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Existence Theorems of Quasivariational Inclusion Problems with Applications to Bilevel Problems and Mathematical Programs with Equilibrium Constraint

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Abstract In this paper, we establish existence theorems of quasivariational inclusion problems; from them, we establish existence theorems of mathematical programs with quasivariational inclusion constraint, bilevel problems, mathematical programs with equilibrium constraint and semi-infinite problems.

Keywords Quasivariational inclusions problems · Bilevel problems · Mathematical programs with equilibrium constraints · Quasiequilibrium problem

1 Introduction

Let X and Y be nonempty closed convex subsets of locally convex Hausdorff topological vector spaces (in short t.v.s.) E_1 and E_2, respectively; let $S : X \multimap X$ and $T : X \multimap Y$ be multivalued maps. Let $F : X \times Y \times Y \to \mathbb{R}$, $f : X \times Y \to \mathbb{R}$ and $g : X \times Y \to \mathbb{R}$ be functions. The bilevel problem is the following problem:

$$\text{(BVI)} \quad \min_{(x,y)} f(x, y),$$

$$\text{s.t.} \quad (x, y) \in X \times Y, \quad x \in S(x),\ y \in T(x), \qquad g(x, y) \geq 0,$$

$$y \text{ is a solution of the problem } \min_{u \in T(x)} F(x, y, u).$$

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Lin and Still [1] showed that, if we assume that $F(x,y,y)=0$ for all $(x,y)\in X\times Y$, then (BVI) is equivalent to the following mathematical program with equilibrium constraint:

$$\text{(MPI)}\quad \min_{(x,y)} f(x,y),$$
$$\text{s.t.}\quad (x,y)\in X\times Y,\quad x\in S(x),\ y\in T(x),\qquad g(x,y)\geq 0,$$
$$F(x,y,u)\geq 0,\quad \text{for all } u\in T(x).$$

If $F: X\times Y\to\mathbb{R}$, then (BVI) reduces to the following bilevel problem:

$$\text{(BVII)}\quad \min_{(x,y)} f(x,y),$$
$$\text{s.t.}\quad (x,y)\in X\times Y,\quad x\in S(x),\ y\in T(x),\qquad g(x,y)\geq 0,$$
$$y \text{ is a solution of the problem}\quad \min_{u\in T(x)} F(x,u).$$

If we assume that $F(x,y)=0$ for all $x\in S(x)$, $y\in T(x)$ and $g(x,y)\geq 0$ in (BVII), as shown in Lin and Still [1], problem (BVII) is equivalent to the following semi-infinite problem:

$$\text{(SIPI)}\quad \min_{(x,y)} f(x,y),$$
$$\text{s.t.}\quad (x,y)\in X\times Y,\quad x\in S(x),\ y\in T(x),\qquad g(x,y)\geq 0,$$
$$F(x,u)\geq 0,\quad \text{for all } u\in T(x).$$

These programs represent three important classes of optimization problems which have been investigated in a large number of papers and books when $S(x)=X$ for all $x\in X$ (see e.g. [1–5] and references therein. Lin and Hsu [6] and Lin [7] studied mathematical programs with equilibrium constraints.

Let Z be a t.v.s. with a cone C. We denote $l(C)=C\cap(-C)$. If $l(C)=\{0\}$, we say that C is a pointed cone. We recall the following definitions [8].

Let A be a nonempty subset of a real t.v.s. Z with a cone C. We say that $x\in A$ is an ideal efficient (or ideal minimal) point of A w.r.t. C if $y-x\in C$ for every $y\in A$. The set of ideal efficient points of A is denoted by $\operatorname{IMin}(A/C)$. A point $x\in A$ is an efficient (or Pareto-minimal, or nondominated) point of A w.r.t. C if there is no $y\in A$ with $x-y\in C\setminus l(C)$. The set of efficient points of A is denoted by $\operatorname{Min}(A/C)$. A point $x\in A$ is a (global) properly efficient point of A w.r.t. C if there exists a convex cone $\tilde{C}$ which is not the whole space and contains $C\setminus\ell(C)$ in its interior so that $x\in\operatorname{Min}(A/\tilde{C})$. The set of properly efficient points of A is denoted by $\operatorname{Pr}\operatorname{Min}(A/C)$. Suppose that $\operatorname{int}C$ is nonempty; $x\in A$ is a weakly efficient point of A w.r.t. C if $x\in\operatorname{Min}(A/\{0\}\cup\operatorname{int}C)$. The set of weakly efficient points of A is denoted by $\operatorname{W}\operatorname{Min}(A/C)$. The following inclusions are obvious:

$$\operatorname{IMin}(A/C)\subseteq\operatorname{Pr}\operatorname{Min}(A/C)\subseteq\operatorname{Min}(A/C)\subseteq\operatorname{W}\operatorname{Min}(A/C).$$

Furthermore, if $\operatorname{IMin}(A/C)\neq\emptyset$, then $\operatorname{IMin}(A/C)=\operatorname{Min}(A/C)$ and it is a singleton whenever C is pointed (see Proposition 2.2, Chap. 2 in [8]).

Let E_1, E_2 be t.v.s., let Z be a real t.v.s., let $X \subset E_1$, $Y \subset E_2$ be nonempty subsets. Let $S : X \multimap X$, $T : X \multimap Y$, $F : X \times Y \times Y \multimap Z$, $C : X \times Y \multimap Z$ such that for each $(x, y) \in X \times Y$, $C(x, y)$ is a convex cone and $\operatorname{int} C(x, y) \neq \emptyset$.

In this paper, we are interested in the following problems: Find $\bar{x} \in X$, $\bar{y} \in Y$, $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ such that one of the following relations holds:

(i) $F(\bar{x}, \bar{y}, u) \subseteq F(\bar{x}, \bar{y}, \bar{y}) + C(\bar{x}, \bar{y})$ for all $u \in T(\bar{x})$.
(ii) $F(\bar{x}, \bar{y}, u) \cap [F(\bar{x}, \bar{y}, \bar{y}) + C(\bar{x}, \bar{y})] \neq \emptyset$ for all $u \in T(\bar{x})$.
(iii) $F(\bar{x}, \bar{y}, u) \nsubseteq [F(\bar{x}, \bar{y}, \bar{y}) - \operatorname{int} C(\bar{x}, \bar{y})]$ for all $u \in T(\bar{x})$.
(iv) $F(\bar{x}, \bar{y}, \bar{y}) \subseteq F(\bar{x}, \bar{y}, u) - C(\bar{x}, \bar{y})$ for all $u \in T(\bar{x})$.
(v) $F(\bar{x}, \bar{y}, \bar{y}) \nsubseteq [F(\bar{x}, \bar{y}, u) + \operatorname{int} C(\bar{x}, \bar{y})]$ for all $u \in T(\bar{x})$.

