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# Riccati Equations for the Bolza Problem Arising in Boundary/Point Control Problems Governed by $C_0$ Semigroups Satisfying a Singular Estimate

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**Abstract** We establish solvability of Riccati equations and optimal feedback synthesis in the context of Bolza control problem for a special class of control systems referred to in the literature as *control systems with singular estimate*. Boundary/point control problems governed by analytic semigroups constitute a very special subcategory of this class which was motivated by and encompasses many PDE control systems with both boundary and point controls that involve interactions of different types of dynamics (parabolic and hyperbolic) on an interface. We also discuss two examples from thermoelasticity and structure acoustics.

**Keywords** Bolza boundary control · Riccati equations · Unbounded coefficients · Singular estimates · Feedback control · Thermoelasticity · Structure-wave interactions

## 1 Introduction

Let $Y$ the state space and $U$ the control space be Hilbert spaces, and consider the control problem of minimizing the following cost functional on time interval $[s, T]$ over all $u \in L_2([s, T]; U)$:

$$J(u, y, s, y_s) = \int_s^T [|Ry(t)|_W^2 + |u(t)|_U^2] dt + |Gy(T)|_Z^2, \tag{1}$$

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subject to the dynamics satisfying the abstract differential equation

$$y_t = Ay + Bu, \quad \text{on } [\mathcal{D}(A^\star)]', \tag{2a}$$

$$y(s) = y_s \in Y. \tag{2b}$$

The operators $A$, $B$, $G$, $R$ are linear satisfying the following hypothesis:

**Assumption 1.1**

(a) $A$ generates a strongly continuous semigroup on a Hilbert space $Y$ denoted by $e^{At}$.
(b) $B$ is a linear operator from $U \to [\mathcal{D}(A^\star)]'$, such that $R(\lambda, A)B \in \mathcal{L}(U, Y)$, for some $\lambda \in \rho(A)$. Without loss of generality, we can assume that $\lambda = 0$ and hence $A^{-1}B \in \mathcal{L}(U; Y)$, where $R(\lambda, A)$ is the resolvent of $A$.
(c) There exists $0 < \gamma < 1$ such that $|e^{At}Bu|_Y \le \frac{C}{t^\gamma}|u|_U$ for all $0 < t \le 1$.
(d) $G$ is a bounded linear operator from $Y$ to $Z$ another Hilbert space.
(e) $R$ is a bounded linear operator from $Y$ to $W$ a Hilbert space.

We provide an optimal feedback representation of the optimal control, along with well-posedness of the associated Riccati equation. The optimal quadratic cost control problem and Riccati synthesis, in the context of infinite-dimensional systems, have been extensively studied in the literature. "Classical" theory, which involves bounded control operators $B$, has been in place and well understood for many years [1, 2]. The main distinctive feature of the model considered above is that control operator $B$ is *not assumed bounded* as an operator $U \to Y$ and typically not even densely defined but rather takes its values in $[\mathcal{D}(A^*)]'$. The motivation for studying this class of problems comes from recent advances in technological applications including smart material technologies where control actions are executed via boundary or point controls which are two prototypes of unbounded action controls.

Riccati theory, in the context of "unbounded" control operators $B$, has been recently studied by several authors. However, it turns out that the "satisfactory" results, which reconstruct classical theory, require strong regularity assumptions imposed on the dynamics, such as *analyticity* of semigroups driving the system [3, 4]. On another front, the study of unbounded control operators in the context of hyperbolic dynamics have revealed a number of pathologies and counterexamples to "classical" results [6, 7]. In fact, while the corresponding theory has been developed [5] for infinite horizon problems, this theory requires very special extensions of the operators $B^*$ in order to reconcile optimal synthesis and the Riccati equation.

A natural class of systems with unbounded control actions which are not necessarily analytic but still yield rich optimization theory are SECS systems (singular estimate control systems), i.e.: control systems with $(A, B)$ that satisfy the singular estimate in (c). The singular estimate implies the kernel in the integral operator describing the control-state map is $L_1$. A fortunate fact is that this class has emerged not only as a "mathematical concept" or generalization of analytic systems, but as a right object of study when formulating many important control problems arising at the frontiers of technological developments such as "smart materials", intelligent controls and interactive structures [8, 9]. Indeed, the resulting control actions (point or boundary) are most often unbounded, while hybrid or coupled connections between several components of the system allow for a possibility of transferring smoothing effects from one component to the entire structure [10]. This is the case for structure-wave interaction, structure-fluid interactions, thermoelastic systems, composite materials, etc. Thus, SECS systems, being an extension of analytic systems, arise naturally in the context of hybrid PDE dynamics. Typical configuration is that of a coupled system which consists of an analytic (parabolic) component and a hyperbolic component interacting (often via a boundary condition) at some lower dimensional manifold called interface.

In fact, for control problems *in the absence of the terminal time penalization* $G = 0$, full and satisfactory Riccati theory has been developed already [10–13]. The novelty of the work presented in this paper is that the Bolza problem is studied with *non-smooth final state* penalization in the context of singular estimate control (SEC) systems. It is well known that the presence of non-smooth terminal condition is bound to bring new pathologies resulting in the loss of regularity of the value function. This is particularly pronounced when unbounded controls are considered which bring forward new phenomena of propagating singularities—both forward and backward—which, in turn, change the Riccati theory drastically. This is true even in the case of analytic semigroups, where the Bolza problem has been already considered [4, 11]. We show that although the Riccati equation is well posed for this class of systems, the gain operator exhibits integrable singularity. In addition, we will show that the finite time "transfer" function for the controlled system exhibits double singularity: one at the origin and another at the terminal time. While the singularity of the "transfer" function at the origin has been known in SECS systems, see [10], the qualitative description of singularity at the end point is new and is caused by the nonsmooth terminal condition for the Riccati equation. In this paper, we provide a complete study of the Bolza problem for SECS class of systems, arriving eventually at a well posed Riccati equation in spite of these singularities.We also illustrate the theory by examples arising in control problems of interactive PDE systems in structure acoustics and thermoelasticity.

## 2 Main Results

The operator, critical for the study of optimal control problems with unbounded actions, is the control-state operator $L_s : L_2([s,T];U) \to C([s,T];[D(A^*)]')$,

$$(L_s u)(t) \equiv A \int_s^t e^{A(t-\tau)} A^{-1} B u(\tau) d\tau. \tag{3}$$

By condition (b) in Assumption 1.1 this operator is linear and bounded within the topologies indicated above with the values in a dual space typically related to some distributions. Indeed, the following result which follows from Young's inequality is well known:

**Lemma 2.1** (See [10]) *Under Assumption 1.1, $L_s$ is bounded from $L_p([s,T];U) \to L_p([s,T];Y)$, $1 < p < \infty$, uniformly in $s$.*

We also define $L_{sT}$ from $D(L_{sT}) \subset L_2([s,T];U)$ to $Y$ as

$$L_{sT}u \equiv (L_s u)(T). \tag{4}$$

It is known [10] that under condition (b) in Assumption 1.1, $H_0^1(0,T;U) \subset D(L_{sT})$, so $L_{sT}$ is densely defined and also closed (note that $A^{-1}L_{sT}$ is bounded). The operator $L_{sT}$, describing terminal action of the control, plays a critical role in the study of Bolza problem. The first difficulty encountered is that this operator is *not bounded* on the control space. Thus, the functional cost $J(u,y,s,y_s)$ is *not bounded*. This is in sharp contrast with the SECS theory for problems *without final state penalization*, [10].

