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Duality Results for Generalized Vector Variational Inequalities with Set-Valued Maps

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Abstract In this paper, we introduce new dual problems of generalized vector variational inequality problems with set-valued maps and we discuss a link between the solution sets of the primal and dual problems. The notion of solutions in each of these problems is introduced via the concepts of efficiency, weak efficiency or Benson proper efficiency in vector optimization. We provide also examples showing that some earlier duality results for vector variational inequality may not be true.

Keywords Vector variational inequalities · Set-valued maps · Duality · Conjugate maps · Biconjugate maps

1 Introduction

The theory of variational inequality problems [1] and its extensions such as the theory of vector variational inequality problems [2] and the theory of scalar and vector equilibrium problems [3, 4] were developed extensively in recent years. Most of the papers dealing with these theories are centered around existence results. There are only a few works devoted to the study of duality in these problems. In 1972, Mosco [5] introduced a dual problem of a variational inequality problem and showed a relationship between the solution sets of the primal and dual problems. In 1980, such

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a relationship was considered in [6] for a more general problem, called the implicit variational problem, which includes the variational inequality problem and the equilibrium problem as special cases. The duality results obtained in [5, 6] were extended to the vector case in [7–9]. The original duality results of [8, 9] were also presented in [10] in a finite-dimensional space, together with some applications to the study of gap functions for vector variational inequality problems. In this paper, we give examples proving that the just mentioned dual problems of [8–10] are not suitable for the duality property of vector variational inequalities, and hence, all possible applications of them cannot be seen to be justified. This fact shows that, when dealing with duality in vector variational inequality problems which are generalizations of those considered in [8–10], we must use dual problems different from those of [8–10]. In this paper, such new dual problems are introduced for a set-valued version of the vector variational inequality problem of [8–10], and they are shown to be useful in establishing duality properties. Our duality results are expressed in Theorems 2.2, 2.3, 3.1, 3.2 below. We observe that, in Theorems 3.1 and 3.2, the notion of solutions in both the primal and dual problems is quite different from that used in Theorems 2.2 and 2.3: In Theorems 3.1 and 3.2, the notion of solutions is introduced via that of proper efficiency of Benson [11, 12]; in Theorems 2.2 and 2.3 of the present paper, as well as in [8–10], the notion of solutions is given via the concept of efficiency or weak efficiency. We observe also that our results are valid without any continuity or convexity assumption.

2 Duality in Generalized Vector Variational Inequalities with Fenchel Conjugate Maps

Let Y be a topological vector space and let C be an arbitrary subset of Y with $0 \in C$ (the set C is not necessarily a cone). Let $C_0 = C \setminus \{0\} \neq \emptyset$, where $\emptyset$ denotes the empty set. For a subset $A \subset Y$, let us set

$$\mathrm{Max}_C A = \{a \in A : [A - a] \cap C_0 = \emptyset\},$$

$$\mathrm{BMax}_C A = \{a \in A : \mathrm{cl}\,\mathrm{cone}[A - a - C] \cap C = \{0\}\},$$

where $\mathrm{cone}\,A = \{\lambda a : \lambda \geq 0, a \in A\}$ and $\mathrm{cl}\,A$ is the closure of A. If no confusion can arise we will delete the subscript C in $\mathrm{Max}_C A$ and $\mathrm{BMax}_C A$. These sets can be found e.g. in [10–12] for the case where C is a convex cone. Namely, if

$$C = c, \tag{1}$$

where $c \subset Y$ is a closed convex cone, then $\mathrm{Max} A$ is called [10, 12] the set of efficient points of A. If

$$C = \{0\} \cup \mathrm{int}\, c, \tag{2}$$

where c is a convex cone with $\emptyset \neq \mathrm{int}\, c$ (the interior c) and $Y \neq c$, then $\mathrm{Max} A$ is called [10, 12] the set of weakly efficient points of A. So, our above definition of

$\mathrm{Max}A$ includes as special cases the known notions of the sets of efficient and weakly efficient points of A.

If C is a convex cone, then BMax A corresponds [11, 12] to the set of all the Benson properly efficient points of A. In vector optimization, several notions of proper efficiencies were introduced. The reader is referred to [13] (see also [12]) for a comprehensive survey where all these notions and links between them can be found. The following lemma will be used later.

Lemma 2.1

(i) *BMax* $A \subset$ *Max* A.
(ii) *If* $Q \subset A$ *is a subset such that* $\mathrm{Max}A \subset Q - C$ (*resp. BMax* $A \subset Q - C$) *then* $\mathrm{Max}A \subset Q$ (*resp. BMax* $A \subset Q$).

Proof (i) If $a \in$ BMax A then $[A - a] \cap C_0 = \emptyset$ (i.e., $a \in \mathrm{Max}A$) since

$$[A - a] \cap C = [A - a - 0] \cap C \subset [A - a - C] \cap C = \{0\}.$$

(ii) If $\mathrm{Max}A = \emptyset$ (resp. BMax $A = \emptyset$) then the required conclusion is obvious. In case $\mathrm{Max}A \neq \emptyset$ (resp. BMax $A \neq \emptyset$) we must show that $a \in \mathrm{Max}A$ (resp. $a \in$ BMax A) $\Longrightarrow a \in Q$. By assumption we can find $q \in Q$ such that $a \in q - C$, i.e., $q - a \in C$. On the other hand, $q - a \notin C_0$ since $q \in A$ and $a \in \mathrm{Max}A$. Therefore $a = q \in Q$. □

Let X be an arbitrary set and $\mathcal{U}$ be a (nonempty) family of single-valued maps $u : X \longrightarrow Y$. Thus, for each $u \in \mathcal{U}$ and $x \in X$, $u(x)$ is an element of Y. As an example of $\mathcal{U}$ we can take the set $\mathcal{L}(X, Y)$ of linear continuous maps from X into Y where X is assumed to be a topological vector space. Sometimes we write $\langle u, x\rangle$ instead of $u(x)$ if $u \in \mathcal{U} = \mathcal{L}(X, Y)$.

For each set-valued map $F : X \longrightarrow 2^Y$ and each map $u \in \mathcal{U}$, let us set

$$A(u, F) = \{u(x) - F(x) : x \in \mathrm{dom}\ F\},$$

where $\mathrm{dom}\ F = \{x \in X : F(x) \neq \emptyset\}$.

For each set-valued map $\mathcal{F} : \mathcal{U} \longrightarrow 2^Y$ and each point $x \in X$, let us set

$$A^*(x, \mathcal{F}) = \{u(x) - \mathcal{F}(u) : u \in \mathrm{dom}\ \mathcal{F}\},$$

where $\mathrm{dom}\ \mathcal{F} = \{u \in \mathcal{U} : \mathcal{F}(u) \neq \emptyset\}$.

Remark 2.1 It is obvious that

$$y \in F(x) \quad \Longrightarrow \quad u(x) - y \in A(u, F), \quad \forall u \in \mathcal{U}, \tag{3}$$

$$v \in \mathcal{F}(u) \quad \Longrightarrow \quad u(x) - v \in A^*(x, \mathcal{F}), \quad \forall x \in X. \tag{4}$$

Definition 2.1 The set-valued map $F^* : \mathcal{U} \longrightarrow 2^Y$ defined by

$$F^*(u) = \mathrm{Max}\ A(u, F) = \{v \in A(u, F) : [A(u, F) - v] \cap C_0 = \emptyset\} \tag{5}$$

is called the Fenchel conjugate map of F.

