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Connectedness of the Set of Efficient Solutions for Generalized Systems

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Abstract We introduce the concept of positive proper efficient solutions to the generalized system in this paper. We show that, under some suitable conditions, the set of positive proper efficient solutions is dense in the set of efficient solutions to the generalized system. We discuss also the connectedness of the set of efficient solutions for the generalized system with monotone bifunctions in real locally convex Hausdorff topological vector spaces.

Keywords Generalized systems · Efficient solutions · Density theorems · Connectedness

1 Introduction

It is well known that one of the important problems of vector variational inequalities and vector optimization problems is to study the topological properties of the set of solutions. The importance lies in the fact that the set of solutions, say S, to a vector optimization problem or a vector variational inequality, must considered as the feasible region of a further problem, say P, which is an optimization problem, whose objective function is formulated by the designer. In this order of ideas, the

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achievement of as many properties of S as possible is extremely important. In fact, in general, the feasible region of P, namely S, is not available explicitly.

Among the topological properties of the set of solutions, the connectedness is of interest, as it provides the possibility of continuously moving from one solution to any other solution. But so far as we know, there are few papers which deal with this subject (see [1–6]). Lee et al. [1] discussed the path-connectedness of the set of weakly efficient solutions and the set of efficient solutions for vector variational inequalities in R^n. Cheng [2] discussed the connectedness of the set of weakly efficient solutions for vector variational inequalities in R^n. In [1, 2], the ordering cone is the positive cone of R^n. Gong and Gong et al. [3–5] introduced the concepts of f-efficient solution, Henig efficient solution, globally efficient solution, superefficient solution and cone-Bensen efficient solution, and discussed the connectedness of the weakly efficient solution set, Henig efficient solution set, globally efficient solution set, superefficient solution set, and cone-Bensen efficient solution set of vector equilibrium problems in locally convex spaces. In [6], Gong discussed arcwise connectedness and the closedness of the strong solution set for vector equilibrium problems in locally convex spaces.

We remark that, so far as we know, there are no papers dealing with the connectedness of efficient solution set of the vector equilibrium problems which will be called generalized systems in this paper in locally convex spaces.

In this paper, we introduce the concept of positive proper efficient solutions to the generalized systems. By using this concept, we show that the set of positive proper efficient solutions is dense in the set of efficient solutions to the generalized systems. Then, by the density theorem, we discuss the connectedness of the set of efficient solutions for the generalized systems with monotone bifunctions in locally convex spaces.

Throughout this paper, let X be a real Hausdorff topological vector space and let Y be a real locally convex Hausdorff topological vector space. Let Y^* be the topological dual space of Y. Let C be a closed convex pointed cone in Y. The cone C induces a partial ordering in Y defined by

$$x \le y, \quad \text{if and only if} \quad y - x \in C.$$

Let

$$C^* = \{f \in Y^* : f(y) \ge 0,\ \text{for all } y \in C\}$$

be the dual cone of C. Denote the quasi-interior of C^* by $C^\sharp$, i.e.

$$C^\sharp := \{f \in Y^* : f(y) > 0 \text{ for all } y \in C \backslash \{0\}\}.$$

Let D be a nonempty subset of Y. The cone hull of D is defined as

$$\text{cone}(D) = \{td : t \ge 0,\ d \in D\}.$$

Denote the closure of D by $\text{cl}(D)$ and the interior of D by $\text{int}D$. A nonempty convex subset B of the convex cone C is called a base of C if $C = \text{cone}(B)$ and $0 \notin \text{cl}(B)$. It is easy to see that $C^\sharp \ne \emptyset$ if and only if C has a base. Let A be a nonempty subset of X and let $F : A \times A \to Y$ be a bifunction. We consider the following generalized system (GS): find $x \in A$ such that

$$F(x, y) \notin -K, \quad \text{for all } y \in A,$$

where $K \cup \{0\}$ is a convex cone in Y.

