J Optim Theory Appl (2008) 139: 403–418
DOI 10.1007/s10957-008-9361-y

Implicit Iteration Scheme with Perturbed Mapping for Equilibrium Problems and Fixed Point Problems of Finitely Many Nonexpansive Mappings

L.C. Ceng · S. Schaible · J.C. Yao

Published online: 26 September 2008

Abstract We introduce an implicit iteration scheme with a perturbed mapping for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of finitely many nonexpansive mappings in a Hilbert space. Then, we establish some convergence theorems for this implicit iteration scheme which are connected with results by Xu and Ori (Numer. Funct. Analysis Optim. 22:767–772, 2001), Zeng and Yao (Nonlinear Analysis, Theory, Methods Appl. 64:2507–2515, 2006) and Takahashi and Takahashi (J. Math. Analysis Appl. 331:506–515, 2007). In particular, necessary and sufficient conditions for strong convergence of this implicit iteration scheme are obtained.

Keywords Implicit iteration scheme with a perturbed mapping · Equilibrium problem · Common fixed point · Finitely many nonexpansive mappings

In this research, the first author was partially supported by the National Science Foundation China (10771141), Ph.D. Program Foundation of Ministry of Education of China (20070270004), and Science and Technology Commision of Shanghai Municipality Grant (075105118).

L.C. Ceng
Department of Mathematics, Shanghai Normal University, Shanghai, China

L.C. Ceng
Scientific Computing Key Laboratory of Shanghai Universities, Shanghai 200234, China
e-mail: zenglc@hotmail.com

S. Schaible
A.G. Anderson Graduate School of Management, University of California, Riverside, CA, USA

J.C. Yao (✉)
Department of Applied Mathematics, National Sun Yat-Sen University, Kaohsiung, Taiwan
e-mail: yaojc@math.nsysu.edu.tw

1 Introduction and Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and let $\Phi : C \times C \to \mathcal{R}$ be a bifunction, where $\mathcal{R}$ is the set of real numbers. The equilibrium problem for $\Phi : C \times C \to \mathcal{R}$ is to find $x \in C$ such that

$$\Phi(x, y) \geq 0, \quad \text{for all } y \in C. \tag{1}$$

The set of solutions of (1) is denoted by $EP(\Phi)$. Given a mapping $T : C \to H$, let $\Phi(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in EP(\Phi)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of a variational inequality. A large number of problems arising from physics, optimization, and economics reduce to finding a solution of an equilibrium problem (1). Some methods have been proposed to solve the equilibrium problem; see, for instance, [3–7].

A mapping T with domain $D(T)$ and range $R(T)$ in H is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \text{for all } x, y \in D(T).$$

We denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{x \in D(T) : Tx = x\}$. If $C \subset H$ is nonempty, bounded, closed and convex and T is a nonexpansive self-mapping of C, then $F(T)$ is nonempty; see Ref. 8 for instance.

Very recently, motivated by Combettes and Hirstoaga [4], Moudafi [9], and Tada and Takahashi [3], Takahashi and Takahashi [6] first introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of equilibrium problem (1) and the set of fixed points of a nonexpansive mapping in a Hilbert space, and then proved a strong convergence theorem which is connected with Combettes and Hirstoaga's result [4] and Wittmann's result [10].

Theorem 1.1 [6, Theorem 3.2] *Let C be a nonempty closed convex subset of H. Let $\Phi : C \times C \to \mathcal{R}$ be a bifunction satisfying the following assumptions*:

(A1) $\Phi(x, x) = 0$ *for all* $x \in C$.
(A2) Φ *is monotone*, *i.e.*, $\Phi(x, y) + \Phi(y, x) \leq 0$ *for all* $x, y \in C$.
(A3) *For each* $x, y, z \in C$,

$$\lim_{t \downarrow 0} \Phi(tz + (1 - t)x, y) \leq \Phi(x, y).$$

(A4) *For each* $x \in C$, $y \mapsto \Phi(x, y)$ *is convex and lower semicontinuous.*

Let $S : C \to H$ be a nonexpansive mapping such that $F(S) \cap EP(\Phi) \neq \emptyset$, let $f : H \to H$ be a contraction and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in H$ and

$$\Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$
$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$

$$\liminf_{n\to\infty} r_n > 0, \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty.$$

Then, $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z \in F(S) \cap EP(\Phi)$, *where* $z = P_{F(S)\cap EP(\Phi)} f(z)$.

Let C be a nonempty closed convex subset of H and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of C. In [1], Xu and Ori introduced the following implicit iteration process. For $x_0 \in C$ and $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated as follows:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1) T_1 x_1,\\
x_2 &= \alpha_2 x_1 + (1-\alpha_2) T_2 x_2,\\
&\vdots\\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N) T_N x_N,\\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1}) T_1 x_{N+1}.\\
&\vdots
\end{aligned}$$

The scheme is expressed in a compact form as

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T_n x_n, \quad \text{for all } n \ge 1, \tag{2}$$

where $T_n = T_{n \bmod N}$.

Using the iteration process (2), they proved the following convergence theorem for nonexpansive mappings in a Hilbert space H.

Theorem 1.2 [1, pp. 770] *Let* H *be a Hilbert space and let* C *be a nonempty closed convex subset of* H. *Let* $\{T_i\}_{i=1}^N$ *be* N *nonexpansive self-mappings of* C *such that* $\Omega = \bigcap_{i=1}^N F(T_i) \neq \emptyset$, *where* $F(T_i) = \{x \in C : T_i x = x\}$. *Let* $x_0 \in C$ *and let* $\{\alpha_n\}_{n=1}^{\infty}$ *be a sequence in* $(0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$. *Then, the sequence* $\{x_n\}$ *defined implicitly by* (2) *converges weakly to a common fixed point of the mappings* $\{T_i\}_{i=1}^N$.

