J Optim Theory Appl (2007) 134: 385–398
DOI 10.1007/s10957-007-9214-0

Stability Results for Efficient Solutions of Vector Optimization Problems

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Published online: 18 July 2007

Abstract Using the additive weight method of vector optimization problems and the method of essential solutions, we study some continuity properties of the mapping which associates the set of efficient solutions $S(f)$ to the objective function f. To understand such properties, the key point is to consider the stability of additive weight solutions and the relationship between efficient solutions and additive weight solutions.

Keywords Additive weight method · Vector optimization · Essential solutions · Efficient solutions · Additive weight solutions

1 Introduction

There is an extensive literature on stability, well-posedness, and sensitivity analysis in optimization. However, compared to the case of scalar optimization, stability and well-posedness analysis has not well developed in vector optimization. We refer to [1–7] for related papers on convergence of efficient sets and stability analysis.

The method of essential solutions has been used widely in various fields recently. It plays a crucial role in the study of the stability of solutions including fixed point problems, Nash equilibrium problems, and optimization problems, as discussed in [8–11]. In [11] Yu has proved the upper semicontinuity properties and the generic lower semicontinuity properties of the weakly efficient solutions in vector optimization problems and has also pointed out that most vector optimization problems (in the sense of Baire categories) are essential (or stable), i.e., their weakly efficient solutions are

Communicated by H.P. Benson.

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all essential. We know that the efficient solution sets can not have the desired properties which weakly efficient solution sets have for vector optimization problems. In fact, the sets of efficient solutions are not always upper semicontinuous. Therefore, the results concerned with the stability of efficient solution sets are not so straightforward as those of weakly efficient solutions. The situation with respect to continuity of efficient solutions is more complicated than in the case of weakly efficient solutions.

On the other hand, the additive weight method plays an important role in the study of vector optimization problems. The major concern of this paper is to obtain some results about stability and generic stability of efficient solutions of vector optimization problems. The key ideas are to use the stability of additive weight solutions, the method of essential solution, and the relationship between efficient solutions and additive weight solutions. In [9] authors have proved some stability results for additive weight solutions of vector optimization problems. In this paper, we will use the method of additive weight solution to investigate the stability of the efficient solutions. Firstly, we show that the sets of efficient solutions are partly upper semicontinuous though they are not upper semicontinuous. And then, based on this result, we prove that most vector optimization problems (in the sense of Baire categories) have at least one essential efficient solution. More precisely speaking, the sets of efficient solutions are almost lower semicontinuous for most vector optimization problems. Finally, we give a full characterization of the essential efficient solution.

2 Generic Stability

Throughout this paper, X denotes a nonempty and compact subset of a metric space (or a nonempty, compact and convex subset of a vector metric space), 2^X the collection of nonempty subsets of X, and $C_m(X)$ the space of all of continuous functions from X to R^m with uniform convergence norm $\|f-g\| = \max_{x\in X}\|f(x)-g(x)\|$.

The general vector optimization problem corresponding to X and $f(\cdot) = (f_1(\cdot), \dots, f_m(\cdot)) \in C_m(X)$ is denoted by VP and written as follows:

$$\text{(VP)} \quad \min f(x), \quad \text{s.t.} \quad x \in X.$$

Let us recall some definitions.

Definition 2.1 Let $f(x) = (f_1(x), \dots, f_m(x)) \in C_m(X)$, where $f_1(x), \dots, f_m(x)$ are real-valued functions defined on X. Then $x^* \in X$ is said to be an efficient solution of f, if there exists no $y \in X$ such that:

$$f_i(y) \le f_i(x^*), \quad \text{for all } i = 1, \dots, m,$$
$$f_i(y) < f_i(x^*), \quad \text{for some } i.$$

Let $S(f)$ denote the set of efficient solutions of f. The set of all efficient points is denoted by $\text{Min}(f) = \{f(x^*)\colon x^* \in S(f)\}$. Then S is a set-valued mapping from $C_m(X)$ to 2^X. This mapping is called the efficient solution mapping on $C_m(X)$. An important question arising in vector optimization problems is about the continuity properties (or stability) of the mapping S.