In addition to Assumption 1.1, the following is also a necessary condition for solvability of the Bolza problem.

**Assumption 2.1** The operator $GL_{sT}$ is closed from $L_2([s,T];U) \to Z$.

*Remark 2.1* The lack of closability leads to counterexamples to the very existence of the optimal control [4].

**Theorem 2.1** *Under set of conditions in Assumptions 1.1 and 2.1, for any initial state $y_s \in Y$ there exists a unique optimal control $u^0(t,s,y_s) \in L_2([s,T];U)$ such that $J(u^0,y^0,s,y_s) = \min_{u\in L_2([s,T],U)} J(u,y(u),s,y_s)$, where $y^0(t,s,y_s) \in L_2([s,T];Y)$ is the corresponding optimal trajectory. Moreover*:

- *The optimal control $u^0(t)$ is continuous on $[s,T)$ with values in $U$ and*

$$|u^0(t,s,y_s)|_U \le \frac{C}{(T-t)^\gamma}, \quad s \le t < T \tag{5}$$

- *The optimal output $y^0(t) \in C([s,T],Y)$ when $\gamma < 1/2$, but if $\gamma \ge 1/2$,*

$$|y^0(t,s,y_s)|_U \le \frac{C}{(T-t)^{2\gamma-1+\epsilon}}, \quad s \le t < T,\ \epsilon > 0. \tag{6}$$

- *Singularity of "transfer function" corresponding to feedback dynamics,*

$$|y^0(t,s,Bu)|_H \le \frac{C_{T-s}|u|_U}{(T-t)^\gamma (t-s)^\gamma}, \quad s < t < T, \tag{7}$$

  *where the constant $C_{T-s}$ blows up when $T \to s$.*

Theorem 2.1 provides existence and uniqueness of the optimal solution along with the characterization of the Riccati operator $P(t)$ as a value function. However, the main focus of this paper is on optimal feedback synthesis and solvability of the Riccati equation which provides a direct way of obtaining the optimal control based on real time evaluation of the state, without the need for solving the dynamics. The difficulty is again the unboundedness of $B$ and the danger that the value function may not be sufficiently smooth (typical for Bolza problems). Moreover, the "usual" feedback operator $B^*P(t)$ arising in "smooth" theory may not be even defined (see [6, 7]). The main result of this paper, formulated below, asserts that the feedback operator is well defined on $[0, T)$ and exhibits controlled singularity at the point $t = T$.

**Theorem 2.2** *Under the same assumptions of Theorem 2.1, the following hold*:

- [Optimal Cost and Value Function] *There exists a selfadjoint positive operator* $P(t) \in \mathcal{L}(Y), t \in [0, T)$ *s.t.* $\langle P(t)x, x\rangle_Y = J(u^0, y^0, s, x)$, *where* $J(u^0, y^0, s, y_s) \equiv \min_{u \in L_2([s,T],U)} J(u, y(u), s, y_s)$.
- [Singular Behavior of the Feedback Operator]
  1. $P(t)$ *is continuous on* $[0, T]$ *and* $P(t) \in \mathcal{L}(Y, C([0, T]; Y))$
  2. *Feedback operator* $B^* P(t) \in \mathcal{L}(Y, C([s, T), U))$ *and satisfies*

$$|B^* P(t)x|_U \leq \frac{C|x|_Y}{(T - t)^\gamma}, \quad 0 \leq t < T \tag{8}$$

- [Optimal Synthesis]

$$u^0(t, s; y_s) = -B^* P(t) y^0(t, s; y_s), \quad s \leq t < T. \tag{9}$$

- [Riccati Equation] $P(t)$ *satisfies the differential Riccati equation with* $t < T, x, y \in \mathcal{D}(A)$,

$$\begin{aligned} &\langle P_t x, y\rangle_Y + \langle A^\star P(t)x + P(t)Ax, y\rangle_Y + \langle Rx, Ry\rangle_Z \\ &\quad = \langle B^* P(t)x, B^* P(t)y\rangle_U \end{aligned} \tag{10a}$$

$$\lim_{t \to T} P(t)x = G^* G x \quad \forall x \in Y \tag{10b}$$

- [Uniqueness] *The Riccati equation has a unique solution in the class of positive and selfadjoint operators* $P(t)$ *satisfying* (8) *when* $\gamma < \frac{1}{2}$.

The remainder of the paper is devoted to the proofs of the main theorems.

## 3 Characterization of the Optimal Control

Existence and uniqueness of optimal control $u^0 \in L_2([s, T]; U)$ minimizing the functional $J$ follows from standard variational argument [17]. Our main goal is to establish the operator theoretic characterization of optimal solutions. This characterization will allow, later on, to infer additional regularity of optimal quantities that goes much beyond what comes from optimization alone.

### 3.1 Preliminary Results and Definitions

In this section, we provide a functional analytic framework for studying the problem and we collect a number of properties that are well known.

With the operator $L_s$ defined in (3), it is well known [4, 10] that the trajectory $y$ due to the input $u$ and initial condition $y_s$ is given by

$$y(t, s; y_0) = e^{A(t-s)} y_0 + (L_s u)(t). \tag{11}$$

Thus, by Lemma 2.1, $y \in L_2([s,T];Y)$ whenever $u \in L_2([s,T];U)$. We next introduce the adjoint operators to $L_s$ and $L_{sT}$. The adjoints will always be considered with respect to $L_2(s,T)$ topology.

$$(L_s^\star f)(t) = \int_t^T B^\star e^{A^\star(s-t)} f(s)ds \tag{12}$$

is bounded from $L_2([s,T];Y)$ to $L_2([s,T];U)$ uniformly with respect to $s$. The adjoint of $L_{sT}$ from $D(L_{sT}^\star) \subset Y$ to $L_2([s,T];U)$ is given by

$$(L_{sT}^\star y)(t) = B^\star e^{A^\star(T-t)} y. \tag{13}$$

Clearly $D(A^*) \subset D(L_{sT}^\star)$. We shall denote by $L_T$ the operator $L_T u \equiv L_{0T} u$ acting from $L_2([0,T];U)$ to $Y$. Next, consider the composition $GL_{sT}$ which is densely defined since $G$ is bounded and closable by Assumption 2.1. Thus, we can define a new Hilbert space $V([s,T];U) \equiv \overline{\mathcal{D}(GL_{sT})}$ when equipped with the inner product

$$\langle u, v\rangle_{V([s,T];U)} = \langle u, v\rangle_{L_2([s,T];U)} + \langle GL_{sT}u, GL_{sT}v\rangle_Z. \tag{14}$$