A vector $x \in A$ is called an efficient solution to the GS if

$$F(x, y) \notin -C \setminus \{0\}, \quad \text{for all } y \in A.$$

The set of efficient solutions to the GS is denoted by $V(A, F)$. If $\text{int}C \neq \emptyset$, a vector $x \in A$ is called a weakly efficient solution to the GS if

$$F(x, y) \notin -\text{int}C, \quad \text{for all } y \in A.$$

The set of weakly efficient solutions to the GS is denoted by $V_{\text{W}}(A, F)$.

Let $f \in C^* \setminus \{0\}$. A vector $x \in A$ is called an f-efficient solution to the GS if

$$f(F(x, y)) \geq 0, \quad \text{for all } y \in A.$$

The set of f-efficient solutions to the GS is denote by $V_f(A, F)$.

Definition 1.1 A vector $x \in A$ is called a positive proper efficient solution to the GS if there exists $f \in C^\natural$ such that

$$f(F(x, y)) \geq 0, \quad \text{for all } y \in A.$$

2 Density and Connectedness

In this section, we first give a density theorem. We will see that, under some suitable conditions, the set of positive proper efficient solutions is dense in the set of the efficient solutions to the GS. Then, we give a result of connectedness for the set of the efficient solutions to the GS.

Let $\varphi : A \times A \to Y$. The mapping φ is called C-monotone on $A \times A$ if

$$\varphi(x, y) + \varphi(y, x) \in -C, \quad \text{for all } x, y \in A.$$

The mapping φ is called C-strongly monotone on $A \times A$ if φ is C-monotone and, if $x, y \in A$, $x \neq y$, then

$$\varphi(x, y) + \varphi(y, x) \in -\text{int}C.$$

Let $\psi : A \to Y$. The mapping ψ is called C-lower (C-upper) semicontinuous at $x_0 \in A$ if, for any neighborhood U of 0 in Y, there is a neighborhood $U(x_0)$ of x_0 such that

$$\psi(x) \in \psi(x_0) + U + C, \quad \text{for all } x \in U(x_0) \cap A,$$

$$\psi(x) \in \psi(x_0) + U - C, \quad \text{for all } x \in U(x_0) \cap A.$$

The mapping ψ is called C-convex if, for every $x_1, x_2 \in A, t \in [0,1]$,

$$t\psi(x_1) + (1-t)\psi(x_2) \in \psi(tx_1 + (1-t)x_2) + C.$$

We say that $D \subset Y$ is a C-convex set if $D + C$ is a convex set in Y.

Lemma 2.1 (See Theorem 3.1 of [4]) *Let $A \subset X$ be a nonempty compact convex set. Let $\psi : A \to Y$ and $\varphi : A \times A \to Y$ be mappings. Assume that the following conditions are satisfied*:

- (i) *ψ is C-lower semicontinuous.*
- (ii) *$\varphi(x,x) \geq 0$ for all $x \in A$ and φ is C-monotone.*
- (iii) *For each $x \in A, \varphi(x,y)$ is C-lower semicontinuous in y and, for each $y \in A$, $\varphi(x,y)$ is C-upper semicontinuous in x.*
- (iv) *For each $x \in A$, $\psi(y) + \varphi(x,y)$ is a C-convex mapping in y.*

Then, for each $f \in C^*\backslash\{0\}$, $V_f(A,F)$ is a nonempty compact convex set, where

$$F(x,y) = \psi(y) + \varphi(x,y) - \psi(x), \quad \textit{for } x, y \in A.$$

Lemma 2.2 *Let X, Y, A, C, φ, ψ be as in Lemma 2.1. If φ is C-strongly monotone on $A \times A$, then for each $f \in C^*\backslash\{0\}$, $V_f(A,F)$ is a singleton, where*