If $\mathcal{U} = \mathcal{L}(X, Y)$ and if C is of the form (1) (resp. (2)), then F^* is exactly the Fenchel transform (resp. weak Fenchel transform) of F (see [10, 12]).

Let $F : X \longrightarrow 2^Y$ and $G : X \longrightarrow 2^{\mathcal{U}}$ be set-valued maps. Let F^* be the Fenchel conjugate map of F. In this section we consider the following problem:

Problem (p): Find $x_0 \in \text{dom } F$, $y_0 \in F(x_0)$ and $u_0 \in G(x_0)$ such that

$$\{A(u_0, F) - [u_0(x_0) - y_0]\} \cap C_0 = \emptyset. \tag{6}$$

Let $F(\cdot) = \{f(\cdot)\}$ and $G(\cdot) = \{-g(\cdot)\}$ where $f : X \longrightarrow Y$ and $g : X \longrightarrow \mathcal{U} = \mathcal{L}(X, Y)$ are single-valued maps. Then, Problem (P) is to find a point $x_0 \in X$ with

$$-\langle g(x_0), x - x_0\rangle + f(x_0) - f(x) \notin C_0, \quad \forall x \in X.$$

(As we have remarked above, $\langle g(x_0), x\rangle$ is the evaluation of $g(x_0) \in \mathcal{L}(X, Y)$ at $x \in X$.) We will refer to this problem as Problem (p) which was considered in [8–10] under the assumption (1) or (2).

Denote by g^{-1} the inverse map of g,

$$g^{-1}(u) = \{x \in X : g(x) = u\}, \quad u \in \mathcal{U} = \mathcal{L}(X, Y).$$

Obviously, dom g^{-1} = im $g := \bigcup_{x\in X} g(x)$. Observe that, for each $u \in \text{dom } g^{-1}$, $g^{-1}(u)$ is a singleton if g is one-to-one (injective). Let us consider the following dual problem where g is assumed to be one-to-one:

Problem (d): Find $u_0 \in -\text{im } g$ such that, for all $u \in \mathcal{U} = \mathcal{L}(X, Y)$,

$$[\langle u - u_0, g^{-1}(-u_0)\rangle + f^*(u_0) - f^*(u)] \cap C_0 = \emptyset. \tag{7}$$

The links between the solution sets of the primal Problem (p) and the dual Problem (d) are given in Theorems 9.3.1 and 9.3.2 of [10] (see also [8, 9]). The first part of these theorems can be formulated as follows.

Theorem 2.1 *Assume that $X = \mathbb{R}^n$ (the n-dimensional Euclidean space), $Y = \mathbb{R}^p$, $g : X \longrightarrow \mathcal{U} = \mathcal{L}(X, Y)$ is injective and $f : X \longrightarrow Y$ is continuous.*

- (i) *Let C be of the form (1) and let $f^*(u) \neq \emptyset$ for all $u \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$. If x_0 is a solution of Problem (p) then $u_0 = -g(x_0)$ is a solution of Problem (d).*
- (ii) *Let C be of the form (2) and let $f^*(u) \neq \emptyset$ for all $u \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$. If x_0 is a solution of Problem (p) then $u_0 = -g(x_0)$ is a solution of Problem (d).*

This theorem is incorrect. Before giving Examples 2.1 and 2.2 proving this conclusion let us note that, for $X = \mathbb{R} := \mathbb{R}^1$ and $Y = \mathbb{R}^2$, each map $u \in \mathcal{L}(X, Y)$ can be identified with a 2×1 matrix, i.e.,

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{8}$$

where $u_i \in \mathbb{R}$, $i = 1, 2$, are some real numbers.

Example 2.1 Consider Problems *(p)* and *(d)*, where $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $\mathcal{U}=\mathcal{L}(\mathbb{R},\mathbb{R}^2)$, and C is defined by (1) with $c=\text{cone}\{(1,1)\}\subset\mathbb{R}^2$. Assume that the injective map $g:\mathbb{R}\longrightarrow\mathcal{L}(\mathbb{R},\mathbb{R}^2)$ and the continuous map $f:\mathbb{R}\longrightarrow\mathbb{R}^2$ are given by

$$g(x)=\begin{bmatrix}0\\x\end{bmatrix},\quad x\in\mathbb{R},$$

$$f(x)=\begin{cases}(x,x)\in\mathbb{R}^2, & \text{if } x\geq 0,\\ (-x,x)\in\mathbb{R}^2, & \text{if } x<0.\end{cases}$$

For each $u\in\mathcal{L}(\mathbb{R},\mathbb{R}^2)$ defined by (8), we have

$$\langle u,x\rangle-f(x)=\begin{cases}x(u_1-1,u_2-1)\in\mathbb{R}^2, & \text{if } x\geq 0,\\ -x(-u_1-1,-u_2+1)\in\mathbb{R}^2, & \text{if } x<0,\end{cases}$$

$$\begin{aligned}A(u,f)&=\{\langle u,x\rangle-f(x):x\in\mathbb{R}\}\\&=\text{cone}\{(u_1-1,u_2-1),(-u_1-1,-u_2+1)\}.\end{aligned}\tag{9}$$

Since $C=c=\text{cone}\{(1,1)\}\subset\mathbb{R}^2$, we have

$$C_0=C\setminus\{0\}=\text{cone}\{(1,1)\}\setminus\{(0,0)\}\subset\mathbb{R}^2.\tag{10}$$

We now prove that, for C_0 being defined by (10), $f^*(u)\neq\emptyset$ for all $u\in\mathcal{L}(\mathbb{R},\mathbb{R}^2)$. Indeed, let $u\in\mathcal{L}(\mathbb{R},\mathbb{R}^2)$ be defined by (8). If $u_2=1$, then by (9)

$$A(u,f)=\text{cone}\{(u_1-1,0),(-u_1-1,0)\}.\tag{11}$$

Since C_0 is given by (10), we can check from (11) that

$$f^*(u)=\text{cone}\{(u_1-1,0),(-u_1-1,0)\}\neq\emptyset.\tag{12}$$

If $u_2\neq 1$ then (u_1-1,u_2-1) and $(-u_1-1,-u_2+1)$ are linearly independent vectors of $\mathbb{R}^2$. Using this fact and recalling that $A(u,f)$ and C_0 are defined by (9) and (10) we can see that $f^*(u)\neq\emptyset$. We have thus showed that $f^*(u)\neq\emptyset$ for all $u\in\mathcal{L}(\mathbb{R},\mathbb{R}^2)$.

We now claim that the conclusion of Part (i) of Theorem 2.1 is incorrect. Indeed, setting $x_0=0\in X=\mathbb{R}$, we see that x_0 is a solution of Problem (p). However, $u_0=-g(x_0)=-\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\in-\text{im}\,g$ is not a solution of Problem (d). This is because (7) cannot be satisfied for $u=\widetilde{u}:=\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Indeed, from (12) with $u=\widetilde{u}$ and $u=u_0$ we have $f^*(\widetilde{u})=\text{cone}\{(-1,0)\}$ and $f^*(u_0)=\text{cone}\{(-1,1)\}$. Since $g^{-1}(-u_0)=0$, $(-1,1)\in f^*(u_0)$ and $(-2,0)\in f^*(\widetilde{u})$, we get

$$(1,1)=(0,0)+(-1,1)-(-2,0)\in\langle\widetilde{u}-u_0,g^{-1}(-u_0)\rangle+f^*(u_0)-f^*(\widetilde{u}).$$

Observe now by (10) that $(1,1)\in C_0$. Therefore, condition (7) is violated for $u=\widetilde{u}$. The conclusion of Part (i) of Theorem 2.1 is thus incorrect.