Let $[V([s,T];U)]'$ be the dual of $V([s,T];U)$ with respect to $L_2([s,T];U)$,

$$V([s,T];U) \subset L_2([s,T];U) \subset [V([s,T];U)]', \tag{15}$$

$$\|u\|_{[V([s,T];U)]'} \le \|u\|_{L_2([s,T];U)} \le \|u\|_{V([s,T];U)}. \tag{16}$$

### 3.2 Characterization of the Optimal Control

Having introduced the space $V([s,T];U)$, we alter the problem to minimizing the cost functional (1) over all $u \in V([s,T];U)$ instead of $u \in L_2([s,T];U)$. By standard optimization theory, this new problem has a unique optimal solution since $J(u,y,s,y_s)$ is continuous, and strictly convex in $u$ with respect to the $V$ norm. Since $u^0 \in V([s,T];U)$, then $GL_{sT}u^0 \in Z$ and $L_{sT}G^*GL_{st}u^0 \in [V(s,T);U)]'$. This observation allows us to consider the optimization problem on a smaller space $V(s,T);U)$ on which operator $GL_{sT}$ is bounded. Moreover, the following property follows from the definition of the norm on the space $V$ defined in (14).

**Lemma 3.1** $|GL_{sT}|_{\mathcal{L}(V([s,T];U);Z)} = |L_{sT}^\star G^\star|_{\mathcal{L}(Z;[V([s,T];U)]')} \le 1.$

The explicit characterization of optimal control below comes from standard variational methods:

**Proposition 3.1** *The unique solution $u^0 \in L_2([s,T];U)$ minimizing the cost functional $J(u,y,s,y_s)$ defined in (1) admits the following representation*:

$$-u^0(t,s;y_s) = \Lambda_{sT}^{-1}[L_{sT}^\star G^\star G e^{A(T-s)} y_0 + L_s^\star R^\star R e^{A(\cdot - s)} y_0], \tag{17a}$$

$$\Lambda_{sT} \equiv I + L_s^\star R^\star R L_s + L_{sT}^\star G^\star G L_{sT}, \tag{17b}$$

$$\|\Lambda_{sT}^{-1} u\|_{V([s,T];U)} \le K|u|_{[V([s,T];U)]'}. \tag{17c}$$

*Proof* See [17] and pp. 25, 26 of [4] for details of proof. □

## 4 Regularity of Optimal Solution

Following [3, 16] we begin by defining a Banach space suitable for capturing the singularities in optimal solutions.

**Definition 4.1** $C_\gamma([s,T];H) = \{f \in C[s,T) : \sup_{t\in[s,T]}(T-t)^\gamma |f(t)|_H < \infty\}$.

Equation (11) together with Proposition 3.1 are used to establish regularity of the optimal trajectory in relation to the optimal control. We conclude that the optimal control is continuous on $[s,T]$ when $\gamma < \frac{1}{2}$, and continuous on $[s,T)$ with a singularity of order $2\gamma - 1$ at the endpoint when $\gamma \geq \frac{1}{2}$.

**Proposition 4.1**

(a) $|u^0(.,s;y_0)|_{C_\gamma([s,T];U)} \leq K|y_0|_Y$.
(b) *If* $0 \leq \gamma \leq \frac{1}{2}$, $|y^0(.,s,y_0)|_{C([s,T];Y)} \leq \kappa|y_0|_Y$.
(c) *If* $\frac{1}{2} \leq \gamma < 1$, $|y^0(.,s,y_0)|_{C_{2\gamma-1+\epsilon}([s,T];Y)} \leq \kappa|y_0|_Y$.

*In addition*, $K$, $\kappa$ *are independent of the initial time* $s$.

The remainder of this section is devoted to the proof of Proposition 4.1.

### 4.1 Preliminary Results

Our plan in establishing the regularity results of Proposition 4.1 is to reexpress the optimal control using (17a) and (11) into

$$-u^0(.,s;y_0) = [I + L_s^\star R^\star R L_s]^{-1}[L_s^\star R^\star R e^{A(.-s)} y_0 + L_{sT}^\star G^\star G y^0(T)]. \tag{18}$$

To prove regularity of the control, we utilize (18) and show that the operator $[I + L_s^\star R^\star R L_s]^{-1}$ is boundedly invertible on the space $C_\gamma([s,T];U)$ defined above. To get the desired result, we also need to show $[L_s^\star R^\star R e^{A(.-s)} y_0 + L_{sT}^\star G^\star G y^0(T)]$ lies within this space. This result is stated in the following lemma which follows from optimization arguments in the same way as in [4].

**Lemma 4.1**

(i) $\|u^0(.,s;y_0)\|_{L_2([s,T];U)} \leq \|u^0\|_{V([s,T];U)} \leq \kappa|y_0|_Y$.
(ii) $|Gy^0(T,s;y_0)|_Z \leq \kappa|y_0|_Y$.

*Moreover*, $\kappa$ *does not depend on* $s$.

*Proof*

(i) This follows from (16) and applying (17c) in Proposition 3.1 to the expression for the control in (17a). See p. 29 of [4] for details.

(ii) From (11), $Gy(T) = Ge^{A(T-s)}y_0 + GL_{sT}u$. Then, the inequality follows from Lemma 3.1, part (i) and the boundedness of $G : Y \to Z$. □

**Lemma 4.2** *The operators $L_s$, $L_s^\star$ defined in* (3) *and* (12) *for* $0<\gamma<1$, *satisfy*:

(a) *For $r$ such that* $r+\gamma<1$, $|L_s u|_{C([s,T];Y)} \le \frac{C_{T,\gamma}}{1-\gamma-r}|u|_{C_r([s,T];U)}$.
(b) *For $r$ such that* $r+\gamma\ge 1$, $|L_s u|_{C_{r+\gamma-1+\epsilon}([s,T];Y)} \le C_{T,\gamma}|u|_{C_r([s,T];U)}$.
(c) *For* $0\le r<1$, $|L_s^\star y|_{C_{r+\gamma-1}([s,T];U)} \le \kappa |y|_{C_r([s,T];Y)}$.

*In particular, the constants $C$ and $\kappa$ are independent of the initial time $s$.*

*Proof* See [4, pp. 35–37] for proof. Note that the estimate in the first step comes from the singular estimate assumption and not analyticity of the semigroup. □

**Lemma 4.3** $|L_{sT}^\star G^\star G y(T)|_{C_\gamma([s,T];U)} \le \nu |y_0|_Y$, *where $\nu$ is independent of $s$.*

*Proof* $|L_{sT}^\star G^\star G y(T)|_{C_\gamma([s,T];U)} = |B^\star e^{A^\star(T-.)}G^\star G y(T)|_{C_\gamma([s,T];U)}$.

