T}}(t,x,\theta,\pi)p_i + tr(\sigma^{\mathsf{T}}((t,x,\theta,\pi)q_i) \\ &\qquad + \sum_{k,j=1}^{n} \int_{\mathbb{R}_0} \gamma_{kj}(t,x,\theta,\pi,z) r^i_{kj}(t,z)\nu_j(dz_j), \quad i = 1,2. \end{aligned} \tag{84}$$

We also have the adjoint equations for the game, as follows:

$$\begin{aligned} dp_i(t) &= -\nabla_x H_i(t, X(t), \theta(t), \pi(t), p_i(t), q_i(t), r^i(t,.))dt \\ &\quad + q_i(t)dB(t) + \int_{\mathbb{R}^n} r^i(t^-,z)\widetilde{N}(dt,dz), \quad t < T, \end{aligned} \tag{85a}$$

$$p_i(T) = \nabla g_i(X(T)), \quad i = 1, 2. \tag{85b}$$

The following result is a generalization of Theorem 2.1.

Theorem 5.1 *Let* $(\hat\theta,\hat\pi) \in \Theta \times \Pi$ *with corresponding state process* $\hat X(t) = X^{(\hat\theta,\hat\pi)}(t)$. *Suppose that there exists a solution* $(\hat p_i(t), \hat q_i(t), \hat r^i(t,z))$, $i = 1,2$, *of the corresponding adjoint equation* (85) *such that, for all* $\theta \in \Theta$ *and* $\pi \in \Pi$, *we have*

$$\begin{aligned} &E[H_1(t, \hat X(t), \hat\theta(t), \hat\pi(t), \hat p_1(t), \hat q_1(t), \hat r^1(t,.)) \mid \mathcal{E}_t^1] \\ &\quad \ge E[H_1(t, \hat X(t), \theta(t), \hat\pi(t), \hat p_1(t), \hat q_1(t), \hat r^1(t,.)) \mid \mathcal{E}_t^1] \end{aligned} \tag{86}$$

and

$$\begin{aligned} &E[H_2(t, \hat X(t), \hat\theta(t), \hat\pi(t), \hat p_2(t), \hat q_2(t), \hat r^2(t,.)) \mid \mathcal{E}_t^2] \\ &\quad \ge E[H_2(t, \hat X(t), \hat\theta(t), \pi(t), \hat p_2(t), \hat q_2(t), \hat r^2(t,.)) \mid \mathcal{E}_t^2]. \end{aligned} \tag{87}$$

Moreover, suppose that, for all $t \in [0,T]$, $H_i(t,x,\theta,\pi,\hat p_i(t),\hat q_i(t),\hat r^i(t,.))$, $i = 1,2$, *is concave in* x,θ,π *and* $g_i(x)$, $i = 1,2$, *is concave in* x. *Then,* $(\hat\theta(t),\hat\pi(t))$ *is a Nash equilibrium point for the game and*

$$J_1(\hat{\theta}, \hat{\pi}) = \sup_{\theta \in \Theta} J_1(\theta, \hat{\pi}), \tag{88}$$

$$J_2(\hat{\theta}, \hat{\pi}) = \sup_{\pi \in \Pi} J_2(\hat{\theta}, \pi). \tag{89}$$

Proof As in the proof of Theorem 2.1, we have

$$\begin{aligned} & J_1(\hat{\theta}, \hat{\pi}) - J_1(\theta, \hat{\pi}) \\ &\quad = E\bigg[\int_0^T \big\{f_1(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t)) - f_1(t, X^{(\pi)}(t), \hat{\theta}(t), \pi(t))\big\}dt \\ &\qquad + g_1(\hat{X}(T)) - g_1(X^{(\pi)}(T))\bigg] \\ &\quad = E\bigg[\int_0^T \big\{H_1(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}_1(t), \hat{q}_1(t), \hat{r}^1(t,.)) \\ &\qquad - H_1(t, X^{(\pi)}(t), \hat{\theta}(t), \pi(t), \hat{p}_1(t), \hat{q}_1(t), \hat{r}^1(t,.)) \\ &\qquad - \nabla_x H_1(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}_1(t), \hat{q}_1(t), \hat{r}^1(t,.)))(\hat{X}(t) - X^{(\pi)}(t))^T\big\}dt\bigg]. \end{aligned} \tag{90}$$

From (86) and the concavity of H_1 in x and π, we have

$$\begin{aligned} & E\bigg[\int_0^T \{H_1(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}_1(t), \hat{q}_1(t), \hat{r}^1(t,.)) \\ &\qquad - H_1(t, X^{(\pi)}(t), \hat{\theta}(t), \pi(t), \hat{p}_1(t), \hat{q}_1(t), \hat{r}^1(t,.))\}dt\bigg] \\ &\quad \geq E\bigg[\int_0^T \nabla_x H_1(t, \hat{X}(t), \hat{\theta}(t), \hat{\pi}(t), \hat{p}_1(t), \hat{q}_1(t), \hat{r}^1(t,.))(\hat{X}(t) - X^{(\pi)}(t))^T dt\bigg]. \end{aligned} \tag{91}$$

Hence,

$$J_1(\hat{\theta}, \hat{\pi}) - J_1(\theta, \hat{\pi}) \geq 0. \tag{92}$$

Since this holds for all $\theta \in \Theta$, we have

$$J_1(\hat{\theta}, \hat{\pi}) = \sup_{\theta \in \Theta} J_1(\theta, \hat{\pi}). \tag{93}$$

In the same way, we show that

$$J_2(\theta^*, \pi) \leq J_2(\theta^*, \pi^*)$$

and

$$J_2(\hat{\theta}, \hat{\pi}) = \sup_{\pi \in \pi} J_2(\hat{\theta}, \pi),$$

whence the desired result. □

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