Example 2.2 Consider Problems (p) and (d), where $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $\mathcal{U}=\mathcal{L}(\mathbb{R},\mathbb{R}^2)$ and C is defined by (2) with $c=\operatorname{int}\mathbb{R}^2_+$ (the interior of the nonnegative orthant $\mathbb{R}^2_+$ of $\mathbb{R}^2$). Assume that the injective map $g:\mathbb{R}\longrightarrow\mathcal{L}(\mathbb{R},\mathbb{R}^2)$ is as in Example 2.1, and that the continuous map $f:\mathbb{R}\longrightarrow\mathbb{R}^2$ is given by $f(x)=(x,x^2)\in\mathbb{R}^2$, $x\in\mathbb{R}$. Observe from (2) with $c=\operatorname{int}\mathbb{R}^2_+$ that $C_0=\operatorname{int}\mathbb{R}^2_+$. For each $u\in\mathcal{L}(\mathbb{R},\mathbb{R}^2)$ defined by (8) and for each $x\in X=\mathbb{R}$, we have

$$\langle u,x\rangle - f(x) = (u_1x-x, u_2x-x^2)\in\mathbb{R}^2, \tag{13a}$$

$$A(u,f)=\bigcup_{x\in\mathbb{R}}\{(\xi,\eta)\in\mathbb{R}^2:\xi=u_1x-x,\eta=u_2x-x^2\}. \tag{13b}$$

Since $C_0=\operatorname{int}\mathbb{R}^2_+$ we can prove that $f^*(u)\neq\emptyset$ for all $u\in\mathcal{L}(\mathbb{R},\mathbb{R}^2)$. Indeed, let $u\in\mathcal{L}(\mathbb{R},\mathbb{R}^2)$ be defined by (8). If $u_1=1$, then from (13)

$$\begin{aligned}A(u,f)&=\bigcup_{x\in\mathbb{R}}\{(\xi,\eta)\in\mathbb{R}^2:\xi=0,\eta=u_2x-x^2\}\\&=\{(0,\eta)\in\mathbb{R}^2:\eta\in]-\infty,u_2^2/4]\}.\end{aligned}$$

Since $C_0=\operatorname{int}\mathbb{R}^2_+$ we can verify that

$$f^*(u)=A(u,f)=\{(0,\eta)\in\mathbb{R}^2:\eta\in]-\infty,u_2^2/4]\},$$

which is obviously a nonempty set.

If $u_1\neq 1$, i.e., $u_1-1\neq 0$ then from $\xi=u_1x-x$ and $\eta=u_2x-x^2$, we obtain $\eta=a\xi^2+b\xi$, where $a=-1/(u_1-1)^2<0$ and $b=u_2/(u_1-1)$. Therefore, from (13) $A(u,f)$ can be rewritten as $A(u,f)=\{(\xi,\eta)\in\mathbb{R}^2:\eta=a\xi^2+b\xi\}$. Since the function $\eta=a\xi^2+b\xi$ of the variable $\xi\in\mathbb{R}$ is decreasing for $\xi\geq -b/2a$, and is increasing for $\xi<-b/2a$, and since $C_0=\operatorname{int}\mathbb{R}^2_+$ we can verify that

$$f^*(u)=\{(\xi,\eta)\in\mathbb{R}^2:\eta=a\xi^2+b\xi,\ \xi\geq -b/2a\}\neq\emptyset. \tag{14}$$

We have thus showed that $f^*(u)\neq\emptyset$ for all $u\in\mathcal{L}(\mathbb{R},\mathbb{R}^2)$.

We now claim that the conclusion of Part (ii) of Theorem 2.1 is incorrect. Indeed, setting $x_0=0\in X=\mathbb{R}$, we see that x_0 is a solution of Problem (p). However, $u_0=-g(x_0)\in-\operatorname{im}g$ is not a solution of Problem (d). This is because (7) cannot be satisfied for $u=\widetilde{u}:=\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]$. Indeed, from (14), with $u=\widetilde{u}$ and $u=u_0$ we have $(1/2,-5/4)\in f^*(\widetilde{u})$ and $(1,-1)\in f^*(u_0)$. Since $g^{-1}(-u_0)=0$, we get

$$\begin{aligned}(1/2,1/4)&=(0,0)+(1,-1)-(1/2,-5/4)\\&\in\langle\widetilde{u}-u_0,g^{-1}(-u_0)\rangle+f^*(u_0)-f^*(\widetilde{u}).\end{aligned}$$

But $(1/2,1/4)\in\operatorname{int}\mathbb{R}^2_+=C_0$. Therefore, condition (7) is violated for $u=\widetilde{u}$. The conclusion of Part (ii) of Theorem 2.1 is thus incorrect.

The above counterexamples prove that, to obtain a result similar to that of Theorem 2.1, we must use dual problems which are different from those of [8–10].

Namely, in this section we will consider the following Problem (D) called the dual problem of Problem (P):

Problem (D). Find $u_0 \in \text{dom } F^*$, $v_0 \in F^*(u_0)$ and $x_0 \in G^{-1}(u_0)$ such that

$$\{A^*(x_0, F^*) - [u_0(x_0) - v_0]\} \cap C_0 = \emptyset, \tag{15}$$

where $G^{-1}(u) = \{x \in X : u \in G(x)\}$.

When applying to the case of Problem (p), the dual Problem (D) is the problem of finding $u_0 \in -\text{im } g$ and $v_0 \in f^*(u_0)$ such that

$$[\langle u - u_0, g^{-1}(-u_0)\rangle + v_0 - f^*(u)] \cap C_0 = \emptyset, \quad \forall u \in \text{dom } f^*.$$

Before going further, let us introduce some definitions. A triplet (x_0, y_0, u_0) (resp. (u_0, v_0, x_0)) which satisfies all the requirements formulated in Problem (P) (resp. Problem (D)) is called a solution of Problem (P) (resp. Problem (D)). If (x_0, y_0, u_0) (resp. (u_0, v_0, x_0)) is a solution of Problem (P) (resp. Problem (D)), then by (6) [resp. (15)],

$$u_0(x_0) - y_0 \in \text{Max} A(u_0, F),$$
$$[\text{resp. } u_0(x_0) - v_0 \in \text{Max} A^*(x_0, F^*)].$$

Thus, the definition of solutions of Problems (P) and (D) is introduced via the notion of Max$A$ ($A = A(u_0, F)$ or $A = A^*(x_0, F^*)$) which, as we have seen above, coincides with the notion of efficiency or weak efficiency in vector optimization [10, 12] if C is of the form (1) or (2). The notion of Max$A$ is also used in the definition of the Fenchel conjugate map F^* which appears in Problem (D). In the next section we will consider Problems (P_B) and (D_B) whose solutions are defined via the notion of BMax A instead of Max$A$.

We denote by sol(P) (resp. sol(D)) the set of all the solutions of Problem (P) (resp. Problem (D)). Before giving a link between sol(P) and sol(D), let us introduce the following notion.

Definition 2.2 The set-valued map $F^{**} : X \longrightarrow 2^Y$ defined by

$$F^{**}(x) = \text{Max } A^*(x, F^*) = \{y \in A^*(x, F^*) : [A^*(x, F^*) - y] \cap C_0 = \emptyset\}$$

is called the Fenchel biconjugate map of F.