Let $T : H \to H$ be a nonexpansive mapping and $F : H \to H$ be a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone; i.e., F satisfies the following conditions:

$$\|Fx - Fy\| \le \kappa \|x - y\|, \quad \text{for all } x, y \in H,$$

and

$$\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, \quad \text{for all } x, y \in H,$$

respectively. For any given numbers $\lambda \in [0, 1)$ and $\mu \in (0, 2\eta/\kappa^2)$, we define the mapping $T^\lambda : H \to H$ by

$$T^\lambda x := Tx - \lambda\mu F(Tx), \quad \text{for all } x \in H.$$

Proposition 1.1 [11] *If* $0 \leq \lambda < 1$ *and* $0 < \mu < 2\eta/\kappa^2$, *then we have for* $T^\lambda : H \to H$,

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|, \quad \textit{for all } x, y \in H,$$

where

$$\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1).$$

Observe that, by Proposition 1.1, for every $u \in H$ and $t \in (0, 1)$, the mapping $S_t : H \to H$ defined by $S_t x := tu + (1 - t)T^\lambda x$ satisfies

$$\|S_t x - S_t y\| = (1 - t)\|T^\lambda x - T^\lambda y\| \leq (1 - t)(1 - \lambda\tau)\|x - y\| \leq (1 - t)\|x - y\|,$$

for all $x, y \in H$, where

$$0 \leq \lambda < 1, \qquad 0 < \mu < 2\eta/\kappa^2, \qquad \tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1).$$

By Banach's contraction principle, there exists a unique $x_t \in H$ satisfying the equation

$$x_t = tu + (1 - t)T^\lambda x_t. \tag{3}$$

Furthermore, let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-maps of H and $F : H \to H$ be a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Let $\{\alpha_n\}_{n=1}^\infty \subset (0, 1)$, $\{\lambda_n\}_{n=1}^\infty \subset [0, 1)$ and take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Applying (3), Zeng and Yao [2] introduced and studied the following implicit iteration process with perturbed mapping F for an approximation of common fixed points of $\{T_i\}_{i=1}^N$. For an arbitrary initial point $x_0 \in H$, the sequence $\{x_n\}_{n=1}^\infty$ is generated as follows:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1 - \alpha_1)[T_1 x_1 - \lambda_1\mu F(T_1 x_1)],\\
x_2 &= \alpha_2 x_1 + (1 - \alpha_2)[T_2 x_2 - \lambda_2\mu F(T_2 x_2)],\\
&\vdots\\
x_N &= \alpha_N x_{N-1} + (1 - \alpha_N)[T_N x_N - \lambda_N\mu F(T_N x_N)],\\
x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1})[T_1 x_{N+1} - \lambda_{N+1}\mu F(T_1 x_{N+1})],\\
&\vdots
\end{aligned}$$

The scheme is expressed in a compact form as

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)[T_n x_n - \lambda_n \mu F(T_n x_n)], \quad \forall n \geq 1. \tag{4}$$

It is clear that if $\lambda_n = 0,\ \forall n \geq 1$, then the implicit iteration scheme (4) reduces to the implicit iteration process (2). Utilizing this iteration process (4), they proved the following convergence theorem for nonexpansive self-maps of H.

Theorem 1.3 [2, Theorem 2.1] *Let H be a real Hilbert space and let $F : H \to H$ be a mapping such that, for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-maps of H such that $\Omega = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $\mu \in (0, 2\eta/\kappa^2)$, let $x_0 \in H$, $\{\lambda_n\}_{n=1}^\infty \subset [0,1)$ and $\{\alpha_n\}_{n=1}^\infty \subset (0,1)$ satisfying the conditions $\sum_{n=1}^\infty \lambda_n < \infty$ and $\alpha \leq \alpha_n \leq \beta,\ \forall n \geq 1$, for some $\alpha, \beta \in (0,1)$. Then, the sequence $\{x_n\}_{n=1}^\infty$ defined by*

$$x_n := \alpha_n x_{n-1} + (1-\alpha_n) T_n^{\lambda_n} x_n = \alpha_n x_{n-1} + (1-\alpha_n)[T_n x_n - \lambda_n \mu F(T_n x_n)], \quad \forall n \geq 1,$$

converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.

Motivated and inspired by the above research work of Takahashi and Takahashi [6], Xu and Ori [1] and Zeng and Yao [2], in this paper we will propose a new implicit iteration scheme with a perturbed mapping for finding a common element of the set of solutions of (1) and the set of common fixed points of a finite family of nonexpansive self-maps of H. Also, we will establish some convergence theorems for this implicit iteration scheme which are connected with Xu and Ori's result [1], Zeng and Yao's result [2] and Takahashi and Takahashi's result [6]. In particular, necessary and sufficient conditions for strong convergence of this implicit iteration scheme are obtained. Since our iteration scheme is implicit, our method of proof is very different from the one of Takahashi and Takahashi's in [6]. Moreover, our requirements on the iterative parameters are much weaker than the ones of Takahashi and Takahashi's in [6]. For example, in our results we remove the following conditions:

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty, \qquad \sum_{n=1}^\infty |r_{n+1} - r_n| < \infty.$$

Next, we further give some preliminaries and results which will be used in the rest of this paper. A Banach space E is said to satisfy Opial's property: if $\{x_n\}$ is a sequence in E which converges weakly to x, then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \text{for all } y \in E,\ y \neq x.$$

It is well known that every Hilbert space satisfies Opial's property (see for instance [12]).