Let

$$R_+^m = \{w = (w_1, \dots, w_m) \in R^m \colon\ w_i \geq 0, i = 1, \dots, m\}$$

and

$$R_{++}^m = \{w = (w_1, \dots, w_m) \in R^m \colon\ w_i > 0, i = 1, \dots, m\}.$$

For any $w \in R_+^m$ and $f = (f_1, \dots, f_m) \in C^m(X)$, let

$$F_{f,w}(x) = \sum_{i=1}^m w_i f_i(x), \quad \forall x \in X.$$

Definition 2.2 $x^* \in X$ is said to be an additive weight solution of $f = (f_1, \dots, f_m) \in C^m(X)$ with respect to weight factor $w = (w_1, \dots, w_m)$, if

$$F_{f,w}(x^*) = \min_{x \in X} F_{f,w}(x).$$

Remark 2.1

(i) For each $f \in C_m(X)$, let $\mathrm{T}(f, w)$ denote the set of all additive weight solutions of f with respect to w. Then, T is a set-valued mapping from $C_m(X)$ to 2^X.
(ii) $\mathrm{T}(f, w) \neq \emptyset$ for each $w \in R_+^m$ and $T(f, w) \subset S(f)$ for each $w \in R_{++}^m$.

Definition 2.3 Let Y be a Hausdorff topological space, and $F : Y \mapsto 2^X$ a set-valued mapping. Then:

(i) F is said to be upper semicontinuous (u.s.c.) at $y \in Y$ if, for each open set $G \supset F(y)$, there exists an open neighborhood $O(y)$ of y such that $G \supset F(y')$ for any $y' \in O(y)$. If F is upper semicontinuous on Y and $F(y)$ is compact for each $y \in Y$, we say that F is an usco mapping.
(ii) F is said to be lower semicontinuous (l.s.c.) at $y \in Y$ if, for each open set $G \cap F(y) \neq \emptyset$, there exists an open neighborhood $O(y)$ of y such that $G \cap F(y') \neq \emptyset$ for any $y' \in O(y)$.
(iii) F is said to be almost lower semicontinuous (a.l.s.c.) at $y \in Y$ if there exists $x \in F(y)$ such that, for each open neighborhood $N(x)$ of x, there exists an open neighborhood $O(y)$ of y such that $N(x) \cap F(y') \neq \emptyset$ for any $y' \in O(y)$.

The following example provides a mapping that is a.l.s.c. but not l.s.c.

Example 2.1 Let $X = [0, 1]$ and define $F : X \mapsto 2^X$ by

$$F(x) = \begin{cases} [0, 1], & x = 0, \\ \{0\}, & x \in (0, 1]. \end{cases}$$

It is easy to check that F is a.l.s.c. but not l.s.c. at 0.

Definition 2.4 For each $f \in C_m(X)$, $x^* \in S(f)$ (resp. $x^* \in \mathrm{T}(w, f)$) is said to be an essential efficient solution (resp. essential additive weight solution) of f provided that for any open neighborhood $N(x^*)$ of x^* in X, there exists an open neighborhood

$O(f)$ of f in $C_m(X)$ such that $N(x^*) \cap S(f') \neq \emptyset$ (resp. $N(x^*) \cap \mathrm{T}(f', w) \neq \emptyset$) for all $f' \in O(f)$. Further, f is said to be E-essential or E-stable (resp. WT-essential or WT-sable) if all its efficient solutions (resp. additive weight solutions) are essential.

Remark 2.2 An optimal solution x^* is called *essential* if each objective function sufficiently close to f has a optimal solution arbitrarily close to x^*.

Definition 2.5 For each $f \in C_m(X)$, let $e(f)$ be a nonempty closed subset of $S(f)$. Then $e(f)$ is said to be an essential efficient set of f provided that for any open set $U \supset e(f)$, there exists an open neighborhood $O(f)$ of f in $C_m(X)$ such that $U \cap S(f') \neq \emptyset$ for any $f' \in O(f)$.

Remark 2.3 If $e(f) = \{x^*\}$ is an one point set, then x^* is an essential efficient solution of f.

Let us recall some definitions of proper efficiency (see [12–16]).

Definition 2.6 For each $f \in C_m(X)$, $x^* \in S(f)$ is said to be a Hu-proper efficient solution ($x^* \in \mathrm{Hu}(f)$) provided that

$$\mathrm{cl}\,\mathrm{conv}\,\mathrm{cone}\big[\big(f(X) - f(x^*)\big) \cup R^m_+\big] \cap -R^m_+ = \{0\},$$

where $\mathrm{conv}\, S$ denotes the convex hull of S.

