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Existence of Solutions of Systems of Generalized Implicit Vector Variational Inequalities

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Abstract We consider five different types of systems of generalized vector variational inequalities and derive relationships among them. We introduce the concept of pseudomonotonicity for a family of multivalued maps and prove the existence of weak solutions of these problems under these pseudomonotonicity assumptions in the setting of Hausdorff topological vector spaces as well as real Banach spaces. We also establish the existence of a strong solution of our problems under lower semicontinuity for a family of multivalued maps involved in the formulation of the problems. By using a nonlinear scalar function, we introduce gap functions for our problems by which we can solve systems of generalized vector variational inequalities using optimization techniques.

Keywords Systems of generalized implicit vector variational inequalities · Pseudomonotonicity · Existence results for a solution · Gap functions · Lower semicontinuity · Upper hemicontinuity · $\mathcal{H}$-Hemicontinuity

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1 Introduction

In the last decade, generalized vector variational inequalities (in short, GVVI) have been used as tools to solve vector optimization problems (in short, VOP), also known as multiobjective optimization problems, both for nondifferentiable and (non)convex vector-valued functions. The weak (respectively, strong) solution of Stampacchia type GVVI provides a sufficient condition (respectively, necessary and sufficient conditions) for a solution of VOP. See for example [1–5] and references therein. In the recent past, systems of Stampacchia type GVVI with a weak solution have been used to solve a Nash equilibrium problem for nondifferentiable and (non)convex vector-valued functions. See for example [6, 7] and references therein.

By means of a gap function of a (vector) variational inequality problem (in short, (V)VIP), we convert our (V)VIP to a (vector) optimization problem. Thus the optimization technique for existence of a solution and numerical algorithms for finding approximate solutions of (V)VIP can be applied. For a more comprehensive study of gap functions, we refer to [8] and references therein. Gap functions for generalized variational inequality problems and generalized variational-like inequality problems are defined in [9, 10], respectively. Recently, Yang and Yao [11] and Li and He [12] introduced gap functions for Stampacchia type GVVI with strong solutions.

In this paper, we consider five different types of systems of generalized implicit vector variational inequalities (in short, SGIVVI). We mention that SGIVVI contain Stamppachia type GVVI as a particular case. In Sect. 2, we gather some known definitions and results which will be used in the sequel. In Sect. 3, by using the nonlinear scalarization function introduced by Chen et al. [13] and by extending the technique of Huang et al. [14], we introduce gap functions for our SGIVVI. In the last section, we define several types of generalized pseudomonotonicities for a family of multivalued maps. By using these pseudomonotonicities, we establish some relationships among our SGIVVI. We prove the existence of a general solution of our SGIVVI under lower semi-continuity of the family of multivalued maps involved in the formulation of the problem. We also establish the existence of a strong solution of our SGVVI by using a $\mathcal{H}$-hemicontinuity assumption in the setting of real Banach spaces. We also prove the existence of a weak solution under pseudomonotonicity and upper hemicontinuity assumptions.

2 Formulations

Let I be any (countable or uncountable) index set. For each $i \in I$, let K_i be a nonempty convex subset of a Hausdorff topological vector space X_i. Throughout this paper, unless otherwise specified, $K = \prod_{i\in I} K_i$ and $X = \prod_{i\in I} X_i$. For each $i \in I$, let Y_i be a topological vector space, $L(X_i, Y_i)$ the space of all continuous functions from X_i to Y_i, D_i a nonempty subset of $L(X_i, Y_i)$ and $C_i : K \to 2^{Y_i}$ a multivalued map such that for all $x \in K$, $C_i(x)$ is a proper, closed and convex cone with apex at the origin and int $C_i(x) \neq \emptyset$, where int C_i and 2^{Y_i} denote the interior of C_i and the family of all subsets of Y_i, respectively. For each $i \in I$, let $F_i : K_i \to 2^{Y_i}$ be a multivalued map with nonempty values and $\psi_i : D_i \times K_i \times K_i \to Y_i$ be a function. We consider the following *Systems of Generalized Implicit Vector Variational Inequality Problems* (in short, SGIVVIP):

Problem P1 Find $\bar{x} \in K$ such that for each $i \in I$,

$$\forall \bar{u}_i \in F_i(\bar{x}) : \psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\operatorname{int} C_i(\bar{x}), \ \forall y_i \in K_i.$$

Problem P2 Find $\bar{x} \in K$ such that for each $i \in I$,

$$\exists \bar{u}_i \in F_i(\bar{x}) : \psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\operatorname{int} C_i(\bar{x}), \ \forall y_i \in K_i.$$

Problem P3 Find $\bar{x} \in K$ such that for each $i \in I$,

$$\forall y_i \in K_i, \ \exists \bar{u}_i \in F_i(\bar{x}) \ (\bar{u}_i \text{ depends on } y_i) : \psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\operatorname{int} C_i(\bar{x}).$$

Problem P4 Find $\bar{x} \in K$ such that for each $i \in I$,

$$\forall y \in K \text{ and } \forall v_i \in F_i(y) : \psi_i(v_i, y_i, \bar{x}_i) \notin \operatorname{int} C_i(\bar{x}),$$

where y_i is the ith component of y.

Problem P5 Find $\bar{x} \in K$ such that for each $i \in I$,

$$\forall y \in K, \ \exists v_i \in F_i(y) \ (v_i \text{ depends on } y) : \psi_i(v_i, y_i, \bar{x}_i) \notin \operatorname{int} C_i(\bar{x}),$$

where y_i is the ith component of y.

Remark 2.1 Problem P1 $\Rightarrow$ Problem P2 $\Rightarrow$ Problem P3 and Problem P4 $\Rightarrow$ Problem P5.

The solutions of Problems P1, P2 and P3 are called general solution, strong solution and weak solution, respectively. In view of Remark 2.1, every general solution is a strong solution and every strong solution is a weak solution.

Problem P3 was first considered and studied by Ansari et al. [6]. They established the existence of a solution of Problem P3 without assuming any monotonicity condition. They showed that if for each $i \in I$, $\psi_i(u_i, x_i, y_i) = \langle u_i, \eta_i(y_i, x_i)\rangle$, where $\eta_i : K_i \times K_i \to X_i$ and $\langle s_i, x_i\rangle$ denotes the evaluation of $s_i \in L(X_i, Y_i)$ at $x_i \in X_i$, Problem P3 provides a sufficient condition (which is in general not necessary) for a solution of systems of vector optimization problems which includes Nash equilibrium problems for nondifferentiable and nonconvex functions.

If for each $i \in I$, $Y_i = \mathbb{R}$ and $C_i(x) = \mathbb{R}_-$ for all $x \in K$, Problem P3 was studied by Ansari and Yao [7]. As an application of their results, they established some existence results for solutions of systems of optimization problems and Nash equilibrium problems.