From the singular estimate in Assumption 1.1(c), and the boundedness of $G^\star$,

$$|L_{sT}^\star G^\star G y(T)|_{C_\gamma([s,T];U)} \le \sup_{t\in[s,T]} (T-t)^\gamma \frac{C}{(T-t)^\gamma}|G^\star G y(T)|_Y \le \nu |y_0|_Y.$$

The last inequality follows from Lemma 4.1(ii); so $\nu$ does not depend on $s$. □

**Lemma 4.4** *$I+L_s^\star R^\star R L_s$ is boundedly invertible on the space $C_\gamma([s,T];U)$, i.e.*

$$|[I+L_s^\star R^\star R L_s]^{-1}u|_{C_\gamma([s,T];U)} \le K|u|_{C_\gamma([s,T];U)},$$

*where $K$ is independent of* $s\in[0,T]$.

*Proof* The boundedness of $I+L_s^\star R^\star R L_s$ is well known in the case of analytic semigroups and also for SECS (singular estimate control systems). However, in what follows, it is essential to track dependence of the constants with respect to initial time. For this reason we present a different (than in [4, 11]) proof of this property. Uniform boundedness in $s$ of the operator $I+L_s^\star R^\star R L_s$ on $C_\gamma([s,T];U)$ is straightforward via Lemma 4.2.

Since $R$ and $R^\star$ are bounded and thus $RL_s$ has the same singular estimate property as $L_s$, It suffices to show that operator $I+L_s^\star L_s$ is an isomorphism from $C_\gamma([s,T];U)$ onto itself in order to invoke the inverse mapping theorem.

So given $f\in C_\gamma([s,T];U)$, we show there is a unique $g_s\in C_\gamma([s,T];U)$ such that $g_s+L_s^\star L_s g_s = f$. This is equivalent to showing that there exists a unique fixed point $g_s$ for the map $T_s(g) = f - L_s^\star L_s g$ so that $T(g_s)=g_s$. We claim the map $|[L_s^\star L_s]^n|$ goes to zero as $n$ goes to $\infty$. This permits applying the fixed point contraction mapping theorem on the Banach space $C_\gamma([s,T];U)$ to deduce the existence of a fixed point $g_s$.

*Claim* $\forall\epsilon>0$, there exists $N$ s.t., whenever $n\ge N$, uniformly in $s\in[0,T]$,

$$\|[L_s^\star L_s]^n\|_{\mathcal{L}(C_\gamma([s,T];U))} < \epsilon. \tag{19}$$

The proof of the claim is technical and the reader is referred to [17] for details. This establishes that $I + L_s^\star L_s$ is an isomorphism onto $C_\gamma$. By the inverse mapping theorem, $[I + L_s^\star R^\star R L_s]^{-1}$ exists and is bounded on $C_\gamma([s, T]; U)$.

To establish uniform boundedness with respect to $s$ of operator $[I + L_s^\star R^\star R L_s]^{-1}$, it suffices to show the fixed point $g_s$ of the operator $T_s g \equiv f - L_s^\star L_s g$ for given $f$ (i.e. $T_s g_s = g_s$) satisfy the same bound for all $s$. See [17] for details. □

### 4.2 Proof of Proposition 4.1

*Proof* (a) Let $0 < r < 1$. Use expression in (18) and apply Lemma 4.4, Lemma 4.3, Lemma 4.2(b). See [17] for details.

(b) Let $0 \le \gamma < \frac{1}{2}$. From (11) $|y^0|_{C([s,T];Y)} = |e^{A(.-s)} y_0 + L_s u^0|_{C([s,T];Y)}$. Next, apply Lemma 4.2(i) with $r = \gamma$ to the second term and then result in (a).

(c) Let $\frac{1}{2} \le \gamma < 1$. Again via the optimal dynamics (11):

$$|y^0|_{C_{2\gamma-1+\epsilon}([s,T];Y)} = |e^{A(.-s)} y_0 + L_s u^0|_{C_{2\gamma-1+\epsilon}([s,T];Y)}.$$

The result follows from Lemma 4.2(b) applied to the second term for $r = \gamma$ (i.e. $2\gamma > 1$) and then estimate on $u^0$ in part (a). □

## 5 Properties of the Riccati Operator

We now introduce the Riccati operator $P(t)$ and establish its regularity properties.

**Definition 5.1** Define an evolution on the trajectory of $y$ by $\Phi(t, s)x \equiv y^0(t, s; x)$, so that $y^0(s) = x \in Y$.

By Proposition 4.1, for each $x \in Y$, $\Phi(t, s)x \in C_{2\gamma-1+\epsilon}([s, T]; Y)$ with the bound uniform in $s$. Since $2\gamma - 1 + \epsilon < 1$, $\Phi(t, s)x$ is Bochner integrable on $[s, T]$ with values in $Y$. In addition, by Lemma 4.1 we have $G\Phi(T, t)x = Gy^0(T, t; x) \in Z$. This allows us to define Riccati operator by the following formula:

**Definition 5.2** Define the linear operator $P(t)$ from $Y$ to $L_\infty([0, T]; Y)$ as

$$P(t)x = \int_t^T e^{A^\star(s-t)} R^\star R \Phi(s, t)x ds + e^{A^\star(T-t)} G^\star G \Phi(T, t)x. \tag{20}$$

The following propositions provide more information about $P(t)$ and its relation to the optimization problem.

**Proposition 5.1** *For s fixed, the following properties hold true pointwise*:

(i) $u^0(t, \tau; \Phi(\tau, s)x) = u^0(t, s; x) \quad \forall 0 \le s \le \tau \le t < T,\ x \in Y.$
(ii) $\Phi(t, s) = \Phi(t, \tau)\Phi(\tau, s)x \quad \forall 0 \le s \le \tau \le t < T,\ x \in Y.$
(iii) $G\Phi(T, s)x = G\Phi(T, \tau)\Phi(\tau, s)x \quad \forall 0 \le s \le \tau < T,\ x \in Y.$

*Proof* See [17] for proof. □

**Proposition 5.2**

(a) $P(t) \in \mathcal{L}(Y, L_\infty([0,T]; U))$.
(b) $u^0(t,s;x) = -B^\star P(t) y^0(t,s;x)$.
(c) $|B^\star P(t)x|_U \le \dfrac{C}{(T-t)^\gamma}|x|_Y, \quad 0 \le t < T$.

**Proposition 5.3**

(a) $\langle P(t)x, y\rangle_Y = \int_t^T \langle R\Phi(\tau,t)x, R\Phi(\tau,t)y\rangle_W d\tau + \langle G\Phi(T,t)x,\ G\Phi(T,t)y\rangle_Z + \int_t^T \langle B^\star P(\tau)\Phi(\tau,t)x, B^\star P(\tau)\Phi(\tau,t)y\rangle_U d\tau,\ \forall t < T$.
(b) *As a consequence*, $P(t) = P^\star(t) \ge 0$.
(c) *The optimal cost of control problem on* $[t,T]$ *with initial value* $y_t \in Y$ *is*

$$J^0(u^0, y^0, t, y_0) = \int_t^T |R\Phi(\tau,t)y_0|_W^2 + |B^\star P(\tau)\Phi(\tau,t)y_0|_U^2 d\tau$$
$$+ |G\Phi(T,t)y_0|_Z^2 = \langle P(t)y_0, y_0\rangle_Y.$$

Proofs of the two propositions are very similar to case of analytic semigroup in [4], and can be found in [17].