Our problems contains the following problems as special cases.

If $F(x, y, y) \subseteq C(x, y)$ for each $(x, y) \in X \times Y$, then problems (i) and (ii) reduced to:

(i)′ Find $\bar{x} \in X$, $\bar{y} \in Y$ s.t. $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$F(\bar{x}, \bar{y}, u) \subseteq C(\bar{x}, \bar{y}), \quad \text{for all } u \in T(\bar{x}).$$

(ii)′ Find $\bar{x} \in X$, $\bar{y} \in Y$ s.t. $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$F(\bar{x}, \bar{y}, u) \cap C(\bar{x}, \bar{y}) \neq \emptyset, \quad \text{for all } u \in T(\bar{x}).$$

If $F(x, y, y) = \{0\}$ for each $(x, y) \in X \times Y$, then problem (iii) reduced to:

(iii)′ Find $\bar{x} \in X$, $\bar{y} \in Y$ s.t. $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$F(\bar{x}, \bar{y}, u) \nsubseteq -\operatorname{int} C(\bar{x}, \bar{y}), \quad \text{for all } u \in T(\bar{x}).$$

If F is a single-valued map, problems (i), (ii), (iv) reduce to the following problem:

Find $\bar{x} \in X$, $\bar{y} \in Y$ s.t. $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ and $F(\bar{x}, \bar{y}, u) \in F(\bar{x}, \bar{y}, \bar{y}) + C(\bar{x}, \bar{y})$ for all $u \in T(\bar{x})$, that is, $F(\bar{x}, \bar{y}, \bar{y}) \in \operatorname{IMin}(F(\bar{x}, \bar{y}, T(\bar{x}))/C(\bar{x}, \bar{y}))$.

If F is a single-valued map, problems (iii) and (v) reduce to the following problem:

Find $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ s.t. $F(\bar{x}, \bar{y}, u) \notin F(\bar{x}, \bar{y}, \bar{y}) - \operatorname{int} C(\bar{x}, \bar{y})$, for all $u \in T(\bar{x})$, that is, $F(\bar{x}, \bar{y}, \bar{y}) \in \operatorname{WMin}(F(\bar{x}, \bar{y}, T(\bar{x}))/C(\bar{x}, \bar{y})) \neq \emptyset$.

Recently, Tan [9] used the scalarization method to study the following problems:

(UQVIP) Find $(\bar{x}, \bar{y}) \in X \times Y$, $\quad \bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$\text{s.t.} \quad F(\bar{x}, \bar{y}, x) \subseteq F(\bar{x}, \bar{y}, \bar{x}) + C, \quad \text{for all } x \in S(\bar{x}).$$

(LQVIP) Find $(\bar{x}, \bar{y}) \in X \times Y$, $\quad \bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$\text{s.t.} \quad F(\bar{x}, \bar{y}, \bar{x}) \subseteq F(\bar{x}, \bar{y}, x) - C, \quad \text{for all } x \in S(\bar{x}).$$

As applications, he studied ideal, Pareto, proper and weak quasioptimization problems. In [9], S and T are assumed to satisfy the upper semicontinuous property, and additional conditions as imposed on the polar cone of C. Luc and Tan [10] studied also the variational inclusion problem with the scalarization method, but the form of their problems is different from ours. Lin and Tan [11] studied the following problem:

Find $(\bar{x}, \bar{y}) \in X \times Y$, s.t. $\bar{x} \in S(\bar{x}, \bar{y})$, $\bar{y} \in T(\bar{x}, \bar{y})$,

$$F(\bar{x}, \bar{y}, u) \subseteq F(\bar{x}, \bar{y}, \bar{y}) + C, \quad \text{for all } u \in T(\bar{x}, \bar{y}).$$

In that paper, the existence theorems of this problems were studied and some applications to the study of the vector quasiequilibrium problem and vector quasioptimization problem were given. Note also that S and T were assumed to be u.s.c. functions there.

In this paper, we study existence theorems of variational inclusion problems, but we do not imposed on T any continuity assumptions. The readers should to note the differences between our results and the methods with [9–11].

As application of our result, we study the existence theorem of the following mathematical programs with variational inclusion constraint:

$$\text{(VMPIC)} \quad \mathrm{Min}(f(\mathcal{M})/C_0) \neq \emptyset, \quad \text{where}$$
$$\mathcal{M} = \{(x, y) \in X \times Y \mid x \in S(x),\ y \in T(x),\ g(x, y) \subseteq C(x, y),$$
$$F(x, y, u) \subseteq [F(x, y, y) + C(x, y)] \text{ for all } u \in T(x)\}.$$

From the mathematical programs with variational inclusion constraint, we study the existence theorem for the following bilevel problem:

$$\text{(VBVI)} \quad \mathrm{Min}(f(\mathcal{M})/C_0) \neq \emptyset, \quad \text{where}$$
$$\mathcal{M} = \{(x, y) \in X \times Y \mid x \in S(x),\ y \in T(x),\ g(x, y) \subseteq C(x, y),$$
$$F(x, y, y) \cap \mathrm{I\,Min}\, F(x, y, T(x)) \neq \emptyset\}.$$

We study also the following mathematical program with variational inclusion constraint:

$$\text{(VMPI)} \quad \mathrm{Min}(f(\mathcal{M})\,/\,C_0) \neq \emptyset, \quad \text{where}$$
$$\mathcal{M} = \{(x, y) \in X \times Y \mid x \in S(x),\ y \in T(x),\ g(x, y) \nsubseteq -\mathrm{int}\, C(x, y),$$
$$F(x, y, u) \nsubseteq [F(x, y, y) - \mathrm{int}\, C(x, y)] \text{ for all } u \in T(x)\}.$$

If $Z = \mathbb{R}$, $C(x, y) = [0, \infty)$, and if F, f, g are real-valued maps, then problems (VMPIC), (VBVI), and (VMPI) reduce to (BVI).