Throughout the rest of the paper, we use the following notation:

(i) when $\{x_n\}$ is a sequence in H, then $x_n \to x$ (resp. $x_n \rightharpoonup x$) denotes strong (resp. weak) convergence of the sequence $\{x_n\}$ to x;

(ii) for a given sequence $\{x_n\} \subset H$, $\omega_w(x_n)$ denotes the weak ω-limit set of $\{x_n\}$; that is,

$$\omega_w(x_n) := \{x \in H : x_{n_j} \rightharpoonup x, \text{ for some subsequence } \{n_j\} \text{ of } \{n\}\}.$$

Definition 1.1 [8] Let K be a closed subset of a Banach space E. A mapping $T : K \to K$ is said to be semicompact if, for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ $(n \to \infty)$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^* \in K$ $(i \to \infty)$.

Lemma 1.1 [8] *Assume that T is a nonexpansive self-mappings of a nonempty closed convex subset C of a Hilbert space H. If T has a fixed point, then $I - T$ is demiclosed; that is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$. Here, I is the identity operator of H.*

Lemma 1.2 [13, pp. 80] *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} a_n = 0$.

Corollary 1.1 [14, pp. 303] *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le a_n + b_n, \quad \forall n \ge 1.$$

If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

Recall that the well-known identity

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2 \tag{5}$$

holds for all $x, y \in H$ and $t \in [0, 1]$. Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \le \|x - y\|, \quad \text{for all } y \in C.$$

Such a P_C is called the metric projection of H onto C. We know that P_C is nonexpansive. Further, for $x \in H$ and $z \in C$,

$$z = P_C x \quad \Leftrightarrow \quad \langle x - z, z - y\rangle \ge 0, \quad \text{for all } y \in C.$$

The following lemma appears implicitly in [15].

Lemma 1.3 [15] *Let C be a nonempty closed convex subset of H and let $\Phi : C \times C \to \mathcal{R}$ be a bifunction satisfying* (A1)–(A4). *Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\Phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \textit{for all } y \in C.$$

The following lemma was also given in [4].

Lemma 1.4 [4] *Assume that $\Phi : C \times C \to \mathcal{R}$ satisfies* (A1)–(A4). *For $r > 0$ and $x \in H$, define a mapping $S_r : H \to C$ as follows*:

$$S_r(x) = \left\{ z \in C : \Phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C \right\},$$

for all $x \in H$. Then, the following statements hold:

(i) *S_r is single-valued.*
(ii) *S_r is firmly nonexpansive, i.e., for all $x, y \in H$,*

$$\|S_r x - S_r y\|^2 \leq \langle S_r x - S_r y, x - y\rangle.$$

(iii) *$F(S_r) = EP(\Phi)$.*
(iv) *$EP(\Phi)$ is closed and convex.*

2 Convergence Theorem

In this section, we deal with an implicit iteration process with a perturbed mapping for finding a common element of the set of solutions of (1) and the set of common fixed points of a finite family of nonexpansive self-maps of a real Hilbert space H.

Theorem 2.1 *Let C be a nonempty closed convex subset of H. Let $\Phi : C \times C \to \mathcal{R}$ be a bifunction satisfying* (A1)–(A4) *and let $\{T_i\}_{i=1}^N$ be N nonexpansive self-maps of H such that $\Omega \cap EP(\Phi) \neq \emptyset$, where Ω is the set of common fixed points of the mappings $\{T_i\}_{i=1}^N$, i.e., $\Omega = \bigcap_{i=1}^N F(T_i)$. Let $F : H \to H$ be a mapping such that, for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Let $\mu \in (0, 2\eta/\kappa^2)$ and let $\{\lambda_n\} \subset [0, 1)$, $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfying the conditions*: $\sum_{n=1}^\infty \lambda_n < \infty$, $\liminf_{n\to\infty} r_n > 0$ *and* $\alpha \leq \alpha_n \leq \beta$, $\forall n \geq 1$, *for some* $\alpha, \beta \in (0, 1)$. *Then, the sequences $\{x_n\}$ and $\{u_n\}$ generated by $x_0 \in H$ and*

$$\Phi(u_{n-1}, y) + \frac{1}{r_n}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \geq 0, \quad \forall y \in C,$$
$$x_n = \alpha_n u_{n-1} + (1 - \alpha_n)[T_n x_n - \lambda_n \mu F(T_n x_n)], \quad \forall n \geq 1,$$

converge weakly to the same element of $\Omega \cap EP(\Phi)$ provided $\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0$.

Proof Let q be an arbitrary element of $\Omega \cap EP(\Phi)$. Then, from $u_{n-1} = S_{r_n} x_{n-1}$, we have

$$\|u_{n-1} - q\| = \|S_{r_n} x_{n-1} - S_{r_n} q\| \leq \|x_{n-1} - q\|, \quad \forall n \geq 1.$$