Theorem 2.2

(i) *If* $(x_0, y_0, u_0) \in \text{sol}(P)$, *then*

$$y_0 \in F^{**}(x_0), \tag{16}$$

and there exists a point $v_0 \in F^*(u_0)$ *such that*

$$v_0 = u_0(x_0) - y_0 \tag{17}$$

and $(u_0, v_0, x_0) \in \text{sol}(D)$.

(ii) *If* $(u_0, v_0, x_0) \in \mathrm{sol}(D)$ *and if*

$$F^{**}(x_0) \subset F(x_0), \tag{18}$$

then there exists a point $y_0 \in F(x_0)$ *such that*

$$y_0 = u_0(x_0) - v_0 \tag{19}$$

and $(x_0, y_0, u_0) \in \mathrm{sol}(P)$.

Proof (i) Let $(x_0, y_0, u_0) \in \mathrm{sol(P)}$ and let v_0 be defined by (17). Using (3) with $(x, y, u) = (x_0, y_0, u_0)$ we obtain $v_0 \in A(u_0, F)$. Together with (6), this yields $v_0 \in F^*(u_0)$. From (4) with $(u, v, x) = (u_0, v_0, x_0)$ and $\mathcal{F} = F^*$, it follows that $u_0(x_0) - v_0 \in A^*(x_0, F^*)$. By (17), this means that

$$y_0 \in A^*(x_0, F^*). \tag{20}$$

To prove that $(u_0, v_0, x_0) \in \mathrm{sol(D)}$, since $u_0 \in \mathrm{dom}\, F^*$, $v_0 \in F^*(u_0)$ and $x_0 \in G^{-1}(u_0)$, it remains to show that

$$y \in A^*(x_0, F^*) \quad \Longrightarrow \quad y - [u_0(x_0) - v_0] \notin C_0. \tag{21}$$

Indeed, let $y \in A^*(x_0, F^*)$ and let $u \in \mathrm{dom}\, F^*$ and $v \in F^*(u)$ be such that

$$y = u(x_0) - v. \tag{22}$$

By the very definition of $F^*(u)$, we obtain

$$[A(u, F) - v] \cap C_0 = \emptyset. \tag{23}$$

Using (3) with $(x, y) = (x_0, y_0)$, we get $u(x_0) - y_0 \in A(u, F)$. Hence, by (23),

$$u(x_0) - y_0 - v \notin C_0. \tag{24}$$

Taking (17) and (22) into account, we obtain from (24)

$$y - [u(x_0) - v_0] \notin C_0.$$

Thus, the implication (21) is established, as desired.

Since $(u_0, v_0, x_0) \in \mathrm{sol(D)}$, we get from (15) and (17)

$$[A^*(x_0, F^*) - y_0] \cap C_0 = \emptyset.$$

Together with (20), this yields (16), as required.

(ii) Let $(u_0, v_0, x_0) \in \mathrm{sol(D)}$. Defining y_0 by (19) we see from (4) with $\mathcal{F} = F^*$ that (20) holds. On the other hand, (15) and (19) yield

$$[A^*(x_0, F^*) - y_0] \cap C_0 = \emptyset.$$

Therefore, by the very definition of $F^{**}(x_0)$ we get $y_0 \in F^{**}(x_0)$. Together with (18), this yields $y_0 \in F(x_0)$. In addition, since $(u_0, v_0, x_0) \in \mathrm{sol(D)}$, we have $v_0 \in F^*(u_0)$, which implies that

$$[A(u_0, F) - v_0] \cap C_0 = \emptyset,$$

i.e., (6) holds since $v_0 = u_0(x_0) - y_0$. Thus, $(x_0, y_0, u_0) \in \mathrm{sol(P)}$, as desired. □

We now give some sufficient conditions for the validity of (18).

Proposition 2.1 *Condition* (18) *holds if*

$$A^*(x_0, F^*) \subset [F(x_0) \cap A^*(x_0, F^*)] - C. \tag{25}$$

Proof Since $F^{**}(x_0) = \mathrm{Max} A^*(x_0, F^*) \subset A^*(x_0, F^*)$, we derive from (25) that $\mathrm{Max} A^*(x_0, F^*) \subset Q - C$, where

$$Q = F(x_0) \cap A^*(x_0, F^*) \subset A^*(x_0, F^*). \tag{26}$$

Applying Lemma 2.1 with $A^*(x_0, F^*)$ in place of A, we obtain

$$F^{**}(x_0) = \mathrm{Max} A^*(x_0, F^*) \subset Q \subset F(x_0).$$ □

Proposition 2.2 *Condition* (18) *holds if* $C \cup -C = Y$ *and if*

$$F(x_0) \cap A^*(x_0, F^*) \neq \emptyset. \tag{27}$$

Proof By Proposition 2.1 it is enough to show that (25) holds. Indeed, assume to the contrary that there exists a point $y \in A^*(x_0, F^*)$ such that $y \notin Q - C$ where Q is defined by (26). This implies that $y \notin y_0 - C$ where y_0 is an arbitrary point belonging to the left side of (27). Since $y - y_0 \notin -C$ and since $C \cup -C = Y$ we get $y - y_0 \in C_0$. From $y \in A^*(x_0, F^*)$ it follows that there exist $u \in \mathcal{U}$ and $v \in F^*(u)$ such that $y = u(x_0) - v$. Observing that $y - y_0 = u(x_0) - y_0 - v$ and $y - y_0 \in C_0$, we conclude that

$$u(x_0) - y_0 - v \in C_0. \tag{28}$$

On the other hand, since $u(x_0) - y_0 \in A(u, F)$ and since $v \in F^*(u)$, we must have $u(x_0) - y_0 - v \notin C_0$, a contradiction to (28). Thus, (25) holds, as required. □

The following result is derived from Theorem 2.2 and Proposition 2.2.

Theorem 2.3

(i) *If* $(x_0, y_0, u_0) \in \mathrm{sol}(P)$ *then* (27) *holds and there exists a point* $v_0 \in F^*(u_0)$ *such that* (17) *is satisfied and* $(u_0, v_0, x_0) \in \mathrm{sol}(D)$.
(ii) *Let* $C \cup -C = Y$. *If* $(u_0, v_0, x_0) \in \mathrm{sol}(D)$ *and if* (27) *holds then there exists a point* $y_0 \in F(x_0)$ *such that* (19) *is satisfied and* $(x_0, y_0, u_0) \in \mathrm{sol}(P)$.

Proof

(i) By Theorem 2.2, all we have to prove is the validity of (27). Since (16) holds, we obtain immediately from the very definition of $F^{**}(x_0)$ that $y_0 \in A^*(x_0, F^*)$. This proves (27) since $y_0 \in F(x_0)$.

(ii) Apply Theorem 2.2 and Proposition 2.2. □

Let us observe that condition $C \cup -C = Y$ in Theorem 2.3 is a strong assumption. We now give examples proving that Theorem 2.2 can be applied without assuming that $C \cup -C = Y$.