Our results on existence theorems of quasivariational inclusion problems, mathematical programs with variational inclusion constraints, and bilevel problems are different from any existence result in the literature; see [1–7, 9–11] and references therein.

2 Preliminaries

Let X and Y be nonempty sets. A multivalued map $T : X \multimap Y$ is a function from X into the power set of T. Let $x \in X$ and $y \in Y$, we denote $x \in T^{-}(y)$ if and only if $y \in T(x)$. Let X and Y be topological spaces and $T : X \multimap Y$. One may consult lower semicontinuity (l.s.c.) [12] for the definitions of upper semicontinuouity (in short u.s.c.), continuity, graph and closeness of T.

The following definitions and theorems are needed in this paper.

Theorem 2.1 [12] *Let X and Y be Hausdorff topological spaces, let $T : X \multimap Y$ be a multivalued map.*

(i) *If f is an u.s.c. multivalued map with closed values, then T is closed.*
(ii) *If X is compact and T is an u.s.c. multivalued map with nonempty compact values, then $T(X)$ is compact.*

Theorem 2.2 [13] *Let X be a nonempty compact convex subset of a t.v.s. and let $F : X \multimap Y$ be a multivalued map. Suppose that:*

(i) *For each $x \in X$, $x \notin F(x)$ and $F(x)$ is convex.*
(ii) *For each $y \in X$, $F^{-}(y)$ is open in X.*

Then, there exists $\bar{x} \in X$ such that $F(\bar{x}) = \emptyset$.

Definition 2.1 Let X and Y be convex subsets of a t.v.s. and let Z be a t.v.s. Let $C : X \times Y \multimap Z$ be a multivalued map such that, for each $(x, y) \in X \times Y$, $C(x, y)$ is a closed convex cone. $F : X \times Y \times Y \multimap Z$ is a multivalued map.

(i) For each $(x, y) \in X \times Y$, $u \multimap F(x, y, u)$ is called $C(x, y)$-quasiconvex if, for $u_1, u_2 \in Y$ and $\lambda \in [0, 1]$, we have

$$\text{either} \quad F(x, y, u_1) \subseteq F(x, y, \lambda u_1 + (1 - \lambda)u_2) + C(x, y)$$
$$\text{or} \quad F(x, y, u_2) \subseteq F(x, y, \lambda u_1 + (1 - \lambda)u_2) + C(x, y).$$

(ii) For each $(x, y) \in X \times Y$, $u \multimap F(x, y, u)$ is called $C(x, y)$-quasiconvex-like if, for any $(x, y) \in X \times Y$, $u_1, u_2 \in Y$ and $\lambda \in [0, 1]$, we have

$$\text{either} \quad F(x, y, \lambda u_1 + (1 - \lambda)u_2) \subseteq F(x, y, u_1) - C(x, y)$$
$$\text{or} \quad F(x, y, \lambda u_1 + (1 - \lambda)u_2) \subseteq F(x, y, u_2) - C(x, y).$$

Definition 2.2 Let X be a convex subset of a t.v.s. and let Z be a t.v.s., let $F : X \multimap Z$ be a multivalued map. F is concave (resp. convex) if, for all $x_1, x_2 \in X$, $\lambda \in [0, 1]$, $\lambda F(x_1) + (1 - \lambda)F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2)$; (resp. $F(\lambda x_1 + (1 - \lambda)x_2) \subset \lambda F(x_1) + (1 - \lambda)F(x_2)$).

3 Main Results

In this section, we establish some existence results for quasivariational inclusion problems. Let E_1 and E_2 be locally convex Hausdorff t.v.s., let Z be a real Hausdorff t.v.s., let X and Y be nonempty closed convex subsets of E_1 and E_2 respectively, let $S : X \multimap X$, $T : X \multimap Y$, $C : X \times Y \multimap Z$, $F : X \times Y \times Y \multimap Z$ be multivalued maps. Let $W : X \times Y \multimap Z$ be defined by $W(x, y) = Z \backslash (-\operatorname{int} C(x, y))$. Throughout the following sections, we use these notations unless specified otherwise. Throughout this paper, all topological spaces are assumed to be Hausdorff.

Theorem 3.1 *Suppose that*:

(i) *S is a compact* [12] *u.s.c. multivalued map with nonempty closed convex values and T is a multivalued map with nonempty compact convex values.*
(ii) *C is a closed multivalued map such that, for each (x, y) in $X \times Y$, $C(x, y)$ is a nonempty convex cone.*
(iii) *F is a continuous multivalued map with nonempty compact values such that, for each $(x, y) \in X \times Y$, $u \multimap F(x, y, u)$ is $C(x, y)$-quasiconvex.*

Then, there exists $(\bar{x}, \bar{y}) \in X \times Y$ s.t. $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ and

$$\text{(H1)} \quad F(\bar{x}, \bar{y}, u) \subseteq [F(\bar{x}, \bar{y}, \bar{y}) + C(\bar{x}, \bar{y})], \quad \textit{for all } u \in T(\bar{x}).$$