When I is a singleton set and $\psi_i(u_i, x_i, y_i) = \langle u_i, \eta_i(y_i, x_i)\rangle$ (respectively, $\psi_i(u_i, x_i, y_i) = \langle u_i, y_i - x_i\rangle$), then Problem P2 provides necessary and sufficient conditions for solutions of vector optimization problems for nondifferentiable and nonconvex functions (respectively, for nondifferentiable, but convex functions). See for example [1, 3]. In this case, Problem P1 is considered and studied by Ansari [15] and Lee et al. [16].

When I is a singleton set, Problems P2 and P3 are studied by Kum and Lee [17, 18]. They proved the existence of solutions of these problems under some kind of pseudomonotonicity assumptions.

So far no work has been done on the existence of solutions of the above mentioned Problems P1–P5 under any kind of monotonicity assumption. This paper is the first effort in this direction.

3 Preliminaries

We recall some known definitions and results which will be used in the sequel.

Definition 3.1 (See [19]) Let $\mathcal{X}$ and $\mathcal{Y}$ be topological spaces. A multivalued map $T : \mathcal{X} \to 2^{\mathcal{Y}}$ is called *upper semicontinuous at* $x_0 \in \mathcal{X}$ if for any open set V in $\mathcal{Y}$ containing $T(x_0)$, there exists an open neighborhood U of x_0 in $\mathcal{X}$ such that $T(x) \subseteq V$ for all $x \in U$.

T is called *lower semicontinuous at* $x \in \mathcal{X}$ if for any $y \in T(x)$ and for any $x_n \in \mathcal{X}$ such that $x_n \to x$, there exists $y_n \in T(x_n)$ such that $y_n \to y$.

It is said to be *lower semicontinuous on* $\mathcal{X}$ if it is lower semicontinuous at every point $x \in \mathcal{X}$.

Definition 3.2 (See [13]) Let $\mathcal{X}$ and $\mathcal{Y}$ be locally convex Hausdorff topological vector spaces and $C : \mathcal{X} \to 2^{\mathcal{Y}}$ be a multivalued map such that for all $x \in \mathcal{X}$, $C(x)$ is a proper, closed and convex cone with apex at the origin and $\operatorname{int} C(x) \neq \emptyset$. Let $e : \mathcal{X} \to \mathcal{X}$ be a vector-valued function such that for all $x \in \mathcal{X}$, $e(x) \in \operatorname{int} C(x)$. The *nonlinear scalarization function* $\xi_e : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is defined as follows:

$$\xi_e(x, y) = \inf\{\lambda \in \mathbb{R} : y \in \lambda e(x) - C(x)\}, \quad \forall (x, y) \in \mathcal{X} \times \mathcal{Y}.$$

The following important properties of the nonlinear scalarization function ξ_e play an important role to provide gap functions for Problems P1 and P4.

Lemma 3.1 (See [13]) *Let $\mathcal{X}$ and $\mathcal{Y}$ be locally convex Hausdorff topological vector spaces and $C : \mathcal{X} \to 2^{\mathcal{Y}}$ be a multivalued map such that for all $x \in \mathcal{X}$, $C(x)$ is a proper, closed and convex cone with apex at the origin and $\operatorname{int} C(x) \neq \emptyset$. Let $e : \mathcal{X} \to \mathcal{X}$ be a vector-valued function such that for all $x \in \mathcal{X}$, $e(x) \in \operatorname{int} C(x)$. For every $\lambda \in \mathbb{R}$ and $(x, y) \in \mathcal{X} \times \mathcal{Y}$, we have*

(i) $\xi_e(x, y) \geq \lambda \Leftrightarrow y \notin \lambda e(x) - \operatorname{int} C(x) \Leftrightarrow \lambda e(x) - y \notin \operatorname{int} C(x)$,
(ii) $\xi_e(x, y) = \lambda \Leftrightarrow y \in \lambda e(x) - \partial C(x)$,

where $\partial C(x)$ is the topological boundary of $C(x)$.

Lemma 3.2 (See [20]) *Let $(E, \|\cdot\|)$ be a normed vector space and $\mathcal{H}$ be a Hausdorff metric on the collection $\mathcal{CB}(E)$ of all nonempty, closed and bounded subsets of E,* *induced by a metric d in terms of $d(x,y)=\|x-y\|$, which is defined as*

$$\mathcal{H}(U,V)=\max\left\{\sup_{x\in U}\inf_{y\in V}\|x-y\|,\ \sup_{y\in V}\inf_{x\in U}\|x-y\|\right\},$$

for all $U,V\in\mathcal{C}B(E)$. If U and V are compact sets in E, then for all $x\in U$, there exists $y\in V$ such that

$$\|x-y\|\le\mathcal{H}(U,V).$$

Definition 3.3 (See [21]) Let Ω be a nonempty convex subset of a normed space $(E,\|\cdot\|)$ and Υ be a normed linear space. A nonempty compact-valued multifunction $T:\Omega\to 2^{L(E,\Upsilon)}$ is said to be *$\mathcal{H}$-hemicontinuous* if for any $x,y\in\Omega$, the mapping $\alpha\mapsto\mathcal{H}(T(x+\alpha(y-x)),T(x))$ is continuous at 0^+, where $\mathcal{H}$ is the Hausdorff metric defined on $\mathcal{C}B(E)$.

The following particular form of a maximal element theorem for a family of multivalued maps due to Lin and Ansari (Corollary 4.4 in [22]) is the main tool to establish the existence of solutions of Problems P1–P5.

Theorem 3.1 (See [22]) *For each $i\in I$, let K_i be a nonempty convex subset of a Hausdorff topological vector space X_i. For each $i\in I$, let $P_i,Q_i:K\to 2^{K_i}$ be multivalued maps satisfying the following conditions*:

(i) *For each $i\in I$ and for all $x\in K$, $\mathrm{co}P_i(x)\subseteq Q_i(x)$, where $\mathrm{co}P_i(x)$ denotes the convex hull of $P_i(x)$.*
(ii) *For each $i\in I$ and for all $x=(x_i)_{i\in I}\in K$, $x_i\notin Q_i(x)$, where x_i is the ith component of x.*
(iii) *For each $i\in I$ and for all $y_i\in K_i$, $P_i^{-1}(y_i)=\{x\in K: y_i\in P_i(x)\}$ is open in K.*
(iv) *There exist a nonempty compact subset M of K and a nonempty compact convex subset N_i of K_i for each $i\in I$ such that for all $x\in K\setminus M$, there exists $i\in I$ such that $P_i(x)\cap N_i\neq\emptyset$.*

Then, there exists $\bar{x}\in K$ such that $P_i(\bar{x})=\emptyset$ for all $i\in I$.

4 Gap Functions

Throughout this section, for each $i\in I$, we assume that X_i and Y_i are locally convex Hausdorff topological vector spaces and K_i is a nonempty convex subset of X_i, and C_i is the same as defined in the previous section. We set $K=\prod_{i\in I}K_i$, $X=\prod_{i\in I}X_i$, and $Y=\prod_{i\in I}Y_i$.

Let us recall the definition of a gap function. For a comprehensive study of gap functions, we refer to [8] and references therein.

Definition 4.1 A function $p:K\to\mathbb{R}$ is said to be a *gap function* for Problem P1 (respectively, Problem P4) if

(a) $p(x)\le 0$, for all $x\in K$;
(b) $p(\bar{x})=0$ if and only if $\bar{x}$ is a solution of Problem P1 (respectively, Problem P4).