Definition 2.7 For each $f \in C_m(X)$, $x^* \in S(f)$ is said to be a Ge-proper efficient solution ($x^* \in \mathrm{Ge}(f)$) provided that there exists $M > 0$ such that, for each $x \in X$ and $i \in \{1, \dots, m\}$ with $f_i(x) < f_i(x^*)$, there is at least one $j \in \{1, \dots, m\}$ ($j \neq i$) satisfying

$$\frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \leq M.$$

Remark 2.4

(i) Let $\mathrm{Be}(f)$ and $\mathrm{Bo}(f)$ denote the sets of proper efficient solutions in the sense of Benson and Borwein, respectively. Then, $\mathrm{Hu}(f) \subset \mathrm{Be}(f) \subset \mathrm{Bo}(f)$ and $\mathrm{Ge}(f) \subset \mathrm{Be}(f) \subset \mathrm{Bo}(f)$ (see [16]).
(ii) For each $w \in R^m_{++}$, $\mathrm{T}(w, f) \subset \mathrm{Hu}(f) \subset \mathrm{Be}(f) \subset \mathrm{Bo}(f)$ and $\mathrm{T}(w, f) \subset \mathrm{Ge}(f) \subset \mathrm{Be}(f) \subset \mathrm{Bo}(f)$.

Lemma 2.1 *Let* $f \in C_m(X)$.

(i) f *is E-essential* (*resp. WT-essential*) *if and only if the mapping* $S : C_m(X) \mapsto 2^X$ (*resp.* $\mathrm{T}(\cdot, w) : C_m(X) \mapsto 2^X$) *is l.s.c. on* $C_m(X)$.
(ii) *There exists an essential solution* $x^* \in S(f)$(*resp.* $x^* \in \mathrm{T}(\cdot, w)$) *if and only if the mapping* $S : C_m(X) \mapsto 2^X$ (*resp.* $\mathrm{T}(\cdot, w) : C_m(X) \mapsto 2^X$) *is a.l.s.c. on* $C_m(X)$.

Lemma 2.2 [17, *Theorem* 2] *Let* X *be a metric space*, Y *be a Baire space*, *and* $F : Y \mapsto 2^X$ *be an usco mapping. Then*, *there is a dense* G_δ *subset* Q' *of* Y *such that* F *is l.s.c. at each* $y \in Q'$.

Lemma 2.3 *For each* $w \in R_+^m$, $\mathrm{T}(\cdot, w) : C_m(X) \mapsto 2^X$ *is an usco mapping on* $C_m(X)$.

Proof Since $F_{(f,w)}$ is continuous and X is compact, it is easy to see that $\mathrm{T}(f, w)$ is compact for each $f \in C_m(X)$.

Suppose $\mathrm{T}(\cdot, w)$ not to be upper semicontinuous at $f \in C_m(X)$. Then, there exists an open set U of X with $U \supset \mathrm{T}(f, w)$ and a sequence $\{f^n\} \subset C_m(X)$ with $f^n \to f$ such that for each $n \in N$, one can find $x_n \in \mathrm{T}(w, f^n)$ satisfying $x_n \notin U$. Since X is compact and $\{x_n\} \subset X$, without loss of generality, we may assume that $x_n \to x_0$. It follows from $x_n \notin U$ that $x_0 \notin U$ and $x_0 \notin \mathrm{T}(f, w)$. Then, there exists some $x' \in X$ such that $F_{f,w}(x') - F_{f,w}(x_0) < 0$. Therefore, for all $x \in X$, we have

$$\begin{aligned}
F_{f^n,w}(x') - F_{f^n,w}(x) &= F_{f^n,w}(x') - F_{f,w}(x') + F_{f,w}(x') - F_{f,w}(x_0) \\
&\quad + F_{f,w}(x_0) - F_{f,w}(x) + F_{f,w}(x) - F_{f^n,w}(x) \\
&= \sum_{i=1}^{m} w_i \big(f_i^n(x') - f_i(x')\big) + F_{f,w}(x') - F_{f,w}(x_0) \\
&\quad + F_{f,w}(x_0) - F_{f,w}(x) + \sum_{i=1}^{m} w_i \big(f_i^n(x) - f_i(x)\big) \\
&\le \sum_{i=1}^{m} w_i \|f^n - f\| + F_{f,w}(x') - F_{f,w}(x_0) \\
&\quad + F_{f,w}(x_0) - F_{f,w}(x) + \sum_{i=1}^{m} w_i \|f^n - f\| \\
&\le 2\|f^n - f\| + F_{f,w}(x') - F_{f,w}(x_0) + F_{f,w}(x_0) - F_{f,w}(x).
\end{aligned}$$