Proof For fixed x in X, define $m_x : T(x) \multimap T(x)$ by

$$m_x(y) = \{u \in T(x) \mid F(x, y, u) \nsubseteq F(x, y, y) + C(x, y)\}.$$

Then for each $y \in T(x)$, it is clear that $y \notin m_x(y)$. We claim that $m_x(y)$ is convex. Indeed, if $u_1, u_2 \in m_x(y)$ and $t \in [0, 1]$, then $u_1, u_2 \in T(x)$, $F(x, y, u_i) \nsubseteq F(x, y, y) + C(x, y)$ for $i = 1, 2$. and $u_t = tu_1 + (1 - t)u_2 \in T(x)$ for all $t \in [0, 1]$. Suppose that there exists $t_0 \in (0, 1)$ such that $u_{t_0} \notin m_x(y)$, then $F(x, y, u_{t_0}) \subseteq F(x, y, y) + C(x, y)$. By (iii), either $F(x, y, u_1) \subseteq F(x, y, y) + C(x, y)$ or $F(x, y, u_2) \subseteq F(x, y, y) + C(x, y)$ hold. This leads to a contradiction. Hence $u_t \in m_x(y)$ for all $t \in [0, 1]$. Therefore $m_x(y)$ is convex for each $y \in T(x)$. Fixed $u \in T(x)$, $m_x^{-1}(u)$ is open in $T(x)$. Indeed, if $y \in \overline{T(x)\backslash m_x^-(u)}$, then there exists a net $\{y_\alpha\}_{\alpha \in \Lambda} \in T(x)\backslash m_x^-(u)$ such that $y_\alpha \longrightarrow y$. Then $y_\alpha \in T(x)$ and $F(x, y_\alpha, u) \subseteq F(x, y_\alpha, y_\alpha) + C(x, y_\alpha)$. Let $z \in F(x, y, u)$, then by (iii), there exists a net $\{z_\alpha\}_{\alpha \in \Lambda}$ such that $z_\alpha \in F(x, y_\alpha, u)$ and $z_\alpha \longrightarrow z$. There exist $v_\alpha \in F(x, y_\alpha, y_\alpha)$ and $c_\alpha \in C(x, y_\alpha)$ such that $z_\alpha = v_\alpha + c_\alpha$. Let $A = \{y_\alpha : \alpha \in \Lambda\} \cup \{y\}$, then A is compact. By (iii) and Theorem 2.1, $F(\{x\} \times A \times A)$ is compact. There exists a subnet $\{v_{\alpha_\lambda}\}$ of $\{v_\alpha\}$ such that $v_{\alpha_\lambda} \longrightarrow v \in F(\{x\} \times A \times A)$. By (iii) and Theorem 2.1, F is closed, hence $v \in F(x, y, y)$. Then $c_{\alpha_\lambda} = z_{\alpha_\lambda} - v_{\alpha_\lambda} \longrightarrow z - v = c$. By (ii), $c \in C(x, y)$, then $z = v + c \in F(x, y, y) + C(x, y)$. Therefore $F(x, y, u) \subseteq F(x, y, y) + C(x, y)$, that is, $y \notin m_x^{-1}(u)$. Since $T(x)$ is closed, $y \in T(x)$. Therefore $y \in (T(x)\backslash m_x^-(u))$ and $T(x)\backslash m_x^-(u)$ is closed. Hence $m_x^{-1}(u)$ is open in $T(x)$ for all $u \in T(x)$. By (i), $T(x)$ is compact and convex. Then by Theorem 2.2 that there exists $\hat{y} \in T(x)$ such that $m_x(\hat{y}) = \emptyset$. That is there exists $\hat{y} \in T(x)$ such that $F(x, \hat{y}, u) \subseteq F(x, \hat{y}, \hat{y}) + C(x, \hat{y})$ for all $u \in T(x)$, so $\hat{y} \in M(x) \neq \emptyset$. By (i) and Himmelberg's fixed point theorem [14], there exists $\bar{x} \in X$ such that $\bar{x} \in S(\bar{x})$ and Theorem 3.1 follows. □

Corollary 3.1 *In Theorem 3.1, if we assume further that T is a compact continuous multivalued map, then the set of solutions is compact.*

Proof Let $M = \{(x, y) \in X \times Y | x \in S(x),\ y \in T(x),\ F(x, y, u) \subseteq F(x, y, y) + C(x, y)$ for all $u \in T(x)\}$. Then M is a closed set. Indeed, if $(x, y) \in \overline{M}$, there exists a net $\{(x_\alpha, y_\alpha)\}_{\alpha \in \Lambda}$, $(x_\alpha, y_\alpha) \in M$ such that $(x_\alpha, y_\alpha) \longrightarrow (x, y)$. One has $x_\alpha \in X$, $y_\alpha \in Y$, $y_\alpha \in T(x_\alpha)$ and $F(x_\alpha, y_\alpha, w) \subseteq F(x_\alpha, y_\alpha, y_\alpha) + C(x_\alpha, y_\alpha)$ for all $w \in T(x_\alpha)$. Then $x \in X$ and $y \in Y$. By assumption on T and Theorem 2.1, T is closed and $y \in T(x)$. Fixed $u \in T(x)$, and $z \in F(x, y, u)$. Since T and F are l.s.c., there exist $u_\alpha \in T(x_\alpha)$ $z_\alpha \in F(x_\alpha, y_\alpha, u_\alpha)$ such that $u_\alpha \longrightarrow u$ and $z_\alpha \longrightarrow z$. Therefore, $F(x_\alpha, y_\alpha, w_\alpha) \subseteq F(x_\alpha, y_\alpha, y_\alpha) + C(x_\alpha, y_\alpha)$. It is easy to see that there exist $v_\alpha \in F(x_\alpha, y_\alpha, y_\alpha)$ and $c_\alpha \in C(x_\alpha, y_\alpha)$ such that $z_\alpha = v_\alpha + c_\alpha$. Let $A = \{x_\alpha : \alpha \in \Lambda\} \cup \{x\}$, $B = \{y_\alpha : \alpha \in \Lambda\} \cup \{y\}$, then A, B are compact. We see $F(A \times B \times B)$ is compact. There exists a subnet $\{v_{\alpha_\lambda}\}_{\alpha_\lambda \in \Lambda}$ of $\{v_\alpha\}_{\alpha \in \Lambda}$ such that $v_{\alpha_\lambda} \longrightarrow v \in F(A \times B \times B)$. We see $v \in F(x, y, y)$. Then $c_{\alpha_\lambda} = z_{\alpha_\lambda} - v_{\alpha_\lambda} \longrightarrow z - v$, and $z - v \in C(x, y)$. Hence $z = v + c \in F(x, y, y) + C(x, y)$. Therefore $F(x, y, u) \subseteq F(x, y, y) + C(x, y)$ for all $u \in T(x)$, and $(x, y) \in M$. That is M is a closed set. Since $M \subseteq \overline{S(X)} \times \overline{T(X)}$ and $\overline{S(X)} \times \overline{T(X)}$ is compact, then M is compact. □

Remark 3.1

(i) Lin and Tan [11] study the problem: (i) Find $\bar{x} \in S(\bar{x}, \bar{y})$, $\bar{y} \in T(\bar{x})$ such that $F(\bar{x}, \bar{y}, u) \subseteq F(\bar{x}, \bar{y}, \bar{y}) + C(\bar{x})$ for all $u \in T(\bar{x})$, where S: $X \times Y \multimap X$. In [11], T is assumed to be a compact continuous multivalued map with nonempty closed convex values, but in Theorem 3.1, T is assumed to be a multivalued map with nonempty compact convex values. The proof of Theorem 3.1 is different from [11].