$$F(x,y) = \psi(y) + \varphi(x,y) - \psi(x), \quad \textit{for } x, y \in A.$$

Proof By Lemma 2.1, we have $V_f(A,F) \neq \emptyset$ for each $f \in C^*\backslash\{0\}$. We show that, for each $f \in C^*\backslash\{0\}$, $V_f(A,F)$ is a singleton. If there exists $f \in C^*\backslash\{0\}$ such that $V_f(A,F)$ is not a single point set, then there exist $x_1, x_2 \in V_f(A,F)$ and $x_1 \neq x_2$. By definition, we have

$$f(\psi(y)) + f(\varphi(x_1,y)) \geq f(\psi(x_1)), \quad \text{for all } y \in A, \tag{1}$$

$$f(\psi(y)) + f(\varphi(x_2,y)) \geq f(\psi(x_2)), \quad \text{for all } y \in A. \tag{2}$$

Putting $y = x_2$ in (1), we get

$$f(\psi(x_2)) + f(\varphi(x_1,x_2)) \geq f(\psi(x_1)). \tag{3}$$

Letting $y = x_1$ in (2), we get

$$f(\psi(x_1)) + f(\varphi(x_2,x_1)) \geq f(\psi(x_2)). \tag{4}$$

By (3) and (4), we obtain

$$f(\varphi(x_1,x_2) + \varphi(x_2,x_1)) \geq 0. \tag{5}$$

Since φ is C-strongly monotone, we have

$$\varphi(x_1,x_2) + \varphi(x_2,x_1) \in -\text{int}C. \tag{6}$$

It follows from $f \in C^*\backslash\{0\}$ and (6) that

$$f(\varphi(x_1, x_2) + \varphi(x_2, x_1)) < 0.$$

This contradicts (5). Thus, $V_f(A, F)$ is a singleton as we claimed. □

Now, we state and prove the main results of this paper.

Theorem 2.1 *Let $A \subset X$ be a nonempty compact convex set. Let $\psi : A \to Y$ and $\varphi : A \times A \to Y$ be mappings. Assume that the following conditions are satisfied*:

(i) *ψ is C-lower semicontinuous.*
(ii) *$\varphi(x, x) \geq 0$ for all $x \in A$ and φ is C-strongly monotone.*
(iii) *For each $x \in A$, $\varphi(x, y)$ is C-lower semicontinuous in y and, for each $y \in A$, $\varphi(x, y)$ is C-upper semicontinuous in x.*
(iv) *For each $x \in A$, $\psi(y) + \varphi(x, y)$ is a C-convex mapping in y.*
(v) *$\psi(A)$ and $D = \{\varphi(x, y) : x, y \in A\}$ are bounded subsets of Y.*
(vi) *$C^\sharp \neq \emptyset$ and* $\mathrm{int}C \neq \emptyset$.

Then,

$$\bigcup_{f \in C^\sharp} V_f(A, F) \subset V(A, F) \subset cl\Bigg(\bigcup_{f \in C^\sharp} V_f(A, F)\Bigg),$$

where

$$F(x, y) = \psi(y) + \varphi(x, y) - \psi(x), \quad \text{for } x, y \in A.$$

Proof By Lemma 2.1, $V_f(A, F) \neq \emptyset$ for each $f \in C^*\backslash\{0\}$. By definition, we have

$$\bigcup_{f \in C^\sharp} V_f(A, F) \subset V(A, F) \subset V_{\mathrm{W}}(A, F). \tag{7}$$

Since for each $x \in A$, $F(x, y) = \psi(y) + \varphi(x, y) - \psi(x)$ is C-convex in y, $F(x, A)$ is a C-convex set. By similar argument as in the proof of Lemma 2.1 in [3], we have

$$V_{\mathrm{W}}(A, F) = \bigcup_{f \in C^*\backslash\{0\}} V_f(A, F). \tag{8}$$

By (7) and (8), we have

$$\bigcup_{f \in C^\sharp} V_f(A, F) \subset V(A, F) \subset \bigcup_{f \in C^*\backslash\{0\}} V_f(A, F). \tag{9}$$

To show that

$$\bigcup_{f \in C^*\backslash\{0\}} V_f(A, F) \subset \mathrm{cl}\Bigg(\bigcup_{f \in C^\sharp} V_f(A, F)\Bigg).$$

Let us define the mapping $H : C^*\backslash\{0\} \to A$ by

$$H(f) = V_f(A, F), \quad f \in C^*\backslash\{0\}.$$

By Lemma 2.2, $H(f)$ is a single-valued mapping. In a way similar to the proof of Theorem 4.1 in [4], we can see that H is continuous on $C^*\backslash\{0\}$.