Observe that

$$\begin{aligned}\|x_n - q\|^2 &= \|\alpha_n u_{n-1} + (1-\alpha_n) T_n^{\lambda_n} x_n - q\|^2 \\ &= \alpha_n \|u_{n-1} - q\|^2 + (1-\alpha_n)\|T_n^{\lambda_n} x_n - q\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|u_{n-1} - T_n^{\lambda_n} x_n\|^2 \\ &\leq \alpha_n \|x_{n-1} - q\|^2 + (1-\alpha_n)\|T_n^{\lambda_n} x_n - q\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|u_{n-1} - T_n^{\lambda_n} x_n\|^2, \end{aligned} \tag{6}$$

where

$$T_n^{\lambda_n} x_n = T_n x_n - \lambda_n \mu F(T_n x_n).$$

Utilizing Proposition 1.1, we have

$$\begin{aligned}\|T_n^{\lambda_n} x_n - q\| &= \|T_n^{\lambda_n} x_n - T_n^{\lambda_n} q + T_n^{\lambda_n} q - q\| \\ &\leq \|T_n^{\lambda_n} x_n - T_n^{\lambda_n} q\| + \|T_n^{\lambda_n} q - q\| \\ &\leq (1-\lambda_n \tau)\|x_n - q\| + \lambda_n \mu \|F(q)\|\end{aligned}$$

which implies that

$$\|T_n^{\lambda_n} x_n - q\|^2 \leq (1-\lambda_n\tau)\|x_n - q\|^2 + \lambda_n \cdot \frac{\mu^2\|F(q)\|^2}{\tau}.$$

This together with (6) yields

$$\begin{aligned}\|x_n - q\|^2 &\leq \alpha_n\|x_{n-1} - q\|^2 + (1-\alpha_n)\bigg[(1-\lambda_n\tau)\|x_n - q\|^2 \\ &\qquad + \lambda_n \cdot \frac{\mu^2\|F(q)\|^2}{\tau}\bigg] - \alpha_n(1-\alpha_n)\|u_{n-1} - T_n^{\lambda_n} x_n\|^2 \\ &\leq \alpha_n\|x_{n-1} - q\|^2 + (1-\alpha_n)\|x_n - q\|^2 \\ &\qquad + (1-\alpha_n)\lambda_n \cdot \frac{\mu^2\|F(q)\|^2}{\tau} - \alpha_n(1-\alpha_n)\|u_{n-1} - T_n^{\lambda_n} x_n\|^2\end{aligned}$$

and so

$$\begin{aligned}\|x_n - q\|^2 &\leq \|x_{n-1} - q\|^2 + (1-\alpha_n)\frac{\lambda_n}{\alpha_n} \cdot \frac{\mu^2\|F(q)\|^2}{\tau} \\ &\quad - (1-\alpha_n)\|u_{n-1} - T_n^{\lambda_n} x_n\|^2 \\ &\leq \|x_{n-1} - q\|^2 + \lambda_n \cdot \frac{\mu^2\|F(q)\|^2}{\tau\alpha} - \|x_n - u_{n-1}\|^2. \end{aligned} \tag{7}$$

Since $\sum_{n=1}^{\infty}\lambda_n\cdot(\mu^2\|F(q)\|^2/\tau\alpha)$ converges, from Corollary 1.1 we deduce that $\lim_{n\to\infty}\|x_n-q\|$ exists. As a consequence, the sequence $\{x_n\}$ is bounded. Hence, we have

$$\|x_n-u_{n-1}\|^2\leq\|x_{n-1}-q\|^2-\|x_n-q\|^2+\lambda_n\cdot\frac{\mu^2\|F(q)\|^2}{\tau\alpha}\to 0,\quad \text{as } n\to\infty,$$

and hence,

$$\lim_{n\to\infty}\|x_n-u_{n-1}\|=0. \tag{8}$$

Now, observe that

$$(1-\beta)\|u_{n-1}-T_n^{\lambda_n}x_n\|\leq(1-\alpha_n)\|u_{n-1}-T_n^{\lambda_n}x_n\|=\|x_n-u_{n-1}\|\to 0,\quad \text{as } n\to\infty,$$

and hence,

$$\lim_{n\to\infty}\|u_{n-1}-T_n^{\lambda_n}x_n\|=0. \tag{9}$$

Also note that the boundedness of $\{x_n\}$ implies that $\{T_nx_n\}$ and $\{F(T_nx_n)\}$ are both bounded. Thus, we have

$$\begin{aligned}\|u_{n-1}-T_nx_n\|&\leq\|u_{n-1}-T_n^{\lambda_n}x_n\|+\|T_n^{\lambda_n}x_n-T_nx_n\|\\&\leq\|u_{n-1}-T_n^{\lambda_n}x_n\|+\lambda_n\mu\|F(T_nx_n)\|\to 0,\quad \text{as } n\to\infty,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_n-T_nx_n\|&=\|\alpha_n(u_{n-1}-T_nx_n)-(1-\alpha_n)\lambda_n\mu F(T_nx_n)\|\\&\leq\|u_{n-1}-T_nx_n\|+\lambda_n\mu\|F(T_nx_n)\|\to 0,\quad \text{as } n\to\infty.\end{aligned} \tag{10}$$

Since

$$\lim_{n\to\infty}\|x_n-x_{n-1}\|=0,$$

it is easy to see that

$$\lim_{n\to\infty}\|x_n-x_{n+i}\|=0,\quad \text{for each } i=1,2,\ldots,N. \tag{11}$$

Consequently, from (10) and (11), it follows that, for each $i=1,2,\ldots,N$,

$$\begin{aligned}\|x_n-T_{n+i}x_n\|&\leq\|x_n-x_{n+i}\|+\|x_{n+i}-T_{n+i}x_{n+i}\|+\|T_{n+i}x_{n+i}-T_{n+i}x_n\|\\&\leq 2\|x_n-x_{n+i}\|+\|x_{n+i}-T_{n+i}x_{n+i}\|\to 0,\quad \text{as } n\to\infty,\end{aligned}$$

and so

$$\lim_{n\to\infty}\|x_n-T_{n+i}x_n\|=0,\quad \text{for each } i=1,2,\ldots,N.$$

This implies that

$$\lim_{n\to\infty}\|x_n-T_lx_n\|=0,\quad \text{for each } l=1,2,\ldots,N. \tag{12}$$