## 6 Riccati Equation

We demonstrate that the operator $P$ indeed satisfies the Riccati equation. We first need to establish—in some weak sense—differentiability of $P(t)$ which requires establishing differentiability of the evolution $\Phi(t,s)$ with respect to both $t$ and $s$. It is differentiability with respect to the second variable that presents main technical difficulties. In order to cope with this we shall need to establish a singular estimate for the transfer function which is quantified in terms of a new scale of spaces ${}_\gamma C([s,T]; H)$ defined below.

**Definition 6.1** ${}_\gamma C([s,T]; H) = \{f \in C(s,T] : \sup_{t\in[s,T]}(t-s)^\gamma |f(t)|_H < \infty\}$, ${}_\gamma C_\gamma([s,T]; H) = \{f \in C(s,T] : \sup_{t\in[s,T]}(t-s)^\gamma (T-t)^\gamma |f(t)|_H < \infty\}$.

**Definition 6.2** Define the operator $A_p(t) = A - BB^\star P(t)$ as an unbounded operator on $Y$ with domain $\mathcal{D}(A_p(t)) = \{x : x - A^{-1}BB^\star P(t)x \in \mathcal{D}(A)\}$.

We are now ready to state the results central for deriving the Riccati equation.

**Proposition 6.1**

(a) $|\Phi(t,s)Bw|_Y \le \frac{\rho_{\gamma,T-s}}{(T-t)^\gamma (t-s)^\gamma}|w|_U$, *i.e.*

$$|\Phi(t,s)Bw|_{{}_\gamma C_\gamma([s,T];Y)} \le \rho_{\gamma,T-s}|w|_U \quad \textit{for } s < t < T \textit{ (two singularities)}.$$

(b) *For any* $x \in \mathcal{D}(A)$ *and* $s < t < T$,

$$\frac{\partial}{\partial s}\Phi(t,s)x = -\Phi(t,s)A_p(s)x = -\Phi(t,s)[A - BB^\star P(s)]x \in {}_\gamma C_\gamma([s,T];Y).$$

(c) *For any* $x \in \mathcal{D}(A)$ *and* $s < t \le T$,

$$\frac{\partial}{\partial s}G\Phi(T,s)x = -G\Phi(T,s)A_p(s)x.$$

**Theorem 6.1** *The operator* $P(t)$ *satisfies the following Riccati equation for* $0 \le t < T$ *and* $x, y \in \mathcal{D}(A)$:

$$\begin{aligned}\langle \dot{P}(t)x, y\rangle_Y &= -\langle R^\star Rx, y\rangle - \langle A^\star P(t)x - P(t)Ax, y\rangle_Y \\ &\quad + \langle B^\star P(t)x, B^\star P(t)y\rangle_U, \qquad (21a)\\ P(T)x &= G^\star Gx. \qquad (21b)\end{aligned}$$

Proofs of Proposition 6.1 and Theorem 6.1 are given in the next section.

## 6.1 Preliminary Lemmas

**Lemma 6.1**

(i) *The evolution* $\Phi(t,s)x$ *is continuous in* $t$, *for all* $s \le t < T$.
(ii) *The evolution* $\Phi(t,s)x$ *is continuous in* $s$ *for a fixed* $t$, *for all* $0 \le s \le t < T$.
(iii) *The map* $s \to G\Phi(T,s)x$ *is continuous in* $s$, *for all* $0 \le s \le t < T$.
(iv) $\frac{\partial}{\partial t}\Phi(t,s)x = A_p(t)\Phi(t,s)x$ *in the dual sense on* $[\mathcal{D}(A^\star)]'$ *for all* $s \le t < T$.

*Proof* Part (i) follows immediately from Proposition 4.1(a, b). See [17] for proofs of (ii), (iii) and (iv). □

**Lemma 6.2** $\Psi_s f(.) \equiv [B^\star P(.)\Phi(., \frac{.+s}{2}) f(\frac{.+s}{2})] \in \mathcal{L}({}_\gamma C_\gamma([s,T];Y); {}_\gamma C_\gamma([s,T];U))$.

**Lemma 6.3**

(i) $|I + L_s\Psi_s f|_{{}_\gamma C_\gamma([s,T];Y)} \le \eta_{\gamma,T-s}|f|_{{}_\gamma C_\gamma([s,T];Y)}$.
(ii) $|[I + L_s\Psi_s]^{-1} f|_{{}_\gamma C_\gamma([s,T];Y)} \le \zeta_{\gamma,T-s}|f|_{{}_\gamma C_\gamma([s,T];Y)}$.
(iii) *For* $T_0 < T$; $|[I + L_s B^\star P(.)]f|_{{}_\gamma C([s,T_0];Y)} \le \eta_{\gamma,T-s,T-T_0}|f|_{{}_\gamma C([s,T_0];Y)}$.
(iv) *For* $T > T_0$; $|[I + L_s B^\star P(.)]^{-1} f|_{{}_\gamma C([s,T_0];Y)} \le \zeta_{\gamma,T-T_0,T-s}|f|_{{}_\gamma C([s,T_0];Y)}$.

*Proof* See [17]. □

## 6.2 Proof of Proposition 6.1

*Proof* (a) Fix $x \in Y$

$$\Phi(t,s)x = e^{A(t-s)}x - \int_s^t e^{A(t-z)}BB^\star P(z)\Phi\left(z, \frac{z+s}{2}\right)\Phi\left(\frac{z+s}{2}, s\right)x\,dz$$

$$= e^{A(t-s)}x - \left[L_s B^\star P(.)\Phi\left(.,\frac{.+s}{2}\right)\Phi\left(\frac{.+s}{2},s\right)x\right](t).$$

So $\Phi(.,s)x = [I + L_s\Psi_s]^{-1}e^{A(.-s)}x$ and by Lemma 6.3(ii), $[I + L_s\Psi_s]^{-1}$ exists and is bounded on ${}_\gamma C_\gamma([s,T];Y)$. We then extend the action of the optimal evolution on elements $x$ such that $e^{A(.-s)}x \in {}_\gamma C_\gamma([s,T];Y)$. Given $w \in U$ we have $e^{A(.-s)}Bw \in {}_\gamma C_\gamma([s,T];Y)$ via the singular estimate assumption, and thus the expression is well defined: $\Phi(.,s)Bw = [I + L_s\Psi_s]^{-1}e^{A(.-s)}Bw$.

Therefore, by Lemma 6.3(iii): $|\Phi(.,s)Bw|_{{}_\gamma C_\gamma([s,T];Y)} \le \rho_{\gamma,T-s}|w|_U$. Hence, for $s < t < T$, $|\Phi(t,s)Bw|_Y \le \frac{\rho_{\gamma,T-s}}{(T-t)^\gamma(t-s)^\gamma}|w|_U$.