Let $x_0 \in \bigcup_{f\in C^*\backslash\{0\}} V_f(A, F)$. Then, there exists $f_0 \in C^*\backslash\{0\}$ such that

$$\{x_0\} = V_{f_0}(A, F) = H(f_0).$$

Since $C^\sharp \neq \emptyset$, let $g \in C^\sharp$ and set

$$f_n = f_0 + (1/n)g.$$

Then, $f_n \in C^\sharp$. We show that $\{f_n\}$ converges to f_0 with respect to the topology $\beta(Y^*, Y)$.

For any neighborhood U of 0 with respect to $\beta(Y^*, Y)$, there exist bounded subsets $B_i \subset Y$ $(i = 1, 2, \dots, m)$ and $\epsilon > 0$ such that

$$\bigcap_{i=1}^{m} \Big\{ f \in Y^* : \sup_{y\in B_i} |f(y)| < \epsilon \Big\} \subset U.$$

Since B_i is bounded and $g \in Y^*$, $|g(B_i)|$ is bounded for $i = 1, \dots, m$. Thus, there exists N such that

$$\sup_{y\in B_i} |(1/n)g(y)| < \epsilon, \quad i = 1, \dots, m,\ n \geq N.$$

Hence $(1/n)g \in U$, that is, $f_n - f_0 \in U$. This means that $\{f_n\}$ converges to f_0 with respect to $\beta(Y^*, Y)$.

Since $H(f)$ is continuous at f_0, we have $H(f_n) \to H(f_0)$. Set $\{x_n\} = H(f_n)$. Then,

$$\{x_n\} = H(f_n) = V_{f_n}(A, F) \subset \bigcup_{f\in C^\sharp} V_f(A, F).$$

Letting $\{x_0\} = H(f_0)$, we have $x_n \to x_0$. This means that

$$x_0 \in \mathrm{cl}\Big(\bigcup_{f\in C^\sharp} V_f(A, F) \Big).$$

Since $x_0 \in \bigcup_{f\in C^*\backslash\{0\}} V_f(A, F)$ is arbitrary, we have

$$\bigcup_{f\in C^*\backslash\{0\}} V_f(A, F) \subset \mathrm{cl}\Big(\bigcup_{f\in C^\sharp} V_f(A, F) \Big). \tag{10}$$

By (9) and (10), we obtain that

$$\bigcup_{f\in C^\sharp} V_f(A, F) \subset V(A, F) \subset \mathrm{cl}\Big(\bigcup_{f\in C^\sharp} V_f(A, F) \Big)$$

and the proof is complete. □

Theorem 2.2 *Let $X, Y, A, C, \varphi, \psi, F$ be as in Theorem 2.1. Then, the set $V(A,F)$ is connected.*

Proof By Theorem 2.1,

$$\bigcup_{f\in C^\sharp} V_f(A,F) \subset V(A,F) \subset \mathrm{cl}\Big(\bigcup_{f\in C^\sharp} V_f(A,F)\Big). \tag{11}$$

It follows from Theorem 4.1 of [4] that $\bigcup_{f\in C^\sharp} V_f(A,F)$ is a connected set. By (11), $V(A,F)$ is a connected set. □

Now, we give an example illustrating Theorem 2.2.