Further, since $\{x_n\}$ is bounded, it has a subsequence $\{x_{n_j}\}$ which converges weakly to some $\bar{x} \in H$ and hence we have

$$\lim_{j\to\infty} \|x_{n_j} - T_l x_{n_j}\| = 0.$$

Note that, from Lemma 1.1, it follows that $I - T_l$ is demiclosed at zero. Thus $\bar{x} \in F(T_l)$. Since l is an arbitrary element in the finite set $\{1, 2, \dots, N\}$, we get $\bar{x} \in \Omega = \bigcap_{i=1}^N F(T_i)$.

Now, let us show $\bar{x} \in EP(\Phi)$. Indeed, by $u_n = S_{r_{n+1}} x_n$, we have

$$\Phi(u_n, y) + \frac{1}{r_{n+1}} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C.$$

From (A2), we also have

$$\frac{1}{r_{n+1}} \langle y - u_n, u_n - x_n \rangle \geq -\Phi(u_n, y) \geq \Phi(y, u_n),$$

and hence,

$$\left\langle y - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j+1}} \right\rangle \geq \Phi(y, u_{n_j}). \tag{13}$$

Since

$$\begin{aligned} x_n &= \alpha_n u_{n-1} + (1 - \alpha_n)[T_n x_n - \lambda_n \mu F(T_n x_n)] \\ &= \alpha_n u_{n-1} + (1 - \alpha_n) T_n^{\lambda_n} x_n, \end{aligned}$$

so we have

$$x_n - x_{n-1} = u_{n-1} - x_{n-1} + (1 - \alpha_n)(T_n^{\lambda_n} x_n - u_{n-1}).$$

Note that

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0.$$

So, from (9) we get

$$\begin{aligned} \|u_{n-1} - x_{n-1}\| &= \|x_n - x_{n-1} - (1 - \alpha_n)(T_n^{\lambda_n} x_n - u_{n-1})\| \\ &\leq \|x_n - x_{n-1}\| + \|T_n^{\lambda_n} x_n - u_{n-1}\| \to 0, \quad \text{as } n \to \infty, \end{aligned}$$

and hence

$$\lim_{n\to\infty} \|u_n - x_n\| = 0.$$

This implies that $\frac{u_{n_j} - x_{n_j}}{r_{n_j+1}} \to 0$. Since $u_{n_j} \rightharpoonup \bar{x}$ (due to $x_{n_j} \rightharpoonup \bar{x}$), from (A4) we obtain

$$0 \geq \Phi(y, \bar{x}), \quad \text{for all } y \in C.$$

For t with $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1-t)\bar{x}$. Since $y \in C$ and $\bar{x} \in C$ (due to $u_{n_j} \rightharpoonup \bar{x}$), we have $y_t \in C$ and hence $\Phi(y_t, \bar{x}) \le 0$. So, from (A1) and (A4), we have

$$\begin{aligned} 0 &= \Phi(y_t, y_t) \\ &\le t\Phi(y_t, y) + (1-t)\Phi(y_t, \bar{x}) \\ &\le t\Phi(y_t, y) \end{aligned}$$

and hence $0 \le \Phi(y_t, y)$. From (A3), we have

$$0 \le \Phi(\bar{x}, y), \quad \text{for all } y \in C,$$

and hence $\bar{x} \in EP(\Phi)$. Therefore, in terms of the above argument, we conclude that $\bar{x} \in \Omega \cap EP(\Phi)$.

On the other hand, let x^* be an arbitrary element of $\omega_w(x_n)$. Then, there exists another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to $x^* \in H$. Clearly, repeating the same argument, we must have $x^* \in \Omega \cap EP(\Phi)$. Next, we claim that $x^* = \bar{x}$. Indeed, if $x^* \ne \bar{x}$, then according to the Opial's property of H, we conclude that

$$\begin{aligned} \lim_{n\to\infty} \|x_n - x^*\| &= \liminf_{k\to\infty} \|x_{n_k} - x^*\| \\ &< \liminf_{k\to\infty} \|x_{n_k} - \bar{x}\| = \lim_{n\to\infty} \|x_n - \bar{x}\| = \liminf_{j\to\infty} \|x_{n_j} - \bar{x}\| \\ &< \liminf_{j\to\infty} \|x_{n_j} - x^*\| = \lim_{n\to\infty} \|x_n - x^*\|. \end{aligned}$$

This leads to a contradiction and so we get $x^* = \bar{x}$. Therefore, $\omega_w(x_n)$ is a single-point set. Hence, $\{x_n\}$ converges weakly to an element of $\Omega \cap EP(\Phi)$. According to $u_n - x_n \to 0$, we know that both $\{x_n\}$ and $\{u_n\}$ converge weakly to the same element of $\Omega \cap EP(\Phi)$. This completes the proof. □