Since $f^n \to f$ and $F_{f,w}$ is continuous at x_0, we have $\|f^n - f\| \to 0$ when $n \to \infty$ and $F_{f,w}(x) - F_{f,w}(x_0)$ arbitrarily close to 0 when x sufficiently close to x_0. Hence there exists some open neighborhood $O(x_0)$ of x_0 and $n_1 \in N$ such that $F_{f^n,w}(x') - F_{f^n,w}(x) < 0$ for all $x \in O(x_0)$ and $n \ge n_1$. Moreover, since $x_n \to x_0$, there exists $n_2 \ge n_1$ such that $x_{n_2} \in O(x_0)$ and thus, $F_{f^{n_2},w}(x') < F_{f^{n_2},w}(x_{n_2})$. Consequently, $x_{n_2} \notin \mathrm{T}(w, f^{n_2})$, contradicting the assumption $x_n \in \mathrm{T}(w, f^n)$. This lemma is proven. □

We emphasize that, in general, the efficient solution mapping is neither u.s.c. nor l.s.c., as shown in the following two examples.

Example 2.2 Let $X = [0, 1] \times [0, 1]$ and $C_2(X)$ be the space of continuous functions from X to R^2. Define $f, f_n : X \mapsto R^2$ by

$$\begin{aligned}
&f(x, y) = (x, y), \\
&f_n(x, y) = \left(\left(1 - \frac{1}{n}\right)x - \frac{1}{n}y, y\right), \quad \forall (x, y) \in X.
\end{aligned}$$

Then, f and f_n are all linear functions and $f_n \to f$ when $n \to \infty$. In this case, we have

$$\begin{aligned}
&\mathrm{Min}(f) = \{(0,0)\},\\
&\mathrm{Hu}(f) = \mathrm{Ge}(f) = \mathrm{Be}(f) = \mathrm{Bo}(f) = S(f) = \{(0,0)\},\\
&\mathrm{Min}(f_n) = \left\{\left(-\frac{1}{n}y, y\right) : y \in [0,1]\right\},\\
&\mathrm{Hu}(f_n) = \mathrm{Ge}(f_n) = \mathrm{Be}(f_n) = \mathrm{Bo}(f_n) = S(f_n) = \{(0,y) : y \in [0,1]\}.
\end{aligned}$$

It is easy to check that none of S, Hu, Ge, Bo, and Be is u.s.c. at f.

Example 2.3 Let us consider a special case of scalar optimization. Let $C_1(X)$ denote the space of continuous functions from X to R, where $X = [0,1]$. Define $f, f_n : X \mapsto R$ as follows:

$$f(x) = \begin{cases} -4x + 1, & x \in \left[0, \frac{1}{4}\right),\\ 0, & x \in \left[\frac{1}{4}, \frac{3}{4}\right),\\ 4x - 3, & x \in \left[\frac{3}{4}, 1\right], \end{cases}$$

$$f_n(x) = \begin{cases} -4x + 1, & x \in \left[0, \frac{1}{4}\right),\\ \frac{1}{n}x - \frac{1}{4n}, & x \in \left[\frac{1}{4}, \frac{12n-1}{16n-4}\right),\\ 4x - 3, & x \in \left[\frac{12n-1}{16n-4}, 1\right]. \end{cases}$$

Then f and f_n are all convex functions and $f_n \to f$. In this case, we have

$$\mathrm{Min}(f) = \{0\}, \qquad S(f) = \left[\frac{1}{4}, \frac{3}{4}\right].$$

$$\mathrm{Min}(f_n) = \{0\}, \qquad S(f_n) = \left\{\frac{1}{4}\right\}.$$

It is clear that S is not l.s.c. at f. In fact, $x = \frac{1}{4}$ is the unique essential solution in efficient solution set $[\frac{1}{4}, \frac{3}{4}]$. On the other hand, note that $x^* = \frac{1}{2}$ is a optimal solution of this scalar optimization, hence x^* is obviously a proper efficient one (such as positive proper efficiency, Hu-proper efficiency, Ge-proper efficiency, Be-proper efficiency, and Bo-proper efficiency). But x^* is not an essential solution.