For each $i \in I$, let $e_i : K \to Y_i$ be a function such that for all $x \in K$, $e_i(x) \in \operatorname{int} C_i(x)$.

Define two mappings $g_0 : K \times I \to \mathbb{R}$ and $g : K \to \mathbb{R}$ by

$$g_0(x,i) = \sup\{-\xi_{e_i}(x, \psi_i(u_i, x_i, y_i)) : y_i \in K_i, u_i \in F_i(x)\}$$

and

$$g(x) = \inf_{i \in I}\{-g_0(x,i)\}. \tag{1}$$

We also define two other mappings $h_0 : K \times I \to \mathbb{R}$ and $h : K \to \mathbb{R}$ by

$$h_0(x,i) = \sup\{-\xi_{e_i}(x, -\psi_i(v_i, y_i, x_i)) : y \in K, v_i \in F_i(y)\},$$

where y_i is the ith component of y, and

$$h(x) = \inf_{i \in I}\{-h_0(x,i)\}. \tag{2}$$

Theorem 4.1 *If for all $x \in K$, for each $i \in I$ and $\forall u_i \in F_i(x)$, $\psi_i(u_i, x_i, x_i) \in -\partial C_i(x)$, where x_i is the ith component of x, then $g(x)$ defined by (1) is a gap function for Problem P1.*

Proof (a) In view of Lemma 3.1(ii), for all $x \in K$, and for each $i \in I$ and $\forall u_i \in F_i(x)$, we obtain $\xi_{e_i}(x, \psi_i(u_i, x_i, x_i)) = 0$ which implies that

$$g_0(x,i) = \sup\{-\xi_{e_i}(x, \psi_i(u_i, x_i, y_i)) : y_i \in K_i, u_i \in F_i(x)\} \geq 0$$

and hence

$$g(x) = \inf_{i \in I}\{-g_0(x,i)\} \leq 0, \quad \forall x \in K.$$

(b) Assume that $g(\bar{x}) = 0$, then we have

$$\inf_{i \in I}\{-\sup\{-\xi_{e_i}(x, \psi_i(\bar{u}_i, \bar{x}_i, y_i)) : y_i \in K_i, \bar{u}_i \in F_i(\bar{x})\}\} = 0.$$

Therefore, for each $i \in I$,

$$\sup\{-\xi_{e_i}(x, \psi_i(\bar{u}_i, \bar{x}_i, y_i)) : y_i \in K_i, \bar{u}_i \in F_i(\bar{x})\} \leq 0$$

and thus for all $y_i \in K_i$ and all $\bar{u}_i \in F_i(\bar{x})$,

$$\xi_{e_i}(x, \psi_i(\bar{u}_i, \bar{x}_i, y_i)) \geq 0.$$

From Lemma 3.1(i), we have

$$\psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\operatorname{int} C_i(\bar{x}), \quad \forall y_i \in K_i \text{ and } \forall \bar{u}_i \in F_i(\bar{x})$$

which implies that $\bar{x}$ is a solution of Problem P1.

Conversely, let $\bar{x} \in K$ be a solution of Problem P1, then for each $i \in I$,

$$\psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\operatorname{int} C_i(\bar{x}) \quad \forall y_i \in K_i \text{ and } \forall \bar{u}_i \in F_i(\bar{x}).$$

Again from Lemma 3.1(i), we have

$$-\xi_{e_i}(\bar{x}, \psi_i(\bar{u}_i, \bar{x}_i, y_i)) \le 0, \quad \forall y_i \in K_i \text{ and } \forall \bar{u}_i \in F_i(\bar{x}).$$

Then,

$$g_0(\bar{x}, i) = \sup\{-\xi_{e_i}(\bar{x}, \psi_i(\bar{u}_i, \bar{x}_i, y_i)) : y_i \in K_i, \bar{u}_i \in F_i(\bar{x})\} \le 0$$

and so

$$g(\bar{x}) = \inf_{i \in I}\{-g_0(\bar{x}, i)\} \ge 0. \tag{3}$$

Combining (a) and (3), we obtain $g(\bar{x}) = 0$. This completes the proof. □

Corollary 4.1 *If for all $x \in K$, for each $i \in I$ and $\forall u_i \in F_i(x)$, $\psi_i(u_i, x_i, x_i) \in -\partial C_i(x)$, where x_i is the ith component of x, then the set $\{\bar{x} : g(\bar{x}) = 0\}$ is equivalent to the set of solutions of Problem* P1.

Theorem 4.2 *If for all $x \in K$, for each $i \in I$ and $\forall u_i \in F_i(x)$, $-\psi_i(u_i, x_i, x_i) \in -\partial C_i(x)$, where x_i is the ith component of x, then $h(x)$ defined by* (2) *is a gap function for Problem* P4.

Proof It follows the lines of the proof of Theorem 4.1. Therefore we omit it. □

Corollary 4.2 *If for all $x \in K$, for each $i \in I$ and $\forall u_i \in F_i(x)$, $-\psi_i(u_i, x_i, x_i) \in -\partial C_i(x)$, where x_i is the ith component of x, then the set $\{\bar{x} : h(\bar{x}) = 0\}$ is equivalent to the set of solutions of Problem* P4.

5 Existence Results

Throughout this section, unless otherwise specified, for each $i \in I$, we assume that $C_i : K \to 2^{Y_i}$ be a multivalued map such that for all $x \in K$, $C_i(x)$ is a proper closed convex cone with apex at origin and $\operatorname{int} C_i(x) \neq \emptyset$.

Definition 5.1 Let $\{\psi_i\}_{i \in I}$ be a family of mappings $\psi_i : D_i \times K_i \times K_i \to Y_i$. A family $\{F_i\}_{i \in I}$ of multivalued maps $F_i : K \to 2^{K_i}$ with nonempty values is called:

(i) *generalized strongly pseudomonotone w.r.t.* $\{\psi_i\}_{i \in I}$ if, for all $x, y \in K$ and for each $i \in I$,

$$\begin{aligned}&\forall u_i \in F_i(x) : \psi_i(u_i, x_i, y_i) \notin -\operatorname{int} C_i(x)\\ &\quad \Rightarrow \quad \forall v_i \in F_i(y) : \psi_i(v_i, y_i, x_i) \notin \operatorname{int} C_i(x);\end{aligned}$$

(ii) *generalized pseudomonotone w.r.t.* $\{\psi_i\}_{i\in I}$ if, for all $x, y \in K$ and for each $i \in I$,

$$\exists u_i \in F_i(x) : \psi_i(u_i, x_i, y_i) \notin -\operatorname{int} C_i(x)$$
$$\Rightarrow \quad \forall v_i \in F_i(y) : \psi_i(v_i, y_i, x_i) \notin \operatorname{int} C_i(x);$$