Theorem 2.2 *Let C be a nonempty closed convex subset of H. Let $\Phi : C \times C \to \mathcal{R}$ be a bifunction satisfying* (A1)–(A4) *and let $\{T_i\}_{i=1}^N$ be N nonexpansive self-maps of H such that $\Omega \cap EP(\Phi) \ne \emptyset$ where Ω is the set of common fixed points of the mappings $\{T_i\}_{i=1}^N$, i.e., $\Omega = \bigcap_{i=1}^N F(T_i)$. Suppose there exists one map $T \in \{T_1, T_2, \dots, T_N\}$ to be semicompact and $F : H \to H$ is a mapping such that, for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Let $\mu \in (0, 2\eta/\kappa^2)$ and let $\{\lambda_n\} \subset [0,1)$, $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfying the conditions*: $\sum_{n=1}^\infty \lambda_n < \infty$, $\liminf_{n\to\infty} r_n > 0$ *and* $\alpha \le \alpha_n \le \beta$, $\forall n \ge 1$, *for some* $\alpha, \beta \in (0,1)$. *Then the sequences $\{x_n\}$ and $\{u_n\}$ generated by $x_0 \in H$ and*

$$\begin{aligned} &\Phi(u_{n-1}, y) + \frac{1}{r_n}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0, \quad \forall y \in C, \\ &x_n = \alpha_n u_{n-1} + (1-\alpha_n)[T_n x_n - \lambda_n \mu F(T_n x_n)], \quad \forall n \ge 1, \end{aligned}$$

converge strongly to the same element of $\Omega \cap EP(\Phi)$ provided $\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0$.

Proof Recalling the proof of Theorem 2.1, we know that

$$\lim_{n\to\infty} \|x_n - q\| \text{ exists}, \quad \forall q \in \Omega \cap EP(\Phi),$$

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall l \in \{1, 2, \dots, N\}.$$

Thus, $\{x_n\}$ is bounded. Then, by the hypothesis that there exists one map $T \in \{T_1, T_2, \dots, T_N\}$ to be semicompact, we may assume that T_1 is semicompact without loss of generality. Therefore,

$$\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$$

and by the definition of semicompactness there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$x_{n_i} \to p \in H, \quad \text{as } i \to \infty.$$

Hence, $x_{n_i} \rightharpoonup p$. Clearly, repeating the same argument as in the proof of Theorem 2.1, we must have $p \in \Omega \cap EP(\Phi)$. So this implies that $\lim_{n\to\infty} \|x_n - p\|$ exists. Consequently, we have

$$\lim_{n\to\infty} \|x_n - p\| = \lim_{i\to\infty} \|x_{n_i} - p\| = 0.$$

According to $u_n - x_n \to 0$, we deduce that both $\{x_n\}$ and $\{u_n\}$ converge strongly to the same point $p \in \Omega \cap EP(\Phi)$. This completes the proof. □

Theorem 2.3 *Let C be a nonempty closed convex subset of H. Let $\Phi : C \times C \to \mathcal{R}$ be a bifunction satisfying (A1)–(A4) and let $\{T_i\}_{i=1}^N$ be N nonexpansive self-maps of H such that $\Omega \cap EP(\Phi) \neq \emptyset$ where Ω is the set of common fixed points of the mappings $\{T_i\}_{i=1}^N$, i.e., $\Omega = \bigcap_{i=1}^N F(T_i)$. Let $F : H \to H$ be a mapping such that, for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Let $\mu \in (0, 2\eta/\kappa^2)$ and let $\{\lambda_n\} \subset [0, 1)$, $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfying the conditions: $\sum_{n=1}^\infty \lambda_n < \infty$, $\liminf_{n\to\infty} r_n > 0$ and $\alpha \le \alpha_n \le \beta$, $\forall n \ge 1$, for some $\alpha, \beta \in (0, 1)$. Then the sequences $\{x_n\}$ and $\{u_n\}$ generated by $x_0 \in H$ and*

$$\Phi(u_{n-1}, y) + \frac{1}{r_n}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0, \quad \forall y \in C,$$

$$x_n = \alpha_n u_{n-1} + (1 - \alpha_n)[T_n x_n - \lambda_n \mu F(T_n x_n)], \quad \forall n \ge 1,$$

converge strongly to the same element of $\Omega \cap EP(\Phi)$ if and only if $\lim_{n\to\infty} d(x_n, \Omega \cap EP(\Phi)) = 0$.

Proof Recalling the proof of Theorem 2.1, we know that $\lim_{n\to\infty} \|x_n - q\|$ exists for each $q \in \Omega \cap EP(\Phi)$. Hence, $\{x_n\}$ is bounded and so are $\{T_n x_n\}$ and $\{F(T_n x_n)\}$.

First, the necessity is apparent. Second we show the sufficiency. Suppose that

$$\liminf_{n\to\infty} d(x_n, \Omega \cap EP(\Phi)) = 0.$$

Since

$$x_n = \alpha_n u_{n-1} + (1-\alpha_n)[T_n x_n - \lambda_n \mu F(T_n x_n)],$$

we derive

$$\begin{aligned}\|x_n - q\| &\le \alpha_n \|u_{n-1} - q\| + (1-\alpha_n)\|T_n x_n - q - \lambda_n \mu F(T_n x_n)\| \\ &\le \alpha_n \|u_{n-1} - q\| + (1-\alpha_n)[\|T_n x_n - T_n q\| + \lambda_n \mu \|F(T_n x_n)\|] \\ &\le \alpha_n \|x_{n-1} - q\| + (1-\alpha_n)\|x_n - q\| + \lambda_n \mu \|F(T_n x_n)\|,\end{aligned}$$

and hence

$$\begin{aligned}\|x_n - q\| &\le \|x_{n-1} - q\| + \lambda_n \cdot \frac{\mu \|F(T_n x_n)\|}{\alpha_n} \\ &\le \|x_{n-1} - q\| + \lambda_n \cdot \frac{\mu M}{\alpha}, \end{aligned} \tag{14}$$