(ii) In [11], Lin and Tan show that, if $\mathrm{I\,Min}(F(x, y, y)/C(x, y)) \neq \emptyset$, then $(\bar{x}, \bar{y})$ is a solution of the problem in Theorem 3.1 if and only if $(\bar{x}, \bar{y})$ is a solution of the following problem: Find $(x, y) \in X \times Y$ such that $x \in S(x)$, $y \in T(y)$ and $F(x, y, y) \cap \mathrm{I\,Min}(F(x, y, T(x))/C(x, y)) \neq \emptyset$.

Following the same arguments as in [11], we have the following corollary.

Corollary 3.2 *In Theorem 3.1, if we assume further that there exists a multivalued map $\tilde{C} : X \times Y \multimap Z$ such that, for each $(x, y) \in X \times Y$, $\tilde{C}(x, y)$ is a convex cone which is not the whole space and contain $C(x, y)\setminus\{0\}$ in its interior. Then, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ and $F(\bar{x}, \bar{y}, \bar{y}) \cap \mathrm{Pr\,Min}(F(\bar{x}, \bar{y}, T(\bar{x}))/C(\bar{x}, \bar{y})) \neq \emptyset$.*

Following the similar arguments as in Theorem 3.1 and Corollary 3.1, we have the following theorem and corollary.

Theorem 3.2 *In Theorem 3.1, let condition* (iv) *be replaced by*

(iv)′ *$F : X \times Y \times Y \multimap Z$ is an u.s.c. multivalued map with nonempty compact values such that, for each $(x, y) \in X \times Y$, $u \multimap F(x, y, u)$ is $C(x, y)$-quasiconvexlike.*

Then, there exists $(\bar{x}, \bar{y}) \in X \times Y$ s.t. $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ and

$$F(\bar{x}, \bar{y}, u) \cap [F(\bar{x}, \bar{y}, \bar{y}) + C(\bar{x}, \bar{y})] \neq \emptyset, \quad \textit{for all } u \in T(\bar{x}).$$

Corollary 3.3 *In Theorem 3.2, if we assume further that T is a compact continuous multivalued map, then the set of solutions is compact.*

Theorem 3.3 *In Theorem 3.1, let conditions* (*iii*) *and* (*iv*) *be replaced respectively by*:

(iii)′ *C is a multivalued map such that, for each (x, y) in $X \times Y$, $C(x, y)$ is a proper closed convex cone with $\operatorname{int} C(x, y) \neq \emptyset$, and W is an u.s.c. multivalued map.*

(iv)′ *F is a continuous multivalued map with nonempty compact values such that, for each $(x, y) \in X \times Y$, $u \multimap F(x, y, u)$ is $C(x, y)$-quasiconvex-like.*

Then, there exists $(\bar{x}, \bar{y}) \in X \times Y$ s.t. $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ and

$$F(\bar{x}, \bar{y}, u) \nsubseteq [F(\bar{x}, \bar{y}, \bar{y}) - \operatorname{int} C(\bar{x}, \bar{y})], \quad \textit{for all } u \in T(\bar{x}).$$

Proof For fixed x in X, define $m_x : T(x) \multimap T(x)$ by

$$m_x(y) = \{u \in T(x) \mid F(x, y, u) \subseteq F(x, y, y) - \operatorname{int} C(x, y)\}.$$

Since $F(x, y, y)$ is a compact set, $w\operatorname{Max}(F(x, y, y)/C(x, y)) \neq \emptyset$ [8], then there exists $b \in F(x, y, y)$ such that $a - b \notin \operatorname{int} C(x, y)$ for all $a \in F(x, y, y)$, that is $b \notin a - \operatorname{int} C(x, y)$ for all $a \in F(x, y, y)$. Hence $F(x, y, y) \nsubseteq F(x, y, y) - \operatorname{int} C(x, y)$, therefore $y \notin m_x(y)$. We claim that $m_x(y)$ is convex. Indeed, if $u_1, u_2 \in m_x(y)$ and $t \in [0, 1]$, then $u_1, u_2 \in T(x)$, $F(x, y, u_1) \subseteq [F(x, y, y) - \operatorname{int} C(x, y)]$ and $F(x, y, u_2) \subseteq [F(x, y, y) - \operatorname{int} C(x, y)]$. Since $T(x)$ is convex, $u_t = tu_1 + (1-t)u_2 \in T(x)$ for all $t \in [0, 1]$. Suppose that there exists $t_0 \in (0, 1)$ such that $u_{t_0} \notin m_x(y)$, then $F(x, y, u_{t_0}) \nsubseteq [F(x, y, y) - \operatorname{int} C(x, y)]$. By (iv)′,

$$\text{either} \quad F(x, y, u_{t_0}) \subseteq [F(x, y, u_1) - C(x, y)] \tag{1}$$