(iii) *generalized weakly pseudomonotone w.r.t.* $\{\psi_i\}_{i\in I}$ if, for all $x, y \in K$ and for each $i \in I$,

$$\exists u_i \in F_i(x) : \psi_i(u_i, x_i, y_i) \notin -\operatorname{int} C_i(x)$$
$$\Rightarrow \quad \exists v_i \in F_i(y) : \psi_i(v_i, y_i, x_i) \notin \operatorname{int} C_i(x);$$

(iv) *generalized pseudomonotone*$^+$ *w.r.t.* $\{\psi_i\}_{i\in I}$ if, for all $x, y \in K$ and for each $i \in I$,

$$\forall u_i \in F_i(x) : \psi_i(u_i, x_i, y_i) \notin -\operatorname{int} C_i(x)$$
$$\Rightarrow \quad \exists v_i \in F_i(y) : \psi_i(v_i, y_i, x_i) \notin \operatorname{int} C_i(x);$$

(v) *generalized hemicontinuous w.r.t.* $\{\psi_i\}_{i\in I}$ if, for all $x, y \in K$ and $\alpha \in [0, 1]$ and for each $i \in I$, the multivalued map

$$\alpha \mapsto \psi_i(F_i(x + \alpha(y - x)), x_i, y_i)$$

is upper semicontinuous at 0^+, where

$$\psi_i(F_i(x + \alpha(y - x)), x_i, y_i) = \{\psi_i(w_i, x_i, y_i) : w_i \in F_i(x + \alpha(y - x))\}.$$

Remark 5.1 Definition 5.1(i) $\Rightarrow$ Definition 5.1(ii) $\Rightarrow$ Definition 5.1(iii); Definition 5.1(iv) $\Rightarrow$ Definition 5.1(iii); Definition 5.1(i) $\Rightarrow$ Definition 5.1(iv); that is, Definition 5.1(i) $\Rightarrow$ Definition 5.1(iv) $\Rightarrow$ Definition 5.1(iii).

Lemma 5.1

(a) *Problem* P3 $\Rightarrow$ *Problem* P4 *if* $\{F_i\}_{i\in I}$ *is generalized pseudomonotone w.r.t.* $\{\psi_i\}_{i\in I}$.
(b) *Problem* P3 $\Rightarrow$ *Problem* P5 *if* $\{F_i\}_{i\in I}$ *is generalized weakly pseudomonotone w.r.t.* $\{\psi_i\}_{i\in I}$.
(c) *Problem* P1 $\Rightarrow$ *Problem* P5 *if* $\{F_i\}_{i\in I}$ *is generalized pseudomonotone*$^+$ *w.r.t.* $\{\psi_i\}_{i\in I}$.
(d) *Problem* P1 $\Rightarrow$ *Problem* P4 *if* $\{F_i\}_{i\in I}$ *is generalized strongly pseudomonotone w.r.t.* $\{\psi_i\}_{i\in I}$.

Lemma 5.2 *For each* $i \in I$, *assume that the following conditions hold*:

(i) *For all* $x \in K$ *and all* $u_i \in F_i(x)$, $\psi_i(u_i, x_i, x_i) \in \mathcal{C}_i = \bigcap_{x\in K} C_i(x)$;
(ii) *For all* $x \in K$ *and all* $u_i \in F_i(x)$, $\psi_i(u_i, x_i, \cdot)$ *is* $\mathcal{C}_i$*-convex, that is, for all* $s_i \in L(X_i, Y_i)$, $x, y \in X$ *and* $\alpha \in [0, 1]$,

$$\psi_i(s_i, x_i, \alpha x_i + (1 - \alpha)y_i) \in \alpha\psi_i(s_i, x_i, x_i) + (1 - \alpha)\psi_i(s_i, x_i, y_i) - \mathcal{C}_i;$$

(iii) *For all* $s_i \in L(X_i, Y_i)$, $x, y, z \in X$ *and* $\alpha \in [0, 1]$,

$$\psi_i(s_i, x_i + \alpha(y_i - x_i), z_i) = (1-\alpha)\psi_i(s_i, x_i, z_i);$$

(iv) $\{F_i\}_{i\in I}$ *is generalized hemicontinuous w.r.t.* $\{\psi_i\}_{i\in I}$.

Then, Problem P5 $\Rightarrow$ *Problem* P3 *as well as Problem* P4 $\Rightarrow$ *Problem* P3.

Proof We first prove that Problem P5 $\Rightarrow$ Problem P3.

Let $\bar{x} \in K$ be a solution of Problem P5. Suppose to the contrary that $\bar{x}$ is not a solution of Problem P3. Then there exist an $i \in I$ and $\hat{y}_i \in K_i$ such that for all $\bar{u}_i \in F_i(\bar{x})$, we have

$$\psi_i(\bar{u}_i, \bar{x}_i, \hat{y}_i) \in -\operatorname{int} C_i(\bar{x}). \tag{4}$$

Let $x_i^\alpha = \bar{x}_i + \alpha(\hat{y}_i - \bar{x}_i)$ for $\alpha \in [0, 1]$. Since each K_i is convex, we have $x_i^\alpha \in K_i$ and so we can let $x^\alpha = (\bar{x}_1, \dots, x_i^\alpha, \dots) \in K$ such that its ith component is x_i^α and the rest of the components are $\bar{x}_j$ for all $j \in I,\ j \neq i$.

Define a multivalued map $H_i : [0, 1] \to 2^{Y_i}$ by

$$H_i(\alpha) = \{\psi_i(u_i^\alpha, \bar{x}_i, \hat{y}_i) : u_i^\alpha \in F_i(x^\alpha)\}.$$

Then from (4)

$$H_i(0) = \psi_i(F_i(\bar{x}), \bar{x}_i, \hat{y}_i) \subseteq -\operatorname{int} C_i(\bar{x}).$$

Since $\{F_i\}_{i\in I}$ is generalized hemicontinuous w.r.t. $\{\psi_i\}_{i\in I}$, there exists $\delta \in (0, 1]$ such that for all $\alpha \in (0, \delta)$,

$$H_i(\alpha) \subseteq -\operatorname{int} C_i(\bar{x}).$$

Therefore, for all $\alpha \in (0, \delta)$ and all $u_i^\alpha \in F_i(x^\alpha)$, we have

$$\psi_i(u_i^\alpha, \bar{x}_i, \hat{y}_i) \in -\operatorname{int} C_i(\bar{x}). \tag{5}$$

Fix $\alpha \in (0, \delta)$. Then by conditions (i)–(iii), we have for all $u_i^\alpha \in F_i(x^\alpha)$

$$\psi_i(u_i^\alpha, x_i^\alpha, x_i^\alpha) \in \alpha\psi_i(u_i^\alpha, x_i^\alpha, \hat{y}_i) + (1-\alpha)\psi_i(u_i^\alpha, x_i^\alpha, \bar{x}_i) - \mathcal{C}_i,$$

$$\begin{aligned} -(1-\alpha)\psi_i(u_i^\alpha, x_i^\alpha, \bar{x}_i) &\in \alpha\psi_i(u_i^\alpha, x_i^\alpha, \hat{y}_i) - \psi_i(u_i^\alpha, x_i^\alpha, x_i^\alpha) - \mathcal{C}_i \\ &\subseteq \alpha(1-\alpha)\psi_i(u_i^\alpha, \bar{x}_i, \hat{y}_i) - \mathcal{C}_i - \mathcal{C}_i \\ &\subseteq -\operatorname{int} C_i(\bar{x}) - C_i(\bar{x}) - C_i(\bar{x}) \subseteq -\operatorname{int} C_i(\bar{x}). \end{aligned}$$

Thus $\psi_i(u_i^\alpha, x_i^\alpha, \bar{x}_i) \in \operatorname{int} C_i(\bar{x})$ for all $u_i^\alpha \in F_i(x^\alpha)$ which contradicts our supposition that $\bar{x}$ is a solution of Problem P5. This completes the proof.