where $\|F(T_n x_n)\| \le M$ for some $M > 0$. Hence, it follows from (14) that, for all $q \in \Omega \cap EP(\Phi)$,

$$\begin{aligned}\|x_{n+m} - p\| &\le \|x_{n+m-1} - q\| + \lambda_{n+m} \cdot \frac{\mu M}{\alpha} \\ &\le \|x_{n+m-2} - q\| + \lambda_{n+m-1} \cdot \frac{\mu M}{\alpha} + \lambda_{n+m} \cdot \frac{\mu M}{\alpha} \\ &\vdots \\ &\le \|x_n - q\| + \lambda_{n+1} \cdot \frac{\mu M}{\alpha} + \lambda_{n+2} \cdot \frac{\mu M}{\alpha} + \cdots + \lambda_{n+m} \cdot \frac{\mu M}{\alpha} \\ &= \|x_n - q\| + \frac{\mu M}{\alpha} \sum_{i=n+1}^{n+m} \lambda_i,\end{aligned}$$

and so,

$$\begin{aligned}\|x_{n+m} - x_n\| &\le \|x_{n+m} - q\| + \|x_n - q\| \\ &\le 2\|x_n - q\| + \frac{\mu M}{\alpha} \sum_{i=n+1}^{n+m} \lambda_i. \end{aligned} \tag{15}$$

Taking the infimum over all $q \in \Omega \cap EP(\Phi)$, from (15) we obtain

$$\|x_{n+m} - x_n\| \le 2d(x_n, \Omega \cap EP(\Phi)) + \frac{\mu M}{\alpha} \sum_{i=n+1}^{n+m} \lambda_i \to 0, \quad \text{as } n \to \infty.$$

Thus, $\{x_n\}$ is a Cauchy sequence. Suppose that $\lim_{n\to\infty} x_n = \hat{x} \in H$. Then,

$$d(\hat{x}, \Omega \cap EP(\Phi)) = \lim_{n\to\infty} d(x_n, \Omega \cap EP(\Phi)) = 0.$$

As each T_i $(1 \le i \le N)$ is a nonexpansive mapping, we know that $F(T_i)$ is closed. Note that $EP(\Phi)$ is closed according to Lemma 1.4. Thus, $\Omega \cap EP(\Phi)$ is closed. Consequently, $\hat{x} \in \Omega \cap EP(\Phi)$. In view of $u_n - x_n \to 0$, we conclude that both $\{x_n\}$ and $\{u_n\}$ converge strongly to the same element $\hat{x}$ of $\Omega \cap EP(\Phi)$. This completes the proof. □

As direct consequences of Theorems 2.1–2.3, we obtain the following corollaries.

Corollary 2.1 *Let C be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-maps of H such that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $F : H \to H$ be a mapping such that, for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Let $\mu \in (0, 2\eta/\kappa^2)$ and let $\{\lambda_n\} \subset [0, 1)$, and $\{\alpha_n\} \subset (0, 1)$ satisfying the conditions: $\sum_{n=1}^\infty \lambda_n < \infty$ and $\alpha \le \alpha_n \le \beta$, $\forall n \ge 1$, for some $\alpha, \beta \in (0, 1)$. Then, the sequence $\{x_n\}$ generated by $x_0 \in H$ and*

$$x_n = \alpha_n P_C x_{n-1} + (1-\alpha_n)[T_n x_n - \lambda_n \mu F(T_n x_n)], \quad \forall n \ge 1,$$

converges weakly to an element of $\bigcap_{i=1}^N F(T_i)$ provided $\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0$.

Proof Put $\Phi(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \ge 1$ in Theorem 2.1. Then, we have $u_{n-1} = P_C x_{n-1}$. So, in terms of Theorem 2.1, whenever $\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0$, then the sequence $\{x_n\}$ generated by $x_0 \in H$ and

$$x_n = \alpha_n P_C x_{n-1} + (1-\alpha_n)[T_n x_n - \lambda_n \mu F(T_n x_n)], \quad \forall n \ge 1,$$

converges weakly to an element of $\bigcap_{i=1}^N F(T_i)$. □

Corollary 2.2 *Let C be a nonempty closed convex subset of H. Let $\Phi : C \times C \to \mathcal{R}$ be a bifunction satisfying (A1)–(A4) such that $EP(\Phi) \neq \emptyset$. Let $F : H \to H$ be a mapping such that, for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Let $\mu \in (0, 2\eta/\kappa^2)$ and let $\{\lambda_n\} \subset [0, 1)$, $\{\alpha_n\} \subset (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfying the conditions: $\sum_{n=1}^\infty \lambda_n < \infty$, $\liminf_{n\to\infty} r_n > 0$ and $\alpha \le \alpha_n \le \beta$, $\forall n \ge 1$, for some $\alpha, \beta \in (0, 1)$. Then, the sequences $\{x_n\}$ and $\{u_n\}$ generated by $x_0 \in H$ and*

$$\Phi(u_{n-1}, y) + \frac{1}{r_n}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0, \quad \forall y \in C,$$
$$x_n = \alpha_n u_{n-1} + (1-\alpha_n)[x_n - \lambda_n \mu F(x_n)], \quad \forall n \ge 1,$$

converge weakly to the same element of $EP(\Phi)$ provided $\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0$.