Theorem 2.1 *Let $S : C_m(X) \mapsto 2^X$ be the efficient solution mapping. Then*:

(i) *There exists an u.s.c. mapping $S_0 : C_m(X) \mapsto 2^X$ such that $S_0(f) \subset S(f)$ for each $f \in C_m(X)$. We say that S is partly upper semicontinuous.*

(ii) *For each $f \in C_m(X)$ and $w \in R^m_{++}$, the set of additive weight solutions $\mathrm{T}(f, w)$ is an essential efficient set of f.*

Proof (i) For any $w \in R^m_{++}$, let $S_0(f) = \mathrm{T}(f, w)$, $\forall f \in C_m(X)$. Then statement (i) of Theorem 2.1 is directly obtained from Remarks 2.1 and 2.3.

(ii) Let $w \in R^m_{++}$. According to Remark 2.1 and Definition 2.5, $\mathrm{T}(\cdot, w)$ is an u.s.c. mapping and $\mathrm{T}(f, w) \subset S(f)$ for all $f \in C_m(X)$. Then for any open set $U \supset \mathrm{T}(f, w)$ there exists some open neighborhood $O(f)$ of f such that $\mathrm{T}(f', w) \subset U$ for all $f' \in O(f)$. Observing that $\mathrm{T}(f', w) \subset S(f')$, we conclude that $U \cap S(f') \supseteq U \cap \mathrm{T}(f', w) = \mathrm{T}(f', w) \neq \emptyset$. Thus $\mathrm{T}(f, w) \subset S(f)$ is essential efficient set of f. □

Corollary 2.1 *The proper efficient solution mappings* Hu, Ge, Be *and* Bo *are all partly u.s.c.*

Proof It is directly obtained from Remark 2.4 and Lemma 2.3. □

The following results dealing with the essential solutions are directly derived from Lemmas 2.2, 2.3 and Theorem 2.1.

Lemma 2.4 *For each fixed $w \in R^m_+$, there exists a dense G_δ subset Q of $C_m(X)$ such that* $\mathrm{T}(\cdot, w)$ *is l.s.c. at each $f \in Q$.*

Theorem 2.2 *There exists a dense G_δ subset Q of $C_m(X)$ such that, for each $f \in Q$, there is at least one $x^* \in \mathrm{T}(f, w) \in S(f)$ $(w \in R^m_{++})$ such that x^* is an essential efficient solution, i.e., S is a.l.s.c. at every $f \in Q$.*

Proof This assertion follows from Remark 2.1(ii) and Lemma 2.4. Take an arbitrary $w = (w_1, \ldots, w_m) \in R^m_{++}$. It follows from Lemma 2.4 that there exists a dense G_δ subset Q of $C_m(X)$ such that $\mathrm{T}(\cdot, w)$ is l.s.c. at each $f \in Q$. For each $f \in Q$, let $x^* \in \mathrm{T}(f, w) \subset S(f)$. Since $\mathrm{T}(\cdot, w)$ is l.s.c. at f, for any open neighborhood $N(x^*)$ of x^*, there exists an open neighborhood $O(f)$ of f such that $N(x^*) \cap \mathrm{T}(f', w) \neq \emptyset$ for all $f' \in O(f)$. To show that x^* is an essential efficient solution, it is clearly enough to show that $N(x^*) \cap S(f') \neq \emptyset$. Observing that $\mathrm{T}(f', w) \subset S(f')$ and $N(x^*) \cap \mathrm{T}(f', w) \neq \emptyset$, we complete the proof. □

Remark 2.5

(i) Theorem 2.2 shows that in the sense of Baire categories, most vector optimization problems have at least one essential efficient solution.
(ii) Note the proof of Theorem 2.2 and Remark 2.4(ii). It follows that the essential solution x^* in Theorem 2.2 is also a proper efficient solution, i.e., $x^* \in \mathrm{T}(w, f) \subset \mathrm{Hu}(f) \subset \mathrm{Be}(f) \subset \mathrm{Bo}(f)$ and $x^* \in \mathrm{T}(w, f) \subset \mathrm{Ge}(f) \subset \mathrm{Be}(f) \subset \mathrm{Bo}(f)$.