$$\text{or} \quad F(x, y, u_{t_0}) \subseteq [F(x, y, u_2) - C(x, y)]. \tag{2}$$

Without loss of generality, we assume (1) holds, then $F(x, y, u_{t_0}) \subseteq F(x, y, y) - \operatorname{int} C(x, y)$, this leads to a contradiction. Hence $u_t \in m_x(y)$ for all $t \in [0, 1]$, that is $m_x(y)$ is convex for all $y \in T(x)$. Fixed $u \in T(x)$, let $y \in \overline{T(x)\backslash m_x^-(u)}$, then there exists a net $\{y_\alpha\}_{\alpha \in \Lambda} \in T(x)\backslash m_x^-(u)$ such that $y_\alpha \longrightarrow y$. Then $y_\alpha \in T(x)$ and $F(x, y_\alpha, u) \nsubseteq F(x, y_\alpha, y_\alpha) - \operatorname{int} C(x, y_\alpha)$. Hence there exists $z_\alpha \in F(x, y_\alpha, u)$ such that $z_\alpha \notin F(x, y_\alpha, y_\alpha) - \operatorname{int} C(x, y_\alpha)$. Let $A = \{y_\alpha : \alpha \in \Lambda\} \cup \{y\}$, then A and $F(\{x\} \times A \times \{u\})$ are compact. There exists a subnet $\{z_{\alpha_\lambda}\}$ of $\{z_\alpha\}$ such that $z_{\alpha_\lambda} \longrightarrow z \in F(\{x\} \times A \times \{u\})$. By (iv)′, $z \in F(x, y, u)$. Let $v \in F(x, y, y)$, then exists a net v_α such that $v_\alpha \in F(x, y_\alpha, y_\alpha)$, $v_\alpha \longrightarrow v$ and $z_\alpha - v_\alpha \notin -\operatorname{int} C(x, y_\alpha)$, therefore $z_\alpha - v_\alpha \in W(x, y_\alpha) = Z\backslash(-\operatorname{int} C(x, y_\alpha))$. By (iii)′ W is closed, and $z - v \in W(x, y)$. This shows that $z - v \notin -\operatorname{int} C(x, y)$ for all $v \in F(x, y, y)$. Hence $F(x, y, u) \nsubseteq F(x, y, y) - \operatorname{int} C(x, y)$ and $y \notin m_x^{-1}(u)$. Since $T(x)$ is closed, $y \in T(x)$. Therefore $y \in T(x)\backslash m_x^-(u)$ and $T(x)\backslash m_x^-(u)$ is closed, that is $m_x^{-1}(u)$ is open in $T(x)$.

By (ii), $T(x)$ is compact and convex. Then by Theorem 2.2, there exists $\hat{y} \in T(x)$ such that $m_x(\hat{y}) = \emptyset$. That is for each $x \in X$, there exists $y_x \in T(x)$ such that $F(x, y_x, u) \nsubseteq F(x, y_x, y_x) - \operatorname{int} C(x, y_x)$ for all $u \in T(x)$. By (i) and Himmelberg fixed point theorem, there exists $\bar{x} \in X$ such that $\bar{x} \in S(\bar{x})$. Theorem 3.3 follows. □

Remark 3.2 If $W(x, y) = Z\backslash(-\operatorname{int} C)$ for all $(x, y) \in X \times Y$, then W is an u.s.c. multivalued map, where C is a constant cone in Z.

Corollary 3.4 *In Theorem 3.3, if we assume further that T is a compact continuous multivalued map, then the set of solutions is compact.*

Proof Let

$$M = \{(x, y) \in X \times Y | x \in S(x), y \in T(x), F(x, y, u) \nsubseteq F(x, y, y) - \operatorname{int} C(x, y),$$
$$\text{for all } u \in T(x)\}.$$

Then M is a closed set. Indeed, if $(x, y) \in \overline{M}$, there exists $(x_\alpha, y_\alpha) \in M$ such that $(x_\alpha, y_\alpha) \longrightarrow (x, y)$. One has $x_\alpha \in X$, $y_\alpha \in Y$, $x_\alpha \in S(x_\alpha))$, $y_\alpha \in T(x_\alpha)$ and $F(x_\alpha, y_\alpha, w) \nsubseteq F(x_\alpha, y_\alpha, y_\alpha) - \operatorname{int} C(x_\alpha, y_\alpha)$ for all $w \in T(x_\alpha)$. Then $x \in X$, $y \in Y$, $x \in S(x)$, and $y \in T(x)$. Fixed $u \in T(x)$ and $v \in F(x, y, y)$. There exists $u_\alpha \in T(x_\alpha)$ such that $u_\alpha \longrightarrow u$ and $v_\alpha \in F(x_\alpha, y_\alpha, y_\alpha)$ such that $v_\alpha \longrightarrow v$. Since $u_\alpha \in T(x_\alpha)$, $F(x_\alpha, y_\alpha, u_\alpha) \nsubseteq F(x_\alpha, y_\alpha, y_\alpha) - \operatorname{int} C(x_\alpha, y_\alpha)$, there exists $z_\alpha \in F(x_\alpha, y_\alpha, u_\alpha)$ such that $z_\alpha \notin F(x_\alpha, y_\alpha, y_\alpha) - \operatorname{int} C(x_\alpha, y_\alpha)$, therefore $z_\alpha - v_\alpha \notin -\operatorname{int} C(x_\alpha, y_\alpha)$, that is $z_\alpha - v_\alpha \in W(x_\alpha, y_\alpha)$. Let $A = \{x_\alpha : \alpha \in \Lambda\} \cup \{x\}$, $B = \{y_\alpha : y \in \Lambda\} \cup \{y\}$, $D = \{u_\alpha, \alpha \in \Lambda\} \cup \{u\}$. Then $A \times B \times C$ is compact and $F(A \times B \times C)$ is compact.

Without loss of generality, we say that $z_\alpha \longrightarrow z$ for some $z \in F(x, y, u)$. Since W is closed, $z_\alpha - v_\alpha \longrightarrow z - v \in W(x, y)$, that is $z - v \notin -\operatorname{int} C(x, y)$ for all $v \in F(x, y, y)$. Hence $F(x, y, u) \nsubseteq F(x, y, y) - \operatorname{int} C(x, y)$ $(x, y) \in M$. That is M is a closed set. Following the same argument as in Corollary 3.1, we can show that M is a compact set. □

Following the same arguments as in Theorem 3.1 and Corollary 3.1, we have the following theorem and corollary.

Theorem 3.4 *In Theorem* 3.1, *let condition* (iv) *is replaced by*:

(iv)′ *F is a continuous multivalued map with nonempty compact values such that, for each $(x, y) \in X \times Y$, $u \multimap F(x, y, u)$ is $C(x, y)$-quasiconvex-like.*

Then, there exist $\bar{x} \in X$, $\bar{y} \in Y$ such that $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ and

$$F(\bar{x}, \bar{y}, \bar{y}) \subseteq F(\bar{x}, \bar{y}, u) - C(\bar{x}, \bar{y}), \quad \textit{for all } u \in T(\bar{x}).$$

Corollary 3.5 *In Theorem* 3.4, *let us assume further that T is a compact continuous multivalued map. Then, the set of solutions is compact.*

Remark 3.3 In Theorem 3.4, let us assume further that $F(x, y, y) \cap C(x, y) \neq \emptyset$ for all $(x, y) \in X \times Y$. Then, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ and $F(\bar{x}, \bar{y}, u) \cap C(\bar{x}, \bar{y}) \neq \emptyset$ for all $u \in T(\bar{x})$.