Proof Put $T_i = I$ for each $i = 1, 2, \ldots, N$ in Theorem 2.1, where I is the identity operator of H. Then, in terms of Theorem 2.1 the sequences $\{x_n\}$ and $\{u_n\}$ generated in Corollary 2.2 converge weakly to the same element of $EP(\Phi)$ provided $\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0$. □

Corollary 2.3 *Let C be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-maps of H such that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Suppose that there exists one map $T \in \{T_1, T_2, \dots, T_N\}$ to be semicompact and $F : H \to H$ is a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Let $\mu \in (0, 2\eta/\kappa^2)$ and let $\{\lambda_n\} \subset [0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfying the conditions: $\sum_{n=1}^\infty \lambda_n < \infty$ and $\alpha \le \alpha_n \le \beta$, $\forall n \ge 1$, for some $\alpha, \beta \in (0,1)$. Then, the sequence $\{x_n\}$ generated by $x_0 \in H$ and*

$$x_n = \alpha_n P_C x_{n-1} + (1-\alpha_n)[T_n x_n - \lambda_n \mu F(T_n x_n)], \quad \forall n \ge 1,$$

converges strongly to an element of $\bigcap_{i=1}^N F(T_i)$ provided $\lim_{n\to\infty}\|x_n - x_{n-1}\| = 0$.

Corollary 2.4 *Let C be a nonempty closed convex subset of H. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-maps of H such that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $F : H \to H$ be a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Let $\mu \in (0, 2\eta/\kappa^2)$ and let $\{\lambda_n\} \subset [0,1)$ and $\{\alpha_n\} \subset (0,1)$ satisfying the conditions: $\sum_{n=1}^\infty \lambda_n < \infty$ and $\alpha \le \alpha_n \le \beta$, $\forall n \ge 1$, for some $\alpha, \beta \in (0,1)$. Then the sequence $\{x_n\}$ generated by $x_0 \in H$ and*

$$x_n = \alpha_n P_C x_{n-1} + (1-\alpha_n)[T_n x_n - \lambda_n \mu F(T_n x_n)], \quad \forall n \ge 1,$$

converges strongly to an element of $\bigcap_{i=1}^N F(T_i)$ if and only if $\liminf_{n\to\infty} d(x_n, \bigcap_{i=1}^N F(T_i)) = 0$.

Corollary 2.5 *Let C be a nonempty closed convex subset of H. Let $\Phi : C \times C \to \mathcal{R}$ be a bifunction satisfying (A1)–(A4) such that $EP(\Phi) \neq \emptyset$. Let $F : H \to H$ be a mapping such that, for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Let $\mu \in (0, 2\eta/\kappa^2)$ and let $\{\lambda_n\} \subset [0,1)$, $\{\alpha_n\} \subset (0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfying the conditions: $\sum_{n=1}^\infty \lambda_n < \infty$, $\liminf_{n\to\infty} r_n > 0$ and $\alpha \le \alpha_n \le \beta$, $\forall n \ge 1$, for some $\alpha, \beta \in (0,1)$. Then the sequences $\{x_n\}$ and $\{u_n\}$ generated by $x_0 \in H$ and*

$$\Phi(u_{n-1}, y) + \frac{1}{r_n}\langle y - u_{n-1}, u_{n-1} - x_{n-1}\rangle \ge 0, \quad \forall y \in C,$$
$$x_n = \alpha_n u_{n-1} + (1-\alpha_n)[x_n - \lambda_n \mu F(x_n)], \quad \forall n \ge 1,$$

converge strongly to the same element of $EP(\Phi)$ if and only if $\liminf_{n\to\infty} d(x_n, EP(\Phi)) = 0$.

References

1. Xu, H.K., Ori, R.G.: An implicit iteration process for nonexpansive mappings. Numer. Funct. Analysis Optim. **22**, 767–773 (2001)
2. Zeng, L.C., Yao, J.C.: Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of nonexpansive mappings. Nonlinear Analysis, Theory, Methods Appl. **64**, 2507–2515 (2006)
3. Takahashi, S., Takahashi, W.: Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces. J. Math. Analysis Appl. **331**, 506–515 (2007)

4. Combettes, P.L., Hirstoaga, S.A.: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Analysis **6**, 117–136 (2005)
5. Flam, S.D., Antipin, A.S.: Equilibrium programming using proximal-like algorithms. Math. Program. **78**, 29–41 (2007)
6. Tada, A., Takahashi, W.: Strong convergence theorem for an equilibrium problem and a nonexpansive mapping. In: Takahashi, W., Tanaka, T. (eds.) Nonlinear Analysis and Convex Analysis. Yokohama Publishers, Yokohama (2007)
7. Ceng, L.C., Yao, J.C.: A hybrid iterative scheme for mixed equilibrium problems and fixed point problems. J. Comput. Appl. Math. **214**, 186–201 (2008).
8. Goebel, K., Kirk, W.A.: Topics on Metric Fixed-Point Theory. Cambridge University Press, Cambridge, England (1990)
9. Moudafi, A.: Viscosity approximation methods for fixed-point problems. J. Math. Analysis Appl. **241**, 46–55 (2000)
10. Wittmann, R.: Approximation of fixed points of nonexpansive mappings. Archiv der Mathematik **58**, 486–491 (1992)
11. Xu, H.K., Kim, T.H.: Convergence of hybrid steepest-descent methods for variational inequalities. J. Optim. Theory Appl. **119**, 185–201 (2003)
12. Opial, Z.: Weak convergence of the sequence of successive approximations for nonexpansive mappings. Bull. Am. Math. Soc. **73**, 591–597 (1967)
13. Osilike, M.O., Aniagbosor, S.C., Akuchu, B.G.: Fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces. PanAm. Math. J. **12**, 77–88 (2002)
14. Tan, K.K., Xu, H.K.: Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process. J. Math. Analysis Appl. **178**, 301–308 (1993)
15. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)