Corollary 2.2 *Proper efficient solution mappings* Hu, Ge, Be *and* Bo *are a.l.s.c. at some dense G_δ subset Q of $C_m(X)$.*

3 Further Properties of Essential Efficient Solutions

In this section, we will give some further results concerning with the properties of essential efficient solutions.

Let X be a convex set in a linear space. $f=(f_1(x),\dots,f_m(x))\in C_m(X)$ is said to be strongly quasiconvex if, for any $x_1\neq x_2\in X$ and $t\in(0,1)$, f_i satisfies

$$f_i(tx_1+(1-t)x_2)<\max\{f_i(x_1),f_i(x_2)\},\quad \forall i=1,\dots,m.$$

The next theorem gives a sufficient condition to ensure that an efficient solution is essential; which is an immediate consequence of Theorem 2.1.

Theorem 3.1 *Let $x^*\in S(f)$ be an efficient solution of f. If there is some $w\in R^m_{++}$ such that x^* is the unique additive weight solution of $F_{(w,f)}(x)$, then x^* is essential.*

Proof Let $w\in R^m_{++}$ and x^* be the unique additive weight solution of $F_{(w,f)}(x)$. Then $\mathrm{T}(f,w)=\{x^*\}$ and $\{x^*\}$ is an essential set of f due to Theorem 2.1(ii). Observing Remark 2.3, we conclude that x^* is essential. □

Example 3.1 Let $X=[-1,1]$ and $f:X\mapsto R^2$ be defined by

$$f(x)=\begin{cases}(0,-t), & 0\le t\le 1,\\ (t,0), & -1\le t<0.\end{cases}$$

Then, f is continuous quasiconvex on compact convex set X.

In this case, we have

$$\mathrm{Min}(f)=\{(-1,0),(0,-1)\},\qquad S(f)=\{-1,1\}.$$

Then, $t=-1$ is the unique additive weight solution of f with respect to weighting factor w whenever $w=(w_1,w_2)$ satisfies $0<w_2<w_1$. Hence $t=-1$ is an essential efficient solution by Theorem 3.1. Similarly, $t=1$ is also an essential solution by choosing $0<w_1<w_2$.

The following example show that the converse of Theorem 3.1 is false in general; that is, an essential efficient solution is not always an unique additive weight solution subject to some weight factor.

Example 3.2 Let $X=[0,1]$ and $f:X\mapsto R^2$ be defined by

$$f(x)=(x-1,-x),\quad \forall 0\le x\le 1.$$

Then, f is continuous and affine on compact convex set X.

In this case, we have

$$\mathrm{Min}(f)=\{(x-1,-x)\colon\ x\in[0,1]\},\qquad S(f)=[0,1].$$

It is easy to check that the efficient solution set $[0,1]=T(f,(1/2,1/2))$ and every point in $[0,1]$ is an essential efficient solution. For each $x\in(0,1)$, x is an essential efficient solution. Since $w_0=(1/2,1/2)$ is the unique weight factor up to positive scalars, there is no weight factor w such that x becomes the unique additive solution subject to w.

Theorem 3.2 *Let $f \in C_m(X)$ and $x^* \in S(f)$. Then, x^* is essential if and only if, for each open neighborhood $U(x^*)$ of x^*, there exists $\bar{x} \in U(x^*)$ such that $\bar{x} \in S(f)$ and $f^{-1}(f(\bar{x})) \cap [X \setminus U(x^*)] = \emptyset$, i.e., there is no efficient solution in $X \setminus U(x^*)$ corresponding to the optimal value $f(\bar{x})$.*

Proof (I) We prove the sufficient part. Suppose to the contrary that x^* is not essential. Then, there exists some open neighborhood $U(x^*)$ of x^* and $f^n \to f$ such that $S(f^n) \cap U(x^*) = \emptyset$. Since X is compact, there exists an open neighborhood $V(x^*)$ of x^* such that $\overline{V(x^*)} \subset U(x^*)$. Let $\bar{x} \in U(x^*)$, $\bar{x} \in S(f)$, and $f^{-1}(f(\bar{x})) \cap [X \setminus V(x^*)]) = \emptyset$.