Following the same arguments as in Theorem 3.3 and Corollary 3.4, we have the following theorem and corollary.

Theorem 3.5 *In Theorem* 3.1, *let condition* (iii) *be replaced by*

(iii)′ *C is a multivalued map such that, for each (x, y) in $X \times Y$, $C(x, y)$ is a proper closed convex cone with $\operatorname{int} C(x, y) \neq \emptyset$, and W is u.s.c.*

Then, there exist $\bar{x} \in X$, $\bar{y} \in Y$ *s.t.* $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ *and*

$$F(\bar{x}, \bar{y}, \bar{y}) \nsubseteq [F(\bar{x}, \bar{y}, u) + \operatorname{int} C(\bar{x}, \bar{y})], \quad \textit{for all } u \in T(\bar{x}).$$

4 Applications

4.1 Mathematical Programs with Variational Inclusions Constraints

Applying Theorem 3.1, we establish the existence theorem of mathematical programs with variational inclusion constraints.

Theorem 4.1 *Let* S, T, F *be the same as in Corollary* 3.1, *let* $C : X \times Y \multimap Z$ *be a closed concave multivalued map such that, for each* (x, y) *in* $X \times Y$, $C(x, y)$ *is a nonempty convex cone, and let* Z_0 *be a real t.v.s., let* C_0 *be a closed convex cone in* Z_0, *let* $f : X \times Y \multimap Z_0$ *be an u.s.c. multivalued map with nonempty compact values. Assume that* $g : X \times Y \multimap Z$ *is a convex l.s.c. multivalued map with nonempty values and*

$$\mathcal{F} = \{(x, y) \in X \times Y \mid g(x, y) \subseteq C(x, y)\} \neq \emptyset.$$

Let A *be the projection of* $\mathcal{F}$ *on* X, *let* B *be the projection of* $\mathcal{F}$ *on* Y. *Suppose that* $T|_A : A \multimap B$, $S|_A : A \multimap A$ *and* $\operatorname{Gr}(T|_A) \subseteq \mathcal{F}$. *Then, there exists a solution of the following problem*: $\operatorname{Min}(f(\mathcal{M})/C_0) \neq \emptyset$, *where*

$$\text{(H2)} \quad \mathcal{M} = \{(x, y) \in X \times Y \mid x \in S(x),\ y \in T(x),\ g(x, y) \subseteq C(x, y) \textit{ and } F(x, y, u) \subseteq [F(x, y, y) + C(x, y)], \textit{ for all } u \in T(x)\}.$$

Proof Let $(x, y) \in \overline{\mathcal{F}}$, then there exists a net $(x_\alpha, y_\alpha) \in \mathcal{F}$ such that $(x_\alpha, y_\alpha) \longrightarrow (x, y)$. One has $x_\alpha \in X$, $y_\alpha \in Y$ and $g(x_\alpha, y_\alpha) \subseteq C(x_\alpha, y_\alpha)$. Fixed $u \in g(x, y)$, there exists $u_\alpha \in g(x_\alpha, y_\alpha)$ such that $u_\alpha \longrightarrow u$. Since C is closed, $u \in C(x, y)$ and $g(x, y) \subseteq C(x, y)$. We see that $x \in X$ and $y \in Y$. Hence $(x, y) \in \mathcal{F}$ and $\mathcal{F}$ is closed. Let $(x_1, y_1), (x_2, y_2) \in \mathcal{F}$, $t \in [0, 1]$. Then for $i = 1, 2$ $(x_i, y_i) \in X \times Y$, $g(x_i, y_i) \subseteq C(x_i, y_i)$. We see that $(x_t, y_t) = t(x_1, y_1) + (1 - t)(x_2, y_2) \in X \times Y$ for each $t \in [0, 1]$. Fixed $t \in [0, 1]$, since g is convex and C is concave, $g(x_t, y_t) \subseteq tg(x_1, y_1) + (1 - t)g(x_2, y_2) \subseteq tC(x_1, y_1) + (1 - t)C(x_2, y_2) \subseteq C(x_t, y_t)$. Hence $(x_t, y_t) \in \mathcal{F}$ for all $t \in [0, 1]$. $\mathcal{F}$ is a nonempty closed convex subset of $X \times Y$. Therefore A and B are nonempty closed convex subsets of X and Y respectively. By Theorem 3.1 that there exist $\bar{x} \in A$, $\bar{y} \in B$ such that $\bar{x} \in S(\bar{x})$, $\bar{y} \in T(\bar{x})$ and $F(\bar{x}, \bar{y}, u) \subseteq F(\bar{x}, \bar{y}, \bar{y}) + C(\bar{x}, \bar{y})$ for all $u \in T|_A(\bar{x})$. Let $M = \{(x, y) \in A \times B \mid x \in S(x), y \in T(x), F(x, y, u) \subseteq F(x, y, y) + C(x, y)$ for all $u \in T(x)$. By Corollary 3.1, M is compact. Since f is u.s.c. with compact values and M is a compact set, $f(M)$ is a compact set, there exists $(\bar{x}, \bar{y}) \in M$ such that $f(\bar{x}, \bar{y}) \cap \operatorname{Min}(f(\mathcal{M})/C_0) \neq \emptyset$. Since $\bar{x} \in A$, $\bar{y} \in T(\bar{x})$, $(\bar{x}, \bar{y}) \in \operatorname{Gr}(T|_A) \subseteq \mathcal{F}$, $g(\bar{x}, \bar{y}) \subseteq C(\bar{x}, \bar{y})$. □

Remark 4.1 If $T(x) = \{y \in Y : g(x, y) \subseteq C(x)\}$ for $x \in A$ and $S(x) = \{x\}$ for all $x \in X$, then $S|_A : A \multimap A$, $T|_A : A \multimap B$ and $Gr(T|_A) \subseteq \mathcal{F}$.

4.2 Applications to Vector Bilevel Problem

If we assume further condition on Theorem 4.1, we have the following existence theorem of solution for the vector bilevel problem.