The proof of the second part lies on the lines of the proof of the first part. Therefore we omit it. □

Remark 5.2 If I is a singleton set, then Lemma 5.2 reduces to Lemma 2.3 in [18].

Proposition 5.1 *Under the conditions of Lemmas* 5.1(a) *and* 5.2, *Problems* P3, P4 *and* P5 *are equivalent.*

Lemma 5.3 *For each $i \in I$, let $(X_i, \|\cdot\|)$ and Y_i be real Banach spaces and K_i be a nonempty convex subset of X_i. For each $i \in I$, assume that the following conditions hold*:

(i) *For all $x \in K$ and all $u_i \in F_i(x)$, $\psi_i(u_i, x_i, x_i) \in \mathcal{C}_i = \bigcap_{x\in K} C_i(x)$.*
(ii) *For all $x \in K$ and all $u_i \in F_i(x)$, $\psi_i(u_i, x_i, \cdot)$ is $\mathcal{C}_i$-convex, that is, for all $s_i \in L(X_i, Y_i)$, $x, y \in X$ and $\alpha \in [0, 1]$,*

$$\psi_i(s_i, x_i, \alpha x_i + (1-\alpha)y_i) \in \alpha\psi_i(s_i, x_i, x_i) + (1-\alpha)\psi_i(s_i, x_i, y_i) - \mathcal{C}_i.$$

(iii) *For all $s_i \in L(X_i, Y_i)$, $x, y, z \in X$ and $\alpha \in [0, 1]$,*

$$\psi_i(s_i, x_i + \alpha(y_i - x_i), z_i) = (1-\alpha)\psi_i(s_i, x_i, z_i).$$

(iv) *ψ_i is continuous in the first argument.*
(v) *F_i is $\mathcal{H}$-hemicontinuous and for all $x \in K$, $F_i(x)$ is a nonempty compact set in Y_i.*
(vi) *The family $\{F_i\}_{i\in I}$ is generalized pseudomonotone w.r.t. $\{\psi_i\}_{i\in I}$.*

Then, Problem P2 *and Problem* P4 *are equivalent.*

Proof Problem P2 $\Rightarrow$ Problem P4 follows from condition (vi).

Problem P4 $\Rightarrow$ Problem P2: Let $\bar{x} \in K$ be a solution of Problem P4. Then for each $i \in I$,

$$\forall y \in K \text{ and } \forall v_i \in F_i(y) : \psi_i(v_i, y_i, \bar{x}_i) \notin \operatorname{int} C_i(\bar{x}), \tag{6}$$

where y_i is the ith component of y. For any given $y \in K$, we know that $y^\alpha = \alpha y + (1-\alpha)\bar{x} \in K$ for all $\alpha \in (0, 1)$ since each K_i is convex and so K. Then from (6), we have

$$\forall i \in I,\ \forall v_i^\alpha \in F_i(y^\alpha) : \psi_i(v_i^\alpha, y_i^\alpha, \bar{x}_i) \notin \operatorname{int} C_i(\bar{x}),$$

where y_i^α is the ith component of y^α. Then we have

$$\forall i \in I,\ \forall v_i^\alpha \in F_i(y^\alpha) : \psi_i(v_i^\alpha, \bar{x}_i, y_i) \notin -\operatorname{int} C_i(\bar{x}). \tag{7}$$

Suppose that (7) does not hold. Then there exist an $i \in I$ and $v_i^\alpha \in F_i(y^\alpha)$ such that

$$\psi_i(v_i^\alpha, \bar{x}_i, y_i) \in -\operatorname{int} C_i(\bar{x}). \tag{8}$$

Fix this i and $v_i^\alpha \in F_i(y^\alpha)$. Then by conditions (i)–(iii), we have

$$\psi_i(v_i^\alpha, y_i^\alpha, y_i^\alpha) \in \alpha\psi_i(v_i^\alpha, y_i^\alpha, y_i) + (1-\alpha)\psi_i(v_i^\alpha, y_i^\alpha, \bar{x}_i) - \mathcal{C}_i,$$

$$\begin{aligned}-(1-\alpha)\psi_i(v_i^\alpha, y_i^\alpha, \bar{x}_i) &\in \alpha\psi_i(v_i^\alpha, y_i^\alpha, y_i) - \psi_i(v_i^\alpha, y_i^\alpha, y_i^\alpha) - \mathcal{C}_i \\ &\subseteq \alpha(1-\alpha)\psi_i(v_i^\alpha, \bar{x}_i, y_i) - \mathcal{C}_i - \mathcal{C}_i \\ &\subseteq -\operatorname{int} C_i(\bar{x}) - C_i(\bar{x}) - C_i(\bar{x}) \subseteq -\operatorname{int} C_i(\bar{x}).\end{aligned}$$

Therefore, $\psi_i(v_i^\alpha, y_i^\alpha, \bar{x}_i) \in \operatorname{int} C_i(\bar{x})$ which contradicts to (6), and hence (7) holds.

Since $F_i(y^\alpha)$ and $F_i(\bar{x})$ are compact. From Lemma 3.2 we have that for each fixed $v_i^\alpha \in F_i(y^\alpha)$, there exists $u_i^\alpha \in F_i(\bar{x})$ such that

$$\|v_i^\alpha - u_i^\alpha\| \le \mathcal{H}(F_i(y^\alpha), F_i(\bar{x})).$$

Since each $F_i(\bar{x})$ is compact, without loss of generality, we may assume that $u_i^\alpha \to \bar{u}_i \in F_i(\bar{x})$ as $\alpha \to 0^+$. Since for each $i \in I$, F_i is $\mathcal{H}$-hemicontinuous,

$$\mathcal{H}(F_i(y^\alpha), F_i(\bar{x})) \to 0 \quad \text{as } \alpha \to 0^+.$$

Therefore,

$$\begin{aligned}\|v_i^\alpha - \bar{u}_i\| &\le \|v_i^\alpha - u_i^\alpha\| + \|u_i^\alpha - \bar{u}_i\| \\ &\le \mathcal{H}(F_i(y^\alpha), F_i(\bar{x})) + \|u_i^\alpha - \bar{u}_i\| \to 0 \quad \text{as } \alpha \to 0^+.\end{aligned}$$