(i) First, we prove that, for each $n = 1, \dots,$ there exists $x_n \in V(x^*)$ such that $d(x_n, \bar{x}) < \frac{1}{n}$ and for such x_n we can choose $y_n \in X \setminus V(x^*)$ such that

$$f_i^n(y_n) \leq f_i^n(x_n), \quad \text{for all } i = 1, \dots, m,$$

$$f_i^n(y_n) < f_i^n(x_n), \quad \text{for some } i.$$

If not, then for some n_0, there exists an open neighborhood $O(\bar{x})$ of $\bar{x}$ such that $O(\bar{x}) \subset V(x^*)$ and for each $x \in O(\bar{x})$ there is no element $y \in X \setminus V(x^*)$ satisfying

$$f_i^{n_0}(y) \leq f_i^{n_0}(x), \quad \text{for all } i = 1, \dots, m,$$

$$f_i^{n_0}(y) < f_i^{n_0}(x), \quad \text{for some } i.$$

Let $S(f^{n_0}, D)$ denote the set of efficient solutions of f^{n_0} subject to the feasible region $D \subset X$. By the compactness of $\overline{V(x^*)}$ and the continuity of f^{n_0}, it is clear that $S(f^{n_0}, \overline{V(x^*)}) \neq \emptyset$. If $S(f^{n_0}, \overline{V(x^*)}) \cap O(\bar{x}) \neq \emptyset$, then we can choose $z^* \in S(f^{n_0}, \overline{V(x^*)}) \cap O(\bar{x})$. Hence, there is no element $y \in \overline{V}(x^*)$ such that

$$f_i^{n_0}(y) \leq f_i^{n_0}(z^*), \quad \text{for all } i = 1, \dots, m,$$

$$f_i^{n_0}(y) < f_i^{n_0}(z^*), \quad \text{for some } i.$$

According to the assumption of $O(\bar{x})$ and $z^* \in O(\bar{x})$, it follows that $z^* \in S(f^{n_0})$, contradicting $S(f^{n_0}) \cap U(x^*) = \emptyset$. If $S(f^{n_0}, \overline{V(x^*)}) \cap O(\bar{x}) = \emptyset$, let $x_0 \in O(\bar{x})$. Then $x_0 \notin S(f^{n_0}, \overline{V(x^*)})$. This shows that there is some $y_0 \in \overline{V(x^*)}$ such that

$$f_i^{n_0}(y_0) \leq f_i^{n_0}(x_0), \quad \text{for all } i = 1, \dots, m,$$

$$f_i^{n_0}(y_0) < f_i^{n_0}(x_0), \quad \text{for some } i.$$

Let $K = \{x \in \overline{V(x^*)}\colon\ f_i^{n_0}(x) \leq f_i^{n_0}(y_0),\ i = 1, \dots, m.\}$. It is clear that $K \neq \emptyset$ and $K \subset \overline{V(x^*)} \subset U(x^*)$. Since f^{n_0} is continuous and K is compact, it is easy to see that $S(f^{n_0}, K) \neq \emptyset$. Choose $z^* \in S(f^{n_0}, K)$ and note that $x_0 \in O(\bar{x})$, $f(y_0) \leq f(x_0)$ and $K = \{x \in \overline{V(x^*)}\colon\ f_i^{n_0}(x) \leq f_i^{n_0}(y_0),\ i = 1, \dots, m.\}$. It is routinely to check that $z^* \in S(f^{n_0})$ which contradicts $S(f^{n_0}) \cap U(x^*) = \emptyset$.

(ii) Now, we arrive to a contradiction. By (i), take a sequence $\{x_n\} \subset V(x^*)$ such that $d(x_n, \bar{x}) < \frac{1}{n}$ and choose $y_n \in X \setminus V(x^*)$ such that

$$f_i^n(y_n) \leq f_i^n(x_n), \quad \text{for all } i = 1, \dots, m,$$

$$f_i^n(y_n) < f_i^n(x_n), \quad \text{for some } i.$$

Since X is compact, without loss of generality, we may assume that $y_n \to y^* \in X \setminus V(x^*)$. Then, $y^* \neq \bar{x}$ and

$$\begin{aligned}
&f_i(y^*) - f_i(\bar{x}) \\
&\quad = f_i(y^*) - f_i(y_n) + f_i(y_n) - f_i^n(y_n) + f_i^n(y_n) - f_i^n(x_n) \\
&\qquad + f_i^n(x_n) - f_i(x_n) + f_i(x_n) - f_i(\bar{x}) \\
&\quad \leq f_i^n(y_n) - f_i^n(x_n) + f_i(y^*) - f_i(y_n) + f_i(x_n) - f_i(\bar{x}) + 2\|f - f^n\| \\
&\quad \leq f_i(y^*) - f_i(y_n) + f_i(x_n) - f_i(\bar{x}) + 2\|f - f^n\|,
\end{aligned}$$