(b) Let $x \in \mathcal{D}(A)$ and $s < t < T$:

$$\frac{\partial}{\partial s}\Phi(t,s)x = \frac{d}{ds}\left(e^{A(t-s)}x - \int_s^t e^{A(t-z)}BB^\star P(z)\Phi(z,s)x dz\right),$$

$$\frac{\partial}{\partial s}\Phi(t,s)x = -Ae^{A(t-s)}x + e^{A(t-s)}BB^\star P(s)x - \left[L_s B^\star P(.)\frac{\partial}{\partial s}\Phi(.,s)x\right](t).$$

Note that we can apply the derivative in $s$ to the evolution by the boundedness of the operator $B^\star P(t)$ on $Y$ for a fixed $t$ from Proposition 5.2(c). Thus,

$$[I + L_s B^\star P(.)]\frac{\partial}{\partial s}\Phi(.,s)x(t) = -e^{A(t-s)}[A - BB^\star P(s)]x.$$

Let $T_0 < T$. The right-hand side is then a function in ${}_\gamma C([s,T_0],Y)$, while $[I + L_s B^\star P(.)]^{-1}$ exists and is bounded on ${}_\gamma C([s,T_0],Y)$ by Lemma 6.3(iv). So,

$$\frac{\partial}{\partial s}\Phi(.,s)x(t) = -[I + L_s B^\star P(.)]^{-1}e^{A(t-s)}[A - BB^\star P(s)]x,$$

$$\frac{\partial}{\partial s}\Phi(t,s)x = -\Phi(t,s)[A - BB^\star P(s)]x = -\Phi(t,s)A_p(s)x. \tag{22}$$

$\frac{\partial}{\partial s}\Phi(t,s) \in {}_\gamma C([s,T_0],Y)$ for $T_0$ and $x \in \mathcal{D}(A)$. However, the right-hand side of (22) is an element of ${}_\gamma C_\gamma([s,T],Y)$, so we can extend $\frac{\partial}{\partial s}\Phi(t,s)x$ to be an element of ${}_\gamma C_\gamma([s,T],Y)$ since, for $x \in \mathcal{D}(A)$,

$$\left|\frac{\partial}{\partial s}\Phi(t,s)x\right|_{{}_\gamma C_\gamma([s,T];Y)} \le |\Phi(t,s)Ax|_{{}_\gamma C_\gamma([s,T];Y)} + |\Phi(t,s)BB^\star P(s)x|_{{}_\gamma C_\gamma([s,T];Y)}.$$

We can now apply the estimate in Proposition 4.1(c) (this bound will also apply to the case when $\gamma < \frac{1}{2}$ as well), while for the second term we use the previous result that $\Phi(t,s)B$ is bounded from ${}_\gamma C_\gamma([s,T];U) \to {}_\gamma C_\gamma([s,T];Y)$,

$$\left|\frac{\partial}{\partial s}\Phi(t,s)x\right|_{{}_\gamma C_\gamma([s,T];Y)}$$
$$\le \sup_{t\in[s,T]}(T-t)^\gamma(t-s)^\gamma\frac{K}{(T-t)^{2\gamma-1+\epsilon}}|Ax|_Y + \rho_{\gamma,T-s}|B^\star P(s)x|_U.$$

Finally, we apply the bound for $B^\star P(s)$ in Proposition 5.2(c) to obtain

$$\left|\frac{\partial}{\partial s}\Phi(t,s)x\right|_{\gamma C_\gamma([s,T];Y)} \le K_{\gamma,T-s}|Ax|_Y.$$

(c) Fix $\epsilon > 0$ and let $t \in (s, T-\epsilon]$; then, $\forall x \in \mathcal{D}(A)$,

$$\frac{\partial}{\partial s}G\Phi(T,s)x = \frac{\partial}{\partial s}[G\Phi(T,t)\Phi(t,s)x] \in Y.$$

See [17] for details. □

**Lemma 6.4** *For $t < T$ and $x \in \mathcal{D}(A)$, we have*

$$\left|\int_t^T e^{A^\star(\tau-t)}R^\star R\Phi(\tau,t)A_p(t)x\,d\tau\right|_Y < \infty.$$

*Proof* See [17]. □

6.3 Riccati Equation and Proof of Theorem 6.1

*Proof* Given $t < T$ and $x, y \in \mathcal{D}(A)$,

$$\begin{aligned}\langle P(t)x, y\rangle_Y = &\left\langle \int_t^T A^{\star-1}e^{A^\star(\tau-t)}R^\star R\Phi(\tau,t)x\,d\tau, Ay\right\rangle_Y \\ &+ \langle A^{\star-1}e^{A^\star(T-t)}G^\star G\Phi(T,t)x, Ay\rangle_Y.\end{aligned}$$

We differentiate $P$ with respect to $t$ using results in Proposition 6.1(b, c),

$$\begin{aligned}\langle \dot{P}(t)x, y\rangle_Y = &-\langle R^\star Rx, y\rangle_Y - \langle P(t)x, Ay\rangle_Y \\ &-\left\langle \int_t^T A^{\star-1}e^{A^\star(\tau-t)}R^\star R\Phi(\tau,t)A_p(t)x\,d\tau, Ay\right\rangle_Y \\ &- \langle A^{\star-1}e^{A^\star(T-t)}G^\star G\Phi(T,t)A_p(t)x, Ay\rangle_Y.\end{aligned}$$

Note $\langle R^\star Rx, y\rangle_Y$ is well defined since $R$ is bounded, $\langle P(t)x, Ay\rangle_Y$ is well defined by boundedness of $P(t)$ on $Y$ for fixed $t$ by Proposition 5.2(a), $\langle \int_t^T e^{A^\star(\tau-t)}R^\star R\Phi(\tau,t) A_p(t)x\,d\tau, y\rangle_Y$ is well defined by Lemma 6.4, while since $G\Phi(T,t)A_p(t)x \in Y$ by Proposition 6.1(c), $\langle e^{A^\star(T-t)}G^\star G\Phi(T,t)A_p(t)x, y\rangle_Y = \langle G\Phi(T,t)A_p(t)x, Ge^{A(T-t)}y\rangle_Y$ is well defined. Hence, $\dot{P}(t)$ is well defined from $\mathcal{D}(A) \to [\mathcal{D}(A)]'$. Thus,

$$\begin{aligned}\langle \dot{P}(t)x, y\rangle_Y = &-\langle R^\star Rx, y\rangle_Y - \langle P(t)x, Ay\rangle_Y \\ &- \langle P(t)Ax, y\rangle_Y + \langle B^\star P(t)x, B^\star P(t)y\rangle_U,\end{aligned}$$

where the right-hand side is well defined for all $x, y \in \mathcal{D}(A)$ and $t < T$. □

**Lemma 6.5**

(i) $\lim_{t\to T}\int_t^T e^{A^\star(\tau-t)}R^\star R\Phi(\tau,t)x\,d\tau=0$.
(ii) $\lim_{t\to T}\|u(.,t,y_0)\|_{V([t,T];U)}=0$.
(iii) $\lim_{t\to T}e^{A^\star(T-t)}G^\star G\Phi(T,t)x=G^\star Gx$.