Since for each $i \in I$, ψ_i is continuous in the first argument and $W_i(\bar{x})$ is closed, we have

$$\psi_i(\bar{u}_i, \bar{x}_i, y_i) \in Y_i \setminus \{-\operatorname{int} C_i(\bar{x})\} = W_i(\bar{x}) \quad \Leftrightarrow \quad \psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\operatorname{int} C_i(\bar{x}),$$

that is, $\bar{x}$ is a solution of Problem P1. □

Definition 5.2 For each $i \in I$, let $F_i : K \to 2^{D_i}$ be a multivalued map. A family $\{\psi_i\}_{i\in I}$ of functions $\psi_i : D_i \times K_i \times K_i \to Y_i$ is called:

(i) *$C_i(x)$-quasiconvex-like w.r.t. $\{F_i\}_{i\in I}$* if, for all $x \in K$, $y_i', y_i'' \in K_i$ and $\alpha \in [0, 1]$, we either have $\forall u_i \in F_i(x)$,

$$\psi_i(u_i, x_i, \alpha y_i' + (1-\alpha)y_i'') \in \psi_i(u_i, x_i, y_i') - \operatorname{int} C_i(x),$$

or

$$\psi_i(u_i, x_i, \alpha y_i' + (1-\alpha)y_i'') \in \psi_i(u_i, x_i, y_i'') - \operatorname{int} C_i(x);$$

(ii) *simultaneously $C_i(x)$-quasiconvex-like w.r.t. $\{F_i\}_{i\in I}$* if for all $x \in K$, $y_i', y_i'' \in K_i$ and $\alpha \in [0, 1]$, we either have $\forall u_i', u_i'' \in F_i(x)$,

$$\psi_i(\alpha u_i' + (1-\alpha)u_i'', x_i, \alpha y_i' + (1-\alpha)y_i'') \in \psi_i(u_i', x_i, y_i') - \operatorname{int} C_i(x),$$

or

$$\psi_i(\alpha u_i' + (1-\alpha)u_i'', x_i, \alpha y_i' + (1-\alpha)y_i'') \in \psi_i(u_i'', x_i, y_i'') - \operatorname{int} C_i(x).$$

Theorem 5.1 *For each $i \in I$, let K_i be a nonempty convex subset of a Hausdorff topological vector space X_i and the graph of the multivalued map $W_i : K \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus \{-\operatorname{int} C_i(x)\}$ for all $x \in K$, be closed. For each $i \in I$, let $F_i : K \to 2^{K_i}$ be a lower semicontinuous multivalued map with nonempty convex values and $\psi_i : D_i \times K_i \times K_i \to Y_i$ be a function such that the following conditions are satisfied*:

(i) *For all $x \in K$, the family $\{\psi_i\}_{i\in I}$ of functions ψ_i is simultaneously $C_i(x)$-quasiconvex-like w.r.t. $\{F_i\}_{i\in I}$.*

(ii) *For all* $x \in K$ *and for all* $u_i \in F_i(x)$, $\psi_i(u_i, x_i, x_i) \notin -\operatorname{int} C_i(x)$.
(iii) *For each fixed* y_i, *the map* $(u_i, x_i) \mapsto \psi_i(u_i, x_i, y_i)$ *is continuous on* $D_i \times K_i$.
(iv) *There exist a nonempty compact subset* M *of* K *and a nonempty compact convex subset* N_i *of* K_i *for each* $i \in I$ *such that for all* $x \in K \setminus M$, *there exist* $i \in I$ *and* $\tilde{y}_i \in N_i$ *such that* $\psi_i(u_i, x_i, \tilde{y}_i) \in -\operatorname{int} C_i(x)$ *for all* $u_i \in F_i(x)$.

Then, *Problem* P1 *has a solution*.

Proof For all $x \in K$ and for each $i \in I$, define a multivalued map $P_i : K \to 2^{K_i}$ by

$$P_i(x) = \{y_i \in K_i : \exists u_i \in F_i(x) \text{ such that } \psi_i(u_i, x_i, y_i) \in -\operatorname{int} C_i(x)\}.$$

Since for all $x \in K$ and for all $u_i \in F_i(x)$, $\psi_i(u_i, x_i, x_i) \notin -\operatorname{int} C_i(x)$, we have $x_i \notin P_i(x)$.

For all $x \in K$, $P_i(x)$ is convex. Indeed, let $y_i', y_i'' \in P_i(x)$. Then

$$\exists u_i' \in F_i(x) \quad \text{such that } \psi_i(u_i', x_i, y_i') \in -\operatorname{int} C_i(x) \tag{9}$$

and

$$\exists u_i'' \in F_i(x) \quad \text{such that } \psi_i(u_i'', x_i, y_i'') \in -\operatorname{int} C_i(x). \tag{10}$$

Since $F_i(x)$ is convex, we have $\hat{u}_i = \alpha u_i' + (1-\alpha)u_i'' \in F_i(x)$ for all $\alpha \in [0, 1]$. For all $\alpha \in [0, 1]$, from condition (i) and (9,10) we either have

$$\psi_i(\alpha u_i' + (1-\alpha)u_i'', x_i, \alpha y_i' + (1-\alpha)y_i'') \in \psi_i(u_i', x_i, y_i') - \operatorname{int} C_i(x) \subseteq -\operatorname{int} C_i(x)$$

or

$$\psi_i(\alpha u_i' + (1-\alpha)u_i'', x_i, \alpha y_i' + (1-\alpha)y_i'') \in \psi_i(u_i'', x_i, y_i'') - \operatorname{int} C_i(x) \subseteq -\operatorname{int} C_i(x).$$

In either case, we have

$$\exists \hat{u}_i = \alpha u_i' + (1-\alpha)u_i'' \in F_i(x) : \psi_i(\hat{u}_i, x_i, \alpha y_i' + (1-\alpha)y_i'') \in -\operatorname{int} C_i(x).$$

That is, $\alpha y_i' + (1-\alpha)y_i'' \in P_i(x)$, and so $P_i(x)$ is convex.

The complement of $P_i^{-1}(y_i)$ in K,

$$[P_i^{-1}(y_i)]^c = \{x \in K : \forall u_i \in F_i(x) \text{ such that } \psi_i(u_i, x_i, y_i) \notin -\operatorname{int} C_i(x)\}$$

is closed in K.