Example 2.1 Let

$$X = Y = R^2, \qquad C = R^2_+ = \{x = (x_1, x_2) : x_1 \geq 0, x_2 \geq 0\}.$$

Let

$$A = \{x = (x_1, x_2) \in R^2 : x_1^2 + x_2^2 \leq 1\},$$
$$F_1(x) = (\alpha x_1, x_2), \qquad F_2(x) = (x_1, x_2),$$

for all $x = (x_1, x_2) \in R^2$, where the constant $\alpha > 0$ is fixed. Define the mappings $\varphi : A \times A \to R^2$ and $\psi : A \to R^2$ by

$$\varphi(x,y) = \big(\langle F_1(x), y-x\rangle, \langle F_2(x), y-x\rangle\big), \quad \text{for } x, y \in A,$$
$$\psi(x) = (x_1, x_2^2), \quad \text{for } x = (x_1, x_2) \in A.$$

It is clear that conditions (i), (iii), (v), and (vi) of Theorem 2.1 are satisfied. We have $\varphi(x,x) = (0,0)$ for all $x \in A$ and

$$\begin{aligned}
&\varphi(x,y) + \varphi(y,x)\\
&\quad = \big(\langle F_1(x), y-x\rangle, \langle F_2(x), y-x\rangle\big) + \big(\langle F_1(y), x-y\rangle, \langle F_2(y), x-y\rangle\big)\\
&\quad = \big(\langle F_1(x), y-x\rangle + \langle F_1(y), x-y\rangle, \langle F_2(x), y-x\rangle + \langle F_2(y), x-y\rangle\big)\\
&\quad = -\big(\langle F_1(x) - F_1(y), x-y\rangle, \langle F_2(x) - F_2(y), x-y\rangle\big)\\
&\quad = -\big(\alpha(x_1-y_1)^2 + (x_2-y_2)^2, (x_1-y_1)^2 + (x_2-y_2)^2\big)\\
&\quad \in -\mathrm{int}R^2_+, \quad \text{for all } x = (x_1, x_2),\ y = (y_1, y_2) \in A \text{ with } x \neq y.
\end{aligned}$$

Thus, φ is R^2_+- strongly monotone and the condition (ii) of Theorem 2.1 is satisfied. For any given $x \in A$ and for any $y, z \in A$, $t \in [0,1]$, we have

$$\begin{aligned}
&\varphi(x, ty + (1-t)z)\\
&\quad = \big(\langle F_1(x), ty + (1-t)z - x\rangle, \langle F_2(x), ty + (1-t)z - x\rangle\big)
\end{aligned}$$

$$= \big(\langle F_1(x), t(y-x)+(1-t)(z-x)\rangle, \langle F_2(x), t(y-x)+(1-t)(z-x)\rangle\big)$$
$$= t\big(\langle F_1(x), y-x\rangle, \langle F_2(x), y-x\rangle\big) + (1-t)\big(\langle F_1(x), z-x\rangle, \langle F_2(x), z-x\rangle\big)$$
$$= t\varphi(x,y) + (1-t)\varphi(x,z).$$

Thus, for each $x \in A, \varphi(x,y)$ is R^2_+-convex in y. For any $x=(x_1,x_2), y=(y_1,y_2) \in A$ and $t \in [0,1]$, since $f(t)=t^2$ is a convex function in R, we have

$$\begin{aligned}\psi(tx+(1-t)y) &= \big(tx_1+(1-t)y_1, (tx_2+(1-t)y_2)^2\big)\\ &\le \big(tx_1+(1-t)y_1, tx_2^2+(1-t)y_2^2\big)\\ &= t(x_1,x_2^2)+(1-t)(y_1,y_2^2)\\ &= t\psi(x)+(1-t)\psi(y).\end{aligned}$$

Thus, ψ is R^2_+-convex. We can see that the condition (iv) of Theorem 2.1 is satisfied. By Theorem 2.2, we conclude that $V(A,F)$ is a connected set, where $F(x,y) = \psi(y)+\varphi(x,y)-\psi(x)$ for $x,y \in A$.

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