Theorem 4.2 *In Theorem 4.1, let us assume further that*

$$\operatorname{IMin}(F(x,y,y)/C(x,y)) \neq \emptyset, \quad \textit{for all } (x,y) \in X \times Y.$$

Then, $(\bar{x},\bar{y})$ *satisfies* (*H*2) *if and only if* $(\bar{x},\bar{y})$ *is a solution of the following problem*: $\operatorname{Min}(f(\mathcal{M})/C_0) \neq \emptyset$, *where*

$$\mathcal{M} = \{(x,y) \in X \times Y \mid x \in S(x), y \in T(x), g(x,y) \subseteq C(x,y) \textit{ and } F(x,y,y) \cap \operatorname{IMin}(F(x,y,T(x))/C(x,y)) \neq \emptyset\}.$$

Proof Theorem 4.2 follows from Theorem 4.1 and Remark 3.1. □

In Theorem 4.1, if F, f and g are single real-valued functions, we have the following theorem.

Theorem 4.3 *Let* S *and* T *be the same as in Corollary 3.1. Let* $f : X \times Y \to \mathbb{R}$ *be an l.s.c. real-valued function, let* $F : X \times Y \times Y \to \mathbb{R}$ *be a continuous real-valued function such that, for each* $(x,y) \in X \times Y$, $u \to F(x,y,u)$ *is quasiconvex. Assume that* $g : X \times Y \to \mathbb{R}$ *is a quasiconcave u.s.c. real-valued function, and let* $\mathcal{F} = \{(x,y) \in X \times Y \mid g(x,y) \geq 0\} \neq \emptyset$. *Let* A *be the projection of* $\mathcal{F}$ *on* X, *let* B *be the projection of* $\mathcal{F}$ *on* Y. *Suppose that* $T|_A : A \multimap B$, $S|_A : A \multimap A$ *and* $Gr(T|_A) \subseteq \mathcal{F}$. *Then, there exists a solution to the following problem*:

$$\begin{aligned}
\text{(BVI)} \quad & \min_{(x,y)} f(x,y) \\
& \textit{s.t.} \quad x \in S(x), \quad y \in T(x), \quad g(x,y) \geq 0, \\
& \qquad F(x,y,u) \geq F(x,y,y), \quad \textit{for all } u \in T(x).
\end{aligned}$$

Proof Take $Z = Z_0 = \mathbb{R}$, $C : X \times Y \multimap Z$ by $C(x,y) = [0,\infty)$ for each $(x,y) \in X \times Y$ and $C_0 = [0,\infty)$. Applying Corollary 3.1 and following the same arguments as in Theorem 4.1, we can prove Theorem 4.3. □

4.3 Mathematical Programs with Equilibrium Constraints

For the special case of the bilevel problem, we have the following existence theorem of solution for a mathematical program with equilibrium constraint.

Remark 4.2 (i) In Theorem 4.3, let us assume further that $F(x, y, y) = 0$ for all $(x, y) \in X \times Y$. Then, there exists a solution to the problem:

$$\text{(MPI)} \quad \min_{(x,y)} f(x, y)$$
$$\text{s.t.} \quad x \in S(x), \quad y \in T(x), \quad g(x, y) \geq 0,$$
$$F(x, y, v) \geq 0, \quad \text{for all } v \in T(x).$$

Proof Since $F(x, y, y) = 0$ for all $(x, y) \in X \times Y$, we note from [1] that problems (MPI) and (BVI) are equivalent; then Remark 4.2 follows from Theorem 4.3.

(ii) In Theorem 4.3, let us suppose that X is a compact set and $S(x) = X$ for all $x \in X$, and $F(x, y, y) = 0$ for all $(x, y) \in X \times Y$. Then, Theorem 4.3 reduces to Theorem 3.2 in [7].

(iii) [1] and [6] studied (MPI) by using a coincidence theorem or maximal element theorem, in [1, 6], we show that $\bar{x} \in X$, but in this paper, we want $\bar{x} \in S(\bar{x})$, so a stronger conditions on S is needed. The results and methods of this paper are different from [6].

Applying Corollary 3.4 and following the same arguments as in Theorem 4.1, we have the following theorem. □

Theorem 4.4 *Let C, S, T, F be the same as in Corollary 3.4, let $W : X \times Y \multimap Z$ be defined by $W(x, y) = Z \setminus (-\operatorname{int} C(x, y))$, a concave u.s.c. multivalued map with nonempty values; let Z_0 be a real t.v.s., let C_0 be a closed convex cone in Z_0, let $f : X \times Y \multimap Z_0$ be a u.s.c. multivalued map with nonempty compact values. Assume that $g : X \times Y \multimap Z$ is a concave l.s.c. multivalued map with nonempty compact values and $\mathcal{F} = \{(x, y) \in X \times Y \mid g(x, y) \nsubseteq -\operatorname{int} C(x, y)\} \neq \emptyset$. Let A be the projection of $\mathcal{F}$ on X, let B be the projection of $\mathcal{F}$ on Y. Suppose that $T|_A : A \multimap B$, $S|_A : A \multimap A$ and $Gr(T|_A) \subseteq \mathcal{F}$. Then, there exists a solution for the following problem:* $\operatorname{Min}(f(\mathcal{M})/C_0) \neq \emptyset$, *where*

$$\mathcal{M} = \{(x, y) \in X \times Y \mid x \in S(x),\ y \in T(x),\ g(x, y) \nsubseteq -\operatorname{int} C(x, y) \text{ and}$$
$$F(x, y, u) \nsubseteq [F(x, y, y) - \operatorname{int} C(x, y)],\ \text{for all } u \in T(x)\}.$$

Corollary 4.1 *In Theorem 4.4, let us assume that $F : X \times Y \times Y \to Z$ is a continuous vector-valued map. Then, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $(\bar{x}, \bar{y})$ is a solution of the following problem:* $\operatorname{Min} f(\mathcal{M}/C_0) \neq \emptyset$, *where*

$$\mathcal{M} = \{(x, y) \in X \times Y \mid x \in S(x),\ y \in T(x),\ g(x, y) \nsubseteq -\operatorname{int} C(x, y) \text{ and}$$
$$F(x, y, y) \in \operatorname{W Min} F(x, y, T(x))\}.$$

Proof Corollary 4.1 follows from Corollary 3.5 and Theorem 4.4. □

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