Indeed, let $\{x^n\}$ be a net in $[P_i^{-1}(y_i)]^c$ such that $x^n \to x^* \in K$ (componentwise). Then for each $i \in I$, and $\forall u_i^n \in F_i(x^n)$ we have $\psi_i(u_i^n, x_i^n, y_i) \notin -\operatorname{int} C_i(x^n)$, that is,

$$\psi_i(u_i^n, x_i^n, y_i) \in W_i(x^n) = Y_i \setminus \{-\operatorname{int} C_i(x^n)\}. \tag{11}$$

By lower semicontinuity of F_i, for any $u_i^* \in F_i(x^*)$, there exists $\tilde{u}_i^n \in F_i(x^n)$ such that $\{\tilde{u}_i^n\}$ converges to u_i^*. Since (11) is true for all $u_i^n \in F_i(x^n)$, therefore, (11) also holds for $\tilde{u}_i^n \in F_i(x^n)$, that is,

$$\psi_i(\tilde{u}_i^n, x_i^n, y_i) \in W_i(x^n).$$

Since $\tilde{u}_i^n \to u_i^*$, $x_i^n \to x^*$ and $\psi_i(\cdot,\cdot,y_i)$ is continuous on $D_i \times K_i$, we have

$$\psi_i(\tilde{u}_i^n, x_i^n, y_i) \to \psi_i(u_i^*, x_i^*, y_i).$$

Since the graph of W_i is closed, we have

$$\psi_i(u_i^*, x_i^*, y_i) \in W_i(x^*) \quad \Rightarrow \quad \psi_i(u_i^*, x_i^*, y_i) \notin -\operatorname{int} C_i(x^*),$$

that is,

$$\forall u_i^* \in F_i(x^*), \quad \psi_i(u_i^*, x_i^*, y_i) \notin -\operatorname{int} C_i(x^*).$$

Hence $x^* \in [P_i^{-1}(y_i)]^c$ and thus $[P_i^{-1}(y_i)]^c$ is closed in K. Therefore, $P_i^{-1}(y_i)$ is open in K.

Then all the conditions of Theorem 3.1 with $P_i \equiv Q_i$ are satisfied and hence there exists $\bar{x} \in K$ such that $P_i(\bar{x}) = \emptyset$ for each $i \in I$, that is,

$$\forall \bar{u}_i \in F_i(\bar{x}) \text{ satisfying } \psi_i(\bar{u}_i, \bar{x}_i, y_i) \notin -\operatorname{int} C_i(x), \quad \forall y_i \in K_i,$$

and so $\bar{x} \in K$ is a solution of Problem P1. □

Theorem 5.2 *For each $i \in I$, let K_i be a nonempty convex subset of a Hausdorff topological vector space X_i and the graph of the multivalued map $W_i : K \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus \{\operatorname{int} C_i(x)\}$ for all $x \in K$, be closed. For each $i \in I$, let $F_i : K \to 2^{K_i}$ be a multivalued map with nonempty values and $\psi_i : D_i \times K_i \times K_i \to Y_i$ be a function such that the following conditions are satisfied*:

- (i) *The family $\{F_i\}_{i\in I}$ of multivalued maps F_i is generalized pseudomonotone w.r.t. $\{\psi_i\}_{i\in I}$.*
- (ii) *For all $x \in K$, the family $\{\psi_i\}_{i\in I}$ of functions ψ_i is $C_i(x)$-quasiconvex-like w.r.t. $\{F_i\}_{i\in I}$.*
- (iii) *For all $x \in K$ and for all $u_i \in F_i(x)$, $\psi_i(u_i, x_i, x_i) \notin -\operatorname{int} C_i(x)$.*
- (iv) *For each fixed $(v_i, y_i) \in D_i \times K_i$, the map $x_i \mapsto \psi_i(v_i, y_i, x_i)$ is continuous on K_i.*
- (v) *There exist a nonempty compact subset M of K and a nonempty compact convex subset N_i of K_i for each $i \in I$ such that for all $x \in K \setminus M$, there exist $i \in I$ and $\tilde{y}_i \in N_i$ such that $\psi_i(u_i, x_i, \tilde{y}_i) \in -\operatorname{int} C_i(x)$ for all $u_i \in F_i(x)$.*

Then, *Problem* P4 *has a solution*.

Proof For all $x \in K$ and for each $i \in I$, define two multivalued maps $P_i, Q_i : K \to 2^{K_i}$ by

$$P_i(x) = \{y_i \in K_i : \exists v_i \in F_i(y) \text{ such that } \psi_i(v_i, y_i, x_i) \in \operatorname{int} C_i(x)\}$$

and

$$Q_i(x) = \{y_i \in K_i : \forall u_i \in F_i(x) \text{ such that } \psi_i(u_i, x_i, y_i) \in -\operatorname{int} C_i(x)\}.$$

From condition (iii), we have $\psi_i(u_i, x_i, x_i) \notin -\operatorname{int} C_i(x)$ and so $x_i \notin Q_i(x)$.

For all $x \in K$ and for each $i \in I$, $P_i(x) \subseteq Q_i(x)$ by generalized pseudomonotonicity of $\{F_i\}_{i\in I}$ w.r.t. $\{\psi_i\}_{i\in I}$.

If for all $x \in K$ and for each $i \in I$, $Q_i(x)$ is convex then $\mathrm{co}P_i(x) \subseteq \mathrm{co}Q_i(x) = Q_i(x)$. Indeed, let $y_i', y_i'' \in Q_i(x)$, then $\forall u_i \in F_i(x)$, we have

$$\psi_i(u_i, x_i, y_i') \in -\operatorname{int} C_i(x) \quad \text{and} \quad \psi_i(u_i, x_i, y_i'') \in -\operatorname{int} C_i(x). \tag{12}$$

Since $\{\psi_i\}_{i\in I}$ is $C_i(x)$-quasiconvex-like and from (12), for all $\alpha \in [0, 1]$ and $\forall u_i \in F_i(x)$, we either have

$$\psi_i(u_i, x_i, \alpha y_i' + (1-\alpha) y_i'') \in \psi_i(u_i, x_i, y_i') - \operatorname{int} C_i(x) \subseteq -\operatorname{int} C_i(x)$$

or

$$\psi_i(u_i, x_i, \alpha y_i' + (1-\alpha) y_i'') \in \psi_i(u_i, x_i, y_i'') - \operatorname{int} C_i(x) \subseteq -\operatorname{int} C_i(x).$$

In either case, we have $\alpha y_i' + (1-\alpha) y_i'' \in Q_i(x)$ for all $\alpha \in [0, 1]$ and so $Q_i(x)$ is convex.

The complement of $P_i^{-1}(y_i)$ in K,

$$[P_i^{-1}(y_i)]^c = \{x \in K : \forall v_i \in F_i(y) \text{ such that } \psi_i(v_i, y_i, x_i) \notin \operatorname{int} C_i(x)\}$$

is closed in K.

Indeed, let $\{x^n\}$ be a net in $[P_i^{-1}(y_i)]^c$ such that $x^n \to x^* \in K$ (componentwise). Then for each $i \in I$ and $\forall v_i \in F_i(y)$, we have $\psi_i(v_i, y_i, x_i^n) \notin \operatorname{int} C_i(x^n)$, that is,

$$\psi_i(v_i, y_i, x_i^n) \in W_i(x^n) = Y_i \setminus \{\operatorname{int} C_i(x^n)\}. \tag{13}$$

Since $\psi_i(v_i, y_i, \cdot)$ is continuous on K_i and the graph of W_i is closed, we have,

$$\psi_i(v_i, y_i, x_i^n) \to \psi_i(v_i, y_i, x_i^*) \in W_i(x^*) \quad \Rightarrow \quad \psi_i(v_i, y_i, x_i^*) \notin \operatorname{int} C_i(x^*).$$

That is, $x^* \in [P_i^{-1}(y_i)]^c$ and thus $[P_i^{-1}(y_i)]^c$ is closed in K. Therefore, $P_i^{-1}(y_i)$ is open in K.