Example 2.3 Consider Problems (P) and (D) where $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $\mathcal{U} = \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$, $C = \mathbb{R}^2_+ \subset \mathbb{R}^2$, $C_0 = C \setminus \{(0,0)\}$. Then obviously, $C \cup -C \neq Y = \mathbb{R}^2$. Assume that $G : \mathbb{R} \longrightarrow \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ and $F : \mathbb{R} \longrightarrow 2^{\mathbb{R}^2}$ are given by

$$G(x) = \begin{bmatrix} 0 \\ x \end{bmatrix},$$

$$F(x) = (x, x^2) + \{0\} \times \mathbb{R}_+ \subset \mathbb{R}^2, \quad x \in \mathbb{R}.$$

To compute $F^*(u)$ with $u \in \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ defined by (8), observe that

$$\langle u, x\rangle - F(x) = \bigcup_{\lambda \geq 0} \{(\xi, \eta - \lambda)\},$$

where $\xi = u_1 x - x$, $\eta = u_2 x - x^2$. If $u_1 = 1$, then

$$A(u, F) = \{(0, \eta) \in \mathbb{R}^2 : \eta \in]-\infty, u_2^2/4]\},$$

and hence,

$$F^*(u) = \{(0, u_2^2/4)\} \neq \emptyset. \tag{29}$$

If $u_1 \neq 1$, i.e., $u_1 - 1 \neq 0$, then since $\xi = u_1 x - x$ and $\eta = u_2 x - x^2$, we obtain $\eta = a\xi^2 + b\xi$, where $a = -1/(u_1 - 1)^2 < 0$, $b = u_2/(u_1 - 1)$. Thus,

$$A(u, F) = \bigcup_{\lambda \geq 0} \{(\xi, \eta - \lambda) \in \mathbb{R}^2 : \eta = a\xi^2 + b\xi\},$$

and hence,

$$F^*(u) = \{(\xi, \eta) \in \mathbb{R}^2 : \eta = a\xi^2 + b\xi,\ \xi \geq -b/2a\} \neq \emptyset. \tag{30}$$

Now, setting $x_0 = 0 \in X = \mathbb{R}$, and observing that $\text{dom } F^* = \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$, we have

$$A^*(x_0, F^*) = \{\langle u, x_0\rangle - F^*(u) : u \in \text{dom } F^*\} = \bigcup_{u \in \mathcal{L}(\mathbb{R}, \mathbb{R}^2)} (-F^*(u)). \tag{31}$$

Using (31), where $F^*(u)$ is defined by (29) for $u_1 = 1$ and by (30) for $u_1 \neq 1$, we can conclude by a simple computation that $A^*(x_0, F^*) = \mathbb{R}^2 \setminus C_0$. Since $C_0 = \mathbb{R}^2_+ \setminus$

$\{(0,0)\} \subset \mathbb{R}^2$, this implies that

$$F^{**}(x_0) = \text{Max } A^*(x_0, F^*) = \{(0,0)\} \subset F(x_0).$$

Therefore, (18) holds. Since $(u_0, v_0, x_0) \in \text{sol(D)}$ where $u_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $v_0 = (0,0) \in Y = \mathbb{R}^2$ and $x_0 = 0 \in X = \mathbb{R}$, Theorem 2.2 can be applied.

Example 2.4 Consider Problems (P) and (D) where $X, Y, \mathcal{U}, G$ and F are as in Example 2.3. Assume now that $C = \{(0,0)\} \cup \text{int } \mathbb{R}^2_+ \subset \mathbb{R}^2$. Then obviously, $C \cup -C \neq Y = \mathbb{R}^2$. Let $u \in \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ be defined by (8). Then, we can check that $F^*(u)$ is given by (30) if $u_1 \neq 1$. In the case $u_1 = 1$, we have

$$F^*(u) = \{0\} \times \,]-\infty, u_2^2/4] \neq \emptyset. \tag{32}$$

Now, setting $x_0 = 0 \in X = \mathbb{R}$ and making use of (30) and (32), we can conclude by a simple computation that $A^*(x_0, F^*) = (\mathbb{R}^2 \setminus \mathbb{R}^2_+) \cup (\{0\} \times \mathbb{R}_+)$. Since $C_0 = \text{int } \mathbb{R}^2_+ \subset \mathbb{R}^2$, this implies that

$$F^{**}(x_0) = \text{Max } A^*(x_0, F^*) = \{0\} \times \mathbb{R}_+ = F(x_0).$$

Therefore, (18) holds and Theorem 2.2 can be applied, with u_0, v_0, x_0 being as in Example 2.3.

3 Duality in Generalized Vector Variational Inequalities with Proper Conjugate Maps

In this section, we will introduce a generalized version of primal and dual vector variational inequality problems whose solutions are introduced via the notion of proper efficiency of Benson (see [11, 12]). Observe that in the literature several definitions of proper efficiency were proposed. The reader is referred to [13] for a comprehensive survey where all these definitions and links between them can be found. Let C, F, G and $\mathcal{U}$ be as in the previous section.

Definition 3.1

(i) The set-valued map $F^*_{\text{B}} : \mathcal{U} \longrightarrow 2^Y$ defined by

$$\begin{aligned} F^*_{\text{B}}(u) &= \text{BMax } A(u, F) \\ &:= \{v \in A(u, F) : \text{cl cone}[A(u, F) - v - C] \cap C = \{0\}\} \end{aligned} \tag{33}$$

is called the Benson conjugate map of F.

(ii) The set-valued map $F^{**}_{\text{B}} : X \longrightarrow 2^Y$ defined by

$$\begin{aligned} F^{**}_{\text{B}}(x) &= \text{BMax } A^*(x, F^*_{\text{B}}) \\ &:= \{y \in A^*(x, F^*_{\text{B}}) : \text{cl cone}[A^*(x, F^*_{\text{B}}) - y - C] \cap C = \{0\}\} \end{aligned} \tag{34}$$

is called the Benson biconjugate map of F.

In this section, we are interested in the following two problems:
Problem ($\mathrm{P_B}$). Find $x_0 \in \mathrm{dom}\, F$, $y_0 \in F(x_0)$ and $u_0 \in G(x_0)$ such that

$$\mathrm{cl}\,\mathrm{cone}\{A(u_0, F) - [u_0(x_0) - y_0] - C\} \cap C = \{0\}. \tag{35}$$

Problem ($\mathrm{D_B}$): Find $u_0 \in \mathrm{dom}\, F^*_{\mathrm{B}}$, $v_0 \in F^*_{\mathrm{B}}(u_0)$ and $x_0 \in G^{-1}(u_0)$ such that

$$\mathrm{cl}\,\mathrm{cone}\{A^*(x_0, F^*_{\mathrm{B}}) - [u_0(x_0) - v_0] - C\} \cap C = \{0\}. \tag{36}$$

A triplet (x_0, y_0, u_0) (resp. (u_0, v_0, x_0)) which satisfies all the requirements formulated in Problem ($\mathrm{P_B}$) (resp. Problem ($\mathrm{D_B}$)) is called a solution of Problem ($\mathrm{P_B}$) (resp. Problem ($\mathrm{D_B}$)). If (x_0, y_0, u_0) (resp. (u_0, v_0, x_0)) is a solution of Problem ($\mathrm{P_B}$) (resp. Problem ($\mathrm{D_B}$)), then

$$u_0(x_0) - y_0 \in \mathrm{BMax}\ A(u_0, F),$$
$$(\text{resp. } u_0(x_0) - v_0 \in \mathrm{BMax}\ A^*(x_0, F^*_{\mathrm{B}})).$$

Thus, the notion of solutions of Problems ($\mathrm{P_B}$) and ($\mathrm{D_B}$) is introduced via the concept of BMax A ($A = A(u_0, F)$ or $A = A^*(x_0, F^*_{\mathrm{B}})$) which corresponds to the Benson proper efficiency in vector optimization (see [11, 12]). This notion is also used in the definition of F^*_{B} which appears in Problem ($\mathrm{D_B}$).