*Proof* See [17]. □

**Proposition 6.2** $\lim_{t\to T}P(t)x=G^\star Gx$.

The proof follows from the previous lemma and Definition 5.2 of $P(t)$.

**Proposition 6.3** *The differential Riccati equation* (10) *has a unique solution in the class of positive self-adjoint operators* $P(\cdot)\in\mathcal{L}(Y,C([0,T],Y))$ *such that* $B^\star P(t)$ *satisfies the singular estimate* (8) *when* $0\le\gamma<\frac{1}{2}$.

See [17] for proof.

## 7 Application to Active Noise Control

In this section, we apply our abstract results of Theorems 2.1, 2.2 developed in this paper to an established active noise control system in an acoustic chamber incorporating piezoelectric control methods (see [8, 9]). Mathematically, the system is a wave equation coupled with an elastic plate equation interacting on the boundary. This system can represent a typical example of the abstract SECS system treated in this paper, provided the system possesses a degree of analyticity sufficient to propagate the smoothing effects needed to validate the singular estimate condition. This shall depend on the intensity of structural damping on the plate. We illustrate this by considering the two extreme cases of structurally damped plates: Kelvin Voigt and "square root" damping. Let $\Omega$ be a bounded domain of dimension $n=2,3$ representing the acoustic chamber with a boundary $\Gamma=\bar{\Gamma}_0\cup\bar{\Gamma}_1$ representing a flexible wall and a solid wall respectively. The acoustic dynamics are described by the wave equation in the variable $z$ where $\rho_1 z_t$ captures the acoustic pressure, while the vibration of the flexible wall $\Gamma_0$ is described by an elastic plate equation in the displacement $w$.

$$z_{tt}=c^2\Delta z,\quad \text{in } \Omega\times(0,T), \tag{23a}$$

$$\frac{\partial}{\partial\nu}z+d_1z=0,\quad \text{on } \Gamma_1\times(0,T), \tag{23b}$$

$$\frac{\partial}{\partial\nu}z+d_2\mathcal{A}^{2r_0}z_t=w_t,\quad \text{on } \Gamma_0\times(0,T), \tag{23c}$$

$$w_{tt}+\mathcal{A}^2w+\rho\mathcal{A}^\alpha w_t+\rho_1 z_t|_{\Gamma_0}=\sum_{j=1}^{J}a_ju_j\delta'_{\xi_j},\quad \text{on } \Gamma_0\times(0,T), \tag{23d}$$

$$w=\Delta w=0,\quad \text{on } \partial\Gamma_0\times[0,T], \tag{23e}$$

$$z(0,.) = z_0, \quad z_t(0,.) = z_1, \quad \text{in } \Omega, \tag{23f}$$

$$w(0,.) = w_0, \quad w_t(0,.) = w_1, \quad \text{on } \Gamma_0. \tag{23g}$$

Let $d_1, d_2, c > 0$ and define the positive, self-adjoint operator $\mathcal{A}$, to be:

$$\mathcal{A} \equiv \Delta^2, \quad \mathcal{D}(\mathcal{A}) = \{w \in H^4(\Gamma_0),\ w = \Delta w = 0,\ \text{on } \partial\Gamma_0\}.$$

We have $\mathcal{D}(\mathcal{A}^{1/2}) \sim H^2(\Gamma_0) \cap H_0^1(\Gamma_0)$. We let $\mathcal{B} : U \to [\mathcal{D}(\mathcal{A})]'$ as

$$\mathcal{B}u = \sum_{j=1}^{J} a_j u_j \delta'_{\xi_j}, \quad [u_1, \dots, u_J] \in \mathbb{R}^J = \mathcal{U}. \tag{24}$$

If $\dim \Omega = 3$ ($\dim \Gamma_0 = 2$), then $\xi_j$ are closed regular curves on $\Gamma_0$, $a_j$ are smooth functions and each $\delta'_{\xi_j}$ denotes the normal (distributional) derivative supported at $\xi_j$. The parameter $1/2 \le \alpha \le 1$ captures the strength of the structure damping and is a measure of analyticity exhibited by the system, while parameter $0 < r_0 < 1/4$ represents the degree of boundary damping on the wave component of the system. We assume that $r_0$ and $\alpha$ vary according to the inverse relationship

$$r_0 + \frac{\alpha}{2} > 3/8. \tag{25}$$

This assumption guarantees the singular estimate behavior as we shall see.

We also define $y(t) \equiv [z(t), z_t(t), w(t), w_t(t)]$ and $y_0 = [z_0, z_1, w_0, w_1]$. Thus, we express the model in (23) as an abstract control system,

$$y_t = Ay + Bu, \quad \text{in } [\mathcal{D}(A^\star)]', \tag{26a}$$

$$y(0) = y_0 \in H \equiv H^1(\Omega) \times L_2(\Omega) \times H^2(\Gamma_0) \cap H_0^1(\Gamma_0) \times L_2(\Gamma_0). \tag{26b}$$

The operator $A$ generates a strongly continuous semigroup of contractions (see Appendix A of [15]). Define the control operator $B : U \to [\mathcal{D}(A^\star)]'$ as

$$B = [0, 0, 0, \mathcal{B}]^T. \tag{27}$$

Indeed, $A^{-1}B$ is bounded from $U$ to $\mathcal{D}(\mathcal{A}^{1/2})$; see [17, 18].

The control objective is to minimize over all $u \in \mathbb{R}^J$ the functional

$$J(u, z, w) = \int_0^T |u(t)|_U^2 dt + |\nabla z(T,.)|_{0,\Omega}^2 + |z_t(T,.)|_{0,\Omega}^2. \tag{28}$$

Operator $R$ here is 0 while the operator $G$ in the abstract framework is the projection of the state variable onto the pressure space. We refer the reader to [17, 18] for proof of the essential closability condition of $GL_T$ (2.1).

We adapt the general results in [15] to the system in (23) with our particular choice of the control operator and under the assumption (25) to obtain the following proposition stating the validity of the singular estimate condition (1.1(c)):

**Proposition 7.1** *The system* (26) *corresponding to the acoustic structure interaction model in* (23) *satisfy the following singular estimate for* $0 < t < 1$:

$$|e^{At}Bu|_Y \le C\frac{|u|_{\mathbb{R}^J}}{t^{\gamma+\epsilon}}, \tag{29}$$

$$\gamma = \begin{cases} \dfrac{3}{8\alpha}, & \dfrac{3}{4} \le \alpha, \\ \dfrac{7/8-\alpha}{1-\alpha}, & \dfrac{3}{4} > \alpha. \end{cases} \tag{30}$$

Assumption (25) means that the weaker the analyticity present in the plate equation as signified by a lower value of $\alpha$, the higher the intensity of the boundary damping required on the wave component as captured by the value of $r_0$. For example, with a value of $\alpha > 3/4$, the actual value of $r_0$ becomes an irrelevant parameter, which means no boundary damping is necessary for the system to exhibit the singular estimate behavior described in the proposition. To illustrate, we consider the following two canonical cases occurring often in applications.