Then all the conditions of Theorem 3.1 are satisfied and hence there exists $\bar{x} \in K$ such that $P_i(\bar{x}) = \emptyset$ for each $i \in I$, that is,

$$\forall v_i \in F_i(y) \text{ satisfying } \psi_i(v_i, y_i, \bar{x}_i) \notin \operatorname{int} C_i(x), \quad \forall y_i \in K_i,$$

and so $\bar{x} \in K$ is a solution of Problem P4. □

Theorem 5.3 *For each $i \in I$, let K_i be a nonempty convex subset of a Hausdorff topological vector space X_i and the graph of the multivalued map $W_i : K \to 2^{Y_i}$ defined by $W_i(x) = Y_i \setminus \{\operatorname{int} C_i(x)\}$ for all $x \in K$, be closed. For each $i \in I$, let $F_i : K \to 2^{K_i}$ be a multivalued map with nonempty values and $\psi_i : D_i \times K_i \times K_i \to Y_i$ be a function such that the following conditions are satisfied*:

(i) *The family $\{F_i\}_{i\in I}$ of multivalued maps F_i is upper hemicontinuous and generalized pseudomonotone w.r.t. $\{\psi_i\}_{i\in I}$.*

(ii) *The family* $\{\psi_i\}_{i\in I}$ *of functions* ψ_i *is* $\mathcal{C}_i$*-convex in the third argument.*

(iii) *For all* $s_i \in L(X_i, Y_i)$, $x, y, z \in X$ *and* $\alpha \in [0, 1]$,

$$\psi_i(s_i, x_i + \alpha(y_i - x_i), z_i) = (1-\alpha)\psi_i(s_i, x_i, z_i).$$

(iv) *For all* $x \in K$ *and for all* $u_i \in F_i(x)$, $\psi_i(u_i, x_i, x_i) \in \mathcal{C}_i$.

(v) *For each fixed* $(v_i, y_i) \in D_i \times K_i$, *the map* $x_i \mapsto \psi_i(v_i, y_i, x_i)$ *is continuous on* K_i.

(vi) *There exist a nonempty compact subset* M *of* K *and a nonempty compact convex subset* N_i *of* K_i *for each* $i \in I$ *such that for all* $x \in K \setminus M$, *there exist* $i \in I$ *and* $\tilde{y}_i \in N_i$ *such that* $\psi_i(u_i, x_i, \tilde{y}_i) \in -\operatorname{int} C_i(x)$ *for all* $u_i \in F_i(x)$.

Then, *Problem* P3 *has a solution*.

Proof For each $i \in I$, let P_i and Q_i be the same as defined in the proof of Theorem 5.2. Then by using conditions (ii)–(iv), it is easy to see that for all $x \in K$, $Q_i(x)$ is convex.

From the proof of Theorem 5.2, there exists a solution $\bar{x} \in K$ of Problem P4. In view of Lemmas 3.1 and 3.2, $\bar{x} \in K$ is a solution of Problem P3. □

Remark 5.3 Theorem 5.2 extended and generalized Theorem 3.1 in [17, 18] for a family of mappings.

Now we prove the existence of a strong solution of Problem P2.

Theorem 5.4 *For each* $i \in I$, *let* K_i *be a nonempty convex subset of a real Banach space* X_i, Y_i *a real Banach space and* W_i *be the same as in Theorem* 5.2. *For each* $i \in I$, *let* $F_i : K \to 2^{K_i}$ *be a multivalued map with nonempty compact values and* $\psi_i : D_i \times K_i \times K_i \to Y_i$ *be a function such that the following conditions are satisfied*:

(i) *The family* $\{F_i\}_{i\in I}$ *of multivalued maps* F_i *is* $\mathcal{H}$*-hemicontinuous and generalized pseudomonotone w.r.t.* $\{\psi_i\}_{i\in I}$.

(ii) *The family* $\{\psi_i\}_{i\in I}$ *of functions* ψ_i *is* $\mathcal{C}_i$*-convex in the third argument.*

(iii) *For all* $s_i \in L(X_i, Y_i)$, $x, y, z \in X$ *and* $\alpha \in [0, 1]$,

$$\psi_i(s_i, x_i + \alpha(y_i - x_i), z_i) = (1-\alpha)\psi_i(s_i, x_i, z_i).$$

(iv) *For all* $x \in K$ *and for all* $u_i \in F_i(x)$, $\psi_i(u_i, x_i, x_i) \in \mathcal{C}_i$.

(v) *For each fixed* $(v_i, y_i) \in D_i \times K_i$, *the map* $x_i \mapsto \psi_i(v_i, y_i, x_i)$ *is continuous on* K_i.

(vi) *There exist a nonempty compact subset* M *of* K *and a nonempty compact convex subset* N_i *of* K_i *for each* $i \in I$ *such that for all* $x \in K \setminus M$, *there exist* $i \in I$ *and* $\tilde{y}_i \in N_i$ *such that* $\psi_i(u_i, x_i, \tilde{y}_i) \in -\operatorname{int} C_i(x)$ *for all* $u_i \in F_i(x)$.

Then, *Problem* P2 *has a solution*.

Proof For each $i \in I$, let P_i and Q_i be the same as defined in the proof of Theorem 5.2. Then by using conditions (ii)–(iv), it is easy to see that for all $x \in K$, $Q_i(x)$ is convex.

From the proof of Theorem 5.2, there exists a solution $\bar{x} \in K$ of Problem P4. From Lemma 5.3, $\bar{x} \in K$ is a solution of Problem P2. □

6 Conclusions

In this paper, we considered five different types of systems of generalized implicit vector variational inequalities (in short, SGIVVI), that is, Problems P1–P5. We named a solution of Problems P1–P3 general solution, strong solution and weak solution, respectively. It is pointed out that every general solution is a strong solution and every strong solution is a weak solution. By using a nonlinear scalarization function, we proposed two gap functions for Problems P1 and P4. By means of a gap function, we convert our SGIVVI into a single optimization problem. Therefore, the existence theory and numerical methods of optimization problems can be used to solve SGIVVI. We defined several types of pseudomonotonicities for a family of multivalued maps w.r.t. a family of functions. By using these pseudomonotonicities, we gave some relationships among Problems P1–P5. By using a known maximal element theorem for a family of multivalued maps, we proved the existence of a solution of Problem P1 under lower semicontinuity assumption on the family of multivalued maps involved in the formulation of the problem. By using a pseudomonotonicity assumption, we also established the existence results for solutions of Problems P2, P3 and P4 in the setting of Hausdorff topological vector spaces as well as real Banach spaces.

By using the technique of [2, 4, 6, 17], it is easy to derive the existence of a solution of Nash equilibrium problems for nondifferentiable and nonconvex functions from Theorem 5.3. By using the technique of [1, 3], one can easily establish the equivalence between systems of vector optimization problems and Problem P1 or Problem P2. Since Theorem 5.1 provides the existence of a solution of Problem P1 and so Problem P2, we will have necessary and sufficient conditions for a solution of system of vector optimization problems.

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