Let us denote by sol($\mathrm{P_B}$) (resp. sol($\mathrm{D_B}$)) the set of all the solutions of Problem ($\mathrm{P_B}$) (resp. Problem ($\mathrm{D_B}$)). From Lemma 2.1 it is clear that sol($\mathrm{P_B}$)$\subset$ sol(P). The following result gives a link between sol($\mathrm{P_B}$) and sol($\mathrm{D_B}$).

Theorem 3.1

(i) *If* $(x_0, y_0, u_0) \in \mathrm{sol}(P_{\mathrm{B}})$ *and if*

$$y_0 \in F^{**}_{\mathrm{B}}(x_0), \tag{37}$$

then there exists a point $v_0 \in F^*_{\mathrm{B}}(u_0)$ *such that* (17) *is satisfied and* $(u_0, v_0, x_0) \in \mathrm{sol}(D_{\mathrm{B}})$.

(ii) *If* $(u_0, v_0, x_0) \in \mathrm{sol}(D_{\mathrm{B}})$ *and if*

$$F^{**}_{\mathrm{B}}(x_0) \subset F(x_0), \tag{38}$$

then there exists a point $y_0 \in F(x_0)$ *such that* (19) *is satisfied and* $(x_0, y_0, u_0) \in \mathrm{sol}(P_{\mathrm{B}})$.

Proof

(i) Let $(x_0, y_0, u_0) \in \mathrm{sol}(\mathrm{P_B})$. By (37), we have

$$y_0 \in A^*(x_0, F^*_{\mathrm{B}}), \tag{39}$$
$$\mathrm{cl}\,\mathrm{cone}[A^*(x_0, F^*_{\mathrm{B}}) - y_0 - C] \cap C = \{0\}. \tag{40}$$

Now, defining v_0 by (17) and observing that $v_0 \in A(u_0, F)$, we conclude from (35) that $v_0 \in F^*_{\mathrm{B}}(u_0)$. Since $x_0 \in G^{-1}(u_0)$ and since (36) is exactly condition (40) with $y_0 = u_0(x_0) - v_0$ we see that $(u_0, v_0, x_0) \in \mathrm{sol}(\mathrm{D_B})$.

(ii) Let $(u_0, v_0, x_0) \in \mathrm{sol}(\mathrm{D_B})$. Defining y_0 by (19) and observing that $v_0 \in F_{\mathrm{B}}^*(u_0)$ we obtain (39). Together with (36) and (19), this proves that $y_0 \in F_{\mathrm{B}}^{**}(x_0)$, i.e., $y_0 \in F(x_0)$ (see (38)). Now, from $u_0(x_0) - y_0 = v_0 \in F_{\mathrm{B}}^*(u_0)$ and from (33) with $u = u_0$ it follows that (35) holds. Therefore, $(x_0, y_0, u_0) \in \mathrm{sol}(\mathrm{P_B})$, as desired. □

Remark 3.1 Condition (37) is introduced in [7] when F is single-valued and the biconjugate map is defined via the set of weakly efficient points.

We now consider some sufficient conditions for the validity of (38). We delete the detailed proof of the following Propositions 3.1 and 3.2 since it is quite similar to that of Propositions 2.1 and 2.2. Observe that the assumptions of Proposition 3.2 assure the validity of condition (4.1) below.

Proposition 3.1 *Condition* (38) *holds if*

$$A^*(x_0, F_{\mathrm{B}}^*) \subset [F(x_0) \cap A^*(x_0, F_{\mathrm{B}}^*)] - C. \tag{41}$$

Proposition 3.2 *Condition* (38) *holds if* $C \cup -C = Y$ *and if*

$$F(x_0) \cap A^*(x_0, F_{\mathrm{B}}^*) \neq \emptyset. \tag{42}$$

Theorem 3.2

(i) *If* $(x_0, y_0, u_0) \in \mathrm{sol}(P_{\mathrm{B}})$ *and if* (37) *holds then* (42) *holds and there exists a point* $v_0 \in F_{\mathrm{B}}^*(u_0)$ *such that* (17) *is satisfied and* $(u_0, v_0, x_0) \in \mathrm{sol}(D_{\mathrm{B}})$.
(ii) *Let* $C \cup -C = Y$. *If* $(u_0, v_0, x_0) \in \mathrm{sol}(D_{\mathrm{B}})$ *and if* (42) *holds then there exists a point* $y_0 \in F(x_0)$ *such that* (19) *is satisfied and* $(x_0, y_0, u_0) \in \mathrm{sol}(P_{\mathrm{B}})$.

Proof

(i) We have seen in the proof of Theorem 3.1 that condition (37) implies that (39) holds and $(u_0, v_0, x_0) \in \mathrm{sol}(\mathrm{D_B})$. To complete of our proof it remains to observe from (39) and condition $y_0 \in F(x_0)$ that (42) is satisfied.
(ii) Apply Theorem 3.1 and Proposition 3.2.

□

Example 3.1 Consider Problems $(\mathrm{P_B})$ and $(\mathrm{D_B})$, where $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $\mathcal{U} = \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$, $C = \mathbb{R}^2_+ \subset \mathbb{R}^2$, and $C_0 = C \setminus \{(0,0)\}$. Then, obviously, $C \cup -C \neq Y = \mathbb{R}^2$. Assume that $G : \mathbb{R} \longrightarrow \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ and $F : \mathbb{R} \longrightarrow 2^{\mathbb{R}^2}$ are given by

$$G(x) = \begin{bmatrix} 0 \\ x \end{bmatrix},$$

$$F(x) = \begin{cases} [(0,0), (0,1)], & \text{if } x = 0, \\ ](x,x), (x,1)[, & \text{if } x \in]0,1[, \\ \emptyset, & \text{if } x \notin [0,1[, \end{cases}$$

for all $x \in \mathbb{R}$.

To compute $F_{\mathrm{B}}^*(u)$ with $u \in \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$ defined by (8), observe that

$$A(u,F)=\{\langle u,x\rangle - F(x) : x\in[0,1[\}$$
$$=\left\{\bigcup_{x\in]0,1[}](u_1x-x,u_2x-x),(u_1x-x,u_2x-1)[\right\}\cup[(0,-1),(0,0)].$$

If $u_1=1$, then

$$A(u,F)=\left\{\bigcup_{x\in]0,1[}](0,u_2x-x),(0,u_2x-1)[\right\}\cup[(0,-1),(0,0)]$$

$$=\begin{cases}[(0,-1),(0,u_2-1)[, & \text{if } u_2>1,\\ [(0,-1),(0,0)], & \text{if } u_2\in[0,1],\\ ](0,u_2-1),(0,0)], & \text{if } u_2<0,\end{cases}$$

and hence,

$$F_{\mathrm{B}}^*(u)=\begin{cases}\{(0,0)\}, & \text{if } u_1=1,\ u_2\le 1,\\ \emptyset, & \text{if } u_1=1,\ u_2>1.\end{cases}$$

If $u_1\neq 1$, then we obtain

$$A(u,F)=\mathrm{co}\{(0,-1),(0,0),(u_1-1,u_2-1)\}$$
$$\setminus(](0,-1),(u_1-1,u_2-1)]\cup](0,0),(u_1-1,u_2-1)]),$$

and hence,

$$F_{\mathrm{B}}^*(u)=\begin{cases}\{(0,0)\}, & \text{if } u_1<1 \text{ or } u_2<1,\\ \emptyset, & \text{if } u_1>1, u_2\ge 1.\end{cases}$$