### 7.1 Application to the Kelvin-Voigt Damping Case

In this case, we have the maximal damping on the "wall" (Kelvin-Voigt damping) corresponding to $\alpha = 1$ and $\mathcal{A} = \Delta^2$ in (23). In that case there is no need for any damping in the boundary conditions (i.e. $d = 0$). Applying Proposition 7.1 to this concrete model produces a value of $\gamma = \frac{3}{8} + \epsilon$ and the following result:

**Theorem 7.1** *For every initial data*; $z_0 \in H^1(\Omega), z_1 \in L_2(\Omega)$, $w_0 \in H^2(\Gamma_0) \cap H_0^1(\Gamma_0)$, $w_1 \in L_2(\Gamma_0)$, *there exists a unique control* $u^0 \in L_2([0,T];\mathbb{R}^J)$ *and the corresponding trajectory* $(z^0, z_t^0, w^0, w_t^0) \in L_2([0,T];H)$ *which is continuous on* $[0,T] \to H$ *s.t.*:

(a) $\|u^0(t)\|_{\mathbb{R}^J} \le C(T-t)^{-3/8-\epsilon}$.
(b) $|z^0(t)|^2_{H_1(\Omega)} + |z^0(t)|^2_{L_2(\Omega)} + |w^0(t)|^2_{H^2(\Gamma_0)} + |w_t^0(t)|^2_{L_2(\Gamma_0)} \le C$.

*Moreover, the results in Theorem* 2.2 *apply with* $\gamma = \frac{3}{8} + \epsilon$.

### 7.2 Application to the Square Root Damping case

The second case corresponds to the minimal case which is the so called *square root damping*—$\alpha = 1/2$. In that case a strong structural damping in the boundary conditions seem necessary as the plate equation demonstrates the minimum degree of analyticity. By Assumption (25) it is sufficient to take boundary damping with $r_0 > 1/8$. We consider a value $r_0 = 1/4$ corresponding to nothing other than the Laplace-Beltrami operator $\Delta_{\Gamma_0}$ defined variationally in a weak sense as $\langle \Delta_{\Gamma_0} z, v\rangle = -\langle \nabla_{\Gamma_0} v, \nabla_{\Gamma_0} z\rangle$: Applying Proposition 7.1 to this model results in $\gamma = \frac{3}{4} + \epsilon$ and the theorem:

**Theorem 7.2** *For every initial data in* $H$; $z_0, z_1 \in H^1(\Omega) \times L_2(\Omega)$, $w_0, w_1 \in H^2(\Gamma_0) \bigcap H_0^1(\Gamma_0) \times L_2(\Gamma_0)$, *there exists a unique control* $u^0 \in L_2([0,T]; \mathbb{R}^J)$ *and a corresponding trajectory* $(z^0, z_t^0, w^0, w_t^0) \in L_2([0,T]; H)$ *s.t.*:

(a) $\|u^0(t)\|_{\mathbb{R}^J} \leq C(T-t)^{-3/4-\epsilon}$.
(b) $|z^0(t)|_{H_1(\Omega)} + |z^0(t)|_{L_2(\Omega)} + |w^0(t)|_{H^2(\Gamma_0)} + |w_t^0(t)|_{L_2(\Gamma_0)}]^{1/2} \leq C(T-t)^{-1/2-\epsilon}$.

*Moreover, the results in Theorem* 2.2 *apply with* $\gamma = \frac{3}{4} + \epsilon$.

See [17, 18] for details.

## 8 Application to a Thermoelastic Plate Model

As before we specialize Theorems 2.1, 2.2 to the control problem involving this thermoelastic plate system subject to a quadratic cost functional with terminal time penalization. Let $\Omega$ be a bounded two-dimensional domain with a boundary $\Gamma$. Consider the thermoelasticity system

$$w_{tt} - \rho\Delta w_{tt} + \Delta^2 w + \Delta\theta = 0, \quad \text{in } \Omega \times (0,T), \tag{31a}$$

$$\theta_t - \Delta\theta - \Delta w_t = 0, \quad \text{in } \Omega \times (0,T), \tag{31b}$$

$$w(0,.) = w^0, \quad w_t(0,.) = w^1, \quad \theta(0,.) = \theta^0, \quad \text{in } \Omega, \tag{31c}$$

$$w = \Delta w = 0, \quad \frac{\partial}{\partial\nu}\theta + b\theta = u, \quad \text{on } \Gamma \times (0,T). \tag{31d}$$

This is a coupled system with temperature $\theta$ and displacement $w$. The first equation with $\rho > 0$ introduces a hyperbolic component while the second equation introduces an analytic component to the system. When $\rho = 0$ the system is analytic [4]. With the dynamics above, we wish to minimize the following functional over all $u \in L_2([0,T]; \Gamma)$:

$$J(u,w,\theta) = \int_0^T \int_\Omega [|R_w(w,w_t)|^2 + |\theta|^2] dx dt + \int_0^T \int_\Gamma |u(t,x)|^2 dx dt + \int_\Omega |\Delta w(T)|^2 + |\nabla w_t(T)|^2 dx. \tag{32}$$

With $y = [\theta, w, w_t]$, we can express this model in the format of the control problem: $y_t = Ay + Bu$ with $A$ unbounded operator generating a $c_0$ semigroup [20] on $Y : H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega) \times L_2(\Omega)$ and $B : U \equiv L_2(\Gamma) \to [\mathcal{D}(A)]'$, see [17, 19] for details. In addition, the resolvent condition 1.1(b) stating that $A^{-1}B$ is bounded from $L_2(\Gamma) \to Y$ holds. The proof of closability of $GL_T T$ can be found in [17, 19]. We next turn to the singular estimate condition of 1.1(c).

**Proposition 8.1** (*See* [14]) *The following singular estimate holds for the control system in* (31): *For every* $\epsilon > 0$, *there exists a constant* $C > 0$ *such that*

$$\|e^{At} Bu\|_Y \leq C t^{-1/4-\epsilon} \|u\|_U, \quad 0 < t \leq T. \tag{33}$$

Specializing Theorems 2.1, 2.2 to this thermoelastic plate system yields the following theorem.

**Theorem 8.1** *For every initial data in $Y$, i.e. $w_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, $w_1 \in H_0^1(\Omega)$, $\theta_0 \in L_2(\Omega)$, there exists a unique control $u^0 \in L_2([0,T] \times \Gamma)$ and the corresponding trajectory $(w^0, w_t^0) \in C([0,T]; Y_w)$, $\theta^0 \in C([0,T]; L_2(\Omega))$ and*

$$|u^0(t)|_{L_2(\Gamma)} \leq C(T-t)^{-1/4-\epsilon}. \tag{34}$$

*Moreover, all the results in Theorem 2.2 apply. See* [17, 19] *for details.*

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