Here, co denotes the convex hull. Now, setting $x_0=0$ and observing that

$$\mathrm{dom}\,F_{\mathrm{B}}^*=\{u\in\mathcal{L}(\mathbb{R},\mathbb{R}^2): u_1=u_2=1 \text{ or } u_1<1 \text{ or } u_2<1\},$$

we have

$$A^*(x_0,F_{\mathrm{B}}^*)=\{\langle u,x_0\rangle - F_{\mathrm{B}}^*(u) : u\in\mathrm{dom}\,F_{\mathrm{B}}^*\}=\bigcup_{u\in\mathrm{dom}\,F_{\mathrm{B}}^*}(-F^*(u))=\{(0,0)\}.$$

This implies that

$$F_{\mathrm{B}}^{**}(x_0)=\mathrm{Max}\ A^*(x_0,F_{\mathrm{B}}^*)=\{(0,0)\}\subset F(x_0).$$

Therefore, (38) holds. Since $(u_0,v_0,x_0)\in\mathrm{sol}(\mathrm{D_B})$, where $u_0=\begin{bmatrix}0\\0\end{bmatrix}$, $v_0=(0,0)\in Y=\mathbb{R}^2$ and $x_0=0\in X=\mathbb{R}$, Theorem 3.1 can be applied.

We will give a sufficient condition for the validity of condition (38) via a notion of subdifferentiability of F and the graph of F, denoted by gr F,

$$\operatorname{gr} F = \{(x, y) \in X \times Y : y \in F(x)\}.$$

Definition 3.2 Let $x \in \operatorname{dom} F$. The set-valued map F is called Benson properly subdifferentiable at $(x, y) \in \operatorname{gr} F$ if there exists a map $u \in \mathcal{U}$ such that $u(x) - y \in F_{\mathrm{B}}^*(u)$. If this property holds for each $y \in F(x)$ then the set-valued map F is called Benson properly subdifferentiable at x.

Proposition 3.3 *Let F be Benson properly subdifferentiable at $x_0 \in \operatorname{dom} F$. Then,* (38) *holds if*

$$F(x_0) \supset \{y \in Y : \operatorname{cl}\operatorname{cone}[F(x_0) - y - C] \cap C \cap \operatorname{cl}\operatorname{cone}[y - F(x_0) - C] = \{0\}\}.$$

Proof Observe that

$$\begin{aligned} y \in A^*(x_0, F_{\mathrm{B}}^*) \quad &\Longrightarrow \quad \exists \overline{u} \in \mathcal{U} : [\overline{u}(x_0) - y] \in F_{\mathrm{B}}^*(\overline{u}) \\ &\Longrightarrow \quad \operatorname{cl}\operatorname{cone}[A(\overline{u}, F) - [\overline{u}(x_0) - y] - C] \cap C = \{0\} \\ &\Longrightarrow \quad \operatorname{cl}\operatorname{cone}[\overline{u}(x_0) - F(x_0) - [\overline{u}(x_0) - y] - C] \cap C = \{0\} \\ &\Longrightarrow \quad \operatorname{cl}\operatorname{cone}[y - F(x_0) - C] \cap C = \{0\}. \end{aligned}$$

On the other hand, by the Benson proper subdifferentiability of F at x_0,

$$y_0 \in F(x_0) \quad \Longrightarrow \quad [\exists u \in \mathcal{U} : y_0 \in u(x_0) - F_{\mathrm{B}}^*(u)] \quad \Longrightarrow \quad y_0 \in A^*(x_0, F_{\mathrm{B}}^*).$$

This implies that $F(x_0) \subset A^*(x_0, F_{\mathrm{B}}^*)$. Therefore, for each $y \in Y$,

$$\operatorname{cl}\operatorname{cone}[A^*(x_0, F_{\mathrm{B}}^*) - y - C] \cap C = \{0\} \quad \Longrightarrow \quad \operatorname{cl}\operatorname{cone}[F(x_0) - y - C] \cap C = \{0\}.$$

From the above discussion, we obtain (38) since

$$\begin{aligned} y \in F_{\mathrm{B}}^{**}(x_0) \quad &\Longleftrightarrow \quad [y \in A^*(x_0, F_{\mathrm{B}}^*); \operatorname{cl}\operatorname{cone}[A^*(x_0, F_{\mathrm{B}}^*) - y - C] \cap C = \{0\}] \\ &\Longrightarrow \quad \operatorname{cl}\operatorname{cone}[y - F(x_0) - C] \cap C \cap \operatorname{cl}\operatorname{cone}[F(x_0) - y - C] = \{0\} \\ &\Longrightarrow \quad y \in F(x_0). \end{aligned}$$

This proves that $F_{\mathrm{B}}^{**}(x_0) \subset F(x_0)$, as desired. □

Corollary 3.1 *Let F be Benson properly subdifferentiable at $x_0 \in \operatorname{dom} F$. Then,* (38) *holds if*

$$F(x_0) \supset \{y \in Y : [F(x_0) - y] \cap C_0 \cap -C_0 = \emptyset\}.$$

In Proposition 3.3 it is required that F is Benson properly subdifferentiable at x_0. We will give a sufficient condition for this property of F. In the rest of this paper we will assume that X and Y are locally convex spaces, C is a convex cone of Y

and $\mathcal{U} = \mathcal{L}(X, Y)$. We denote by 0_X and 0_Y the origin of X and Y, respectively. We first give a necessary condition for the Benson proper subdifferentiability of F at $(x_0, y_0) \in \operatorname{gr} F$.

Proposition 3.4 *Let $\mathcal{C} = \{0_X\} \times C \subset X \times Y$. If F is Benson properly subdifferentiable at $(x_0, y_0) \in \operatorname{gr} F$, then*

$$\operatorname{cone}[(x_0, y_0) - \operatorname{gr} F - \mathcal{C}] \cap \mathcal{C} = \{(0_X, 0_Y)\}. \tag{43}$$

Proof We first observe from the proof of Proposition 3.3 that the Benson proper subdifferentiability of F at $(x_0, y_0) \in \operatorname{gr} F$ implies that

$$\operatorname{cl} \operatorname{cone}[F(x_0) - y_0 - C] \cap C = \{0_Y\}. \tag{44}$$

Now, assume to the contrary that (43) is not satisfied. Then, there exist $\lambda > 0$, $(x, y) \in \operatorname{gr} F$, $c' \in C$ and $c'' \in C \setminus \{0\}$ such that

$$\lambda[(x_0, y_0) - (x, y) - (0_X, c')] = \{0_X, c'')\},$$

i.e., $x = x_0$ and

$$\lambda(y_0 - y - c') = c'' \in C \setminus \{0_Y\}. \tag{45}$$

Since $x = x_0$ and $(x, y) \in \operatorname{gr} F$, it is clear that $y \in F(x_0)$. Hence, condition (45) conflicts to (44) and Proposition 3.4 is thus established. □

We will prove in Proposition 3.5 that a condition stronger than (43) is sufficient for the Benson proper subdifferentiability of F at (x_0, y_0). We will need a notion of a base of a cone. A subset Δ of a convex cone $C \subset Y$ is called a base of C if Δ is convex, $0 \notin \operatorname{cl} \Delta$ and $\operatorname{cone} \Delta = C$. Obviously, if C has a base then C is pointed, i.e., $C \cap -C = \{0_Y\}$. The following lemma plays a key role in the proof of Proposition 3.5.

Lemma 3.1 *Let X and Y be locally convex spaces and let $C \subset Y$ be a convex cone with compact base. Let $\mathcal{U} = \mathcal{L}(X, Y)$ and $\mathcal{C} = \{0_X\} \times C$. If for $x_0 \in \operatorname{int} \operatorname{dom} F$ and $y_0 \in F(x_0)$, there exists a closed convex cone $K \subset X \times Y$ such that*

$$(x_0, y_0) - \operatorname{gr} F \subset K, \tag{46}$$

$$K \cap \mathcal{C} = \{(0_X, 0_Y)\}, \tag{47}$$

then F is Benson properly subdifferentiable at (x_0, y_0), i.e., there exists $u \in \mathcal{U} = \mathcal{L}(X, Y)$ such that $u(x_0) - y_0 \in F^*_{\mathrm{B}}(u)$.

Proof Since K is a closed convex cone and since $\mathcal{C}$ is a convex cone with compact base, we derive from (47) and Theorem 2.3 of [14] that there exists a pointed convex cone $D \subset X \times Y$ such that

$$K \cap D = \{(0_X, 0_Y)\}, \tag{48}$$

$$\mathcal{C} \setminus \{(0_X, 0_Y)\} \subset \operatorname{int} D. \tag{49}$$

Since $\mathcal{C} = \{0_X\} \times C$ where C is a convex cone with compact base, it follows from (49) that there exists a convex cone $\widehat{C} \subset Y$ with $0_Y \in \widehat{C}$, $\operatorname{int} \widehat{C} \neq \emptyset$ and

$$\{0_X\} \times C \setminus \{(0_X, 0_Y)\} \subset \{0_X\} \times \operatorname{int} \widehat{C} \subset \operatorname{int} D. \tag{50}$$

Now, observing from (48) that $K \cap \operatorname{int} D = \emptyset$, we can derive by a separation theorem that there exists a nonzero element $(x^*, y^*) \in X^* \times Y^*$ such that

$$\langle x^*, x\rangle + \langle y^*, y\rangle \leq \langle x^*, d_1\rangle + \langle y^*, d_2\rangle$$

for all $(x, y) \in K$ and $(d_1, d_2) \in \operatorname{int} D$ where X^* (resp. Y^*) denotes the space of linear continuous functionals defined on X (resp. Y). Combining this with (46) and (50), and taking account of the continuity of y^*, we obtain

$$\langle x^*, x - x_0\rangle + \langle y^*, y - y_0\rangle \leq \langle y^*, \widehat{c}\rangle, \tag{51}$$

for all $(x, y) \in \operatorname{gr} F$ and $\widehat{c} \in \widehat{C}$. Setting $(x, y) = (x_0, y_0)$ in (51) we get $\langle y^*, \widehat{c}\rangle \geq 0$ for all $\widehat{c} \in \widehat{C}$, i.e., $y^* \in \widehat{C}^+$ (the set of linear continuous functionals $y^* \in Y^*$ which are nonnegative on $\widehat{C}$). In view of (50) $C \subset \widehat{C}$ and hence, $y^* \in C^+$. Now, observe from (51) that $y^* \neq 0_{Y^*}$. Indeed, otherwise we get $\langle x^*, x - x_0\rangle \leq 0$, $\forall x \in \operatorname{dom} F$, which implies that $x^* = 0_{X^*}$ since $x_0 \in \operatorname{int} \operatorname{dom} F$. This contradicts the nontriviality of (x^*, y^*). Thus, $y^* \in C^+ \setminus \{0_{Y^*}\}$ and hence, there exists $c_0 \in C$ with $\langle y^*, c_0\rangle = 1$. Setting $u(x) = -\langle x^*, x\rangle c_0$, for each $x \in X$, we see that $u \in \mathcal{L}(X, Y)$ and $\langle y^*, u(x)\rangle = -\langle x^*, x\rangle$, $\forall x \in X$. Therefore, by (51),

$$\lambda\langle y^*, u(x) - y - u(x_0) + y_0 - c\rangle \leq 0, \quad \forall (x, y) \in \operatorname{gr} F,\ c \in \widehat{C},\ \lambda \geq 0.$$

Since $y^* \in \widehat{C}^+ \setminus \{0_{Y^*}\}$, this proves that

$$\lambda(u(x) - y - [u(x_0) - y_0] - c) \notin \operatorname{int} \widehat{C}, \quad \forall (x, y) \in \operatorname{gr} F,\ c \in \widehat{C},\ \lambda \geq 0.$$

Therefore,

$$\operatorname{cl}\operatorname{cone}[A(u, F) - [u(x_0) - y_0] - \widehat{C}) \cap \operatorname{int} \widehat{C} = \emptyset.$$

This implies that

$$\operatorname{cl}\operatorname{cone}[A(u, F) - [u(x_0) - y_0] - C) \cap C = \{0\}$$

since by (50) $C \setminus \{0_Y\} \subset \operatorname{int} \widehat{C}$. Therefore, $u(x_0) - y_0 \in F^*_{\mathrm{B}}(u)$, as desired. □

Since $\mathcal{C} := \{0_X\} \times C$ is a subset of $X \times Y$, each subset $\mathcal{A} \subset X \times Y$ can be associated to the set $\mathrm{BMax}_{\mathcal{C}}\mathcal{A}$ which is constructed as the set $\mathrm{BMax}_C A$ in Sect. 2 with $\mathcal{A}$ and $\mathcal{C}$ instead of A and C, respectively. Namely,

$$\mathrm{BMax}_{\mathcal{C}}\mathcal{A} = \{a \in \mathcal{A} : \mathrm{cl}\,\mathrm{cone}[\mathcal{A} - a - \mathcal{C}] \cap \mathcal{C} = \{(0_X, 0_Y)\}\}.$$

So, if $\mathcal{A} = -\,\mathrm{gr}\,F$ and $(x_0, y_0) \in \mathrm{gr}\,F$, then the condition

$$-(x_0, y_0) \in \mathrm{BMax}_{\mathcal{C}}[-\,\mathrm{gr}\,F] \tag{52}$$

is equivalent to the following condition which is near to condition (43):

$$\mathrm{cl}\,\mathrm{cone}[(x_0, y_0) - \mathrm{gr}\,F - \mathcal{C}] \cap \mathcal{C} = \{(0_X, 0_Y)\}. \tag{53}$$

Proposition 3.5 *Let X and Y be locally convex spaces and let $C \subset Y$ be a convex cone with compact base. Let $\mathcal{U} = \mathcal{L}(X, Y)$ and $x_0 \in \mathrm{int}\ \mathrm{dom}\,F$. Then*:

(i) *F is Benson properly subdifferentiable at $(x_0, y_0) \in \mathrm{gr}\,F$ if*

$$-(x_0, y_0) \in \mathrm{BMax}_{\mathcal{C}}[-\,\mathrm{gr}\,F]$$

and if $[(x_0, y_0) - \mathrm{gr}\,F]$ is nearly $(-\mathcal{C})$-subconvexlike in the sense of [15, 16], *i.e.*, $\mathrm{cl}\,\mathrm{cone}[(x_0, y_0) - \mathrm{gr}\,F - \mathcal{C}]$ *is a convex set.*

(ii) *F is Benson properly subdifferentiable at x_0 if, for each $y_0 \in F(x_0)$, the conditions formulated in Part* (*i*) *are satisfied.*

Proof Applying Lemma 3.1 with $K = \mathrm{cl}\,\mathrm{cone}[(x_0, y_0) - \mathrm{gr}\,F - \mathcal{C}]$ we obtain the conclusion of Part (i). Part (ii) is an immediate consequence of Part (i). □

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