for all $i = 1, 2, \ldots, m$. Let $n \to \infty$. Observing $f^n \to f$, $y_n \to y^*$ and $x_n \to \bar{x}$, we have that $f_i(y^*) \leq f_i(\bar{x})$ for all $i = 1, 2, \ldots, m$. Since $\bar{x}$ is efficient solution of f, it implies $f(y^*) = f(\bar{x})$ which contradicts the assumption $f^{-1}(f(\bar{x})) \cap [X \setminus V(x^*)] = \emptyset$.

(II) For the necessary part, suppose to the contrary that there exist some essential efficient solution $x^* \in S(f)$ and some open neighborhood $U(x^*)$ such that $f^{-1}(f(x)) \cap [X \setminus U(x^*)] \neq \emptyset$ for each efficient solution $x \in U(x^*)$. Let $y_x \in f^{-1}(f(x)) \cap [X \setminus U(x^*)]$. Then $y_x \neq x$ and $f(y_x) = f(x)$. Choose an open neighborhood $V(x^*)$ of x^* such that $\overline{V(x^*)} \subset U(x^*)$.

Since X is compact, by the Urysohn lemma we can construct continuous function β such that $\beta(x) = 0$ if $x \in \overline{V(x^*)}$ and $\beta(x) = 1$ if $x \in X \setminus U(x^*)$ and $0 \leq \beta(x) \leq 1$. For each $n \in N$, let $\vec{\mathbf{n}} = (1/n, \ldots, 1/n)$. Define $f^n \in C_m(X)$ by

$$f^n(x) = f(x) - \beta(x)\vec{\mathbf{n}}, \quad \forall x \in X.$$

Then, $f^n \to f$. Take an arbitrary $x \in V(x^*)$. If $x \in S(f)$, then there exists $y_x \in X \setminus U(x^*)$ such that $f(y_x) = f(x)$. It follows that

$$f_i^n(y_x) = f_i(y_x) - \frac{1}{n} < f_i(y_x) = f_i(x) = f_i^n(x), \quad \forall i = 1, \ldots, m.$$

This shows that $x \notin S(f^n)$. If $x \notin S(f)$, then there is $y \in X$ such that:

$$\begin{aligned}
&f_i(y) \leq f_i(x), && \text{for all } i = 1, \ldots, m. \\
&f_{i_0}(y) < f_{i_0}(x), && \text{for some } i_0.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
f_i^n(y) &= f_i(y) - \frac{1}{n}\beta(y) \\
&\leq f_i(x) - \frac{1}{n}\beta(y) \leq f_i(x) = f_i^n(x), \quad \text{for all } i = 1, \ldots, m, \\
f_{i_0}^n(y) &= f_{i_0}(y) - \frac{1}{n}\beta(y) \\
&< f_{i_0}(x) - \frac{1}{n}\beta(y) \leq f_{i_0}(x) = f_{i_0}^n(x), \quad \text{for some } i_0.
\end{aligned}$$

Consequently, $x \notin S(f^n)$. Hence, $V(x^*) \cap S(f_n) = \emptyset$, which contradicts the fact that x^* is an essential solution. The proof is complete. □

Below we provide a necessary and sufficient condition under which the efficient solution mapping S is l.s.c.

Theorem 3.3 *Let $f \in C_m(X)$. S is l.s.c. at f if and only if, for each $x^* \in S(f)$ and each open neighborhood $U(x^*)$ of x^*, there exists $\bar{x} \in U(x^*)$ such that $\bar{x} \in S(f)$ and $f^{-1}(f(\bar{x})) \cap [X \setminus U(x^*)] = \emptyset$.*

Proof This is immediate from Definition 2.4, Lemma 2.1 and Theorem 3.2. □

The following corollaries are immediate consequences of Theorem 3.3.

Corollary 3.1 *Let $f \in C_m(X)$ and $x^* \in S(f)$. Then, x^* is essential if there exists a sequence $\{\bar{x}_n\} \subset S(f)$ such that $\bar{x}_n \to x^*$ and $f^{-1}(f(\bar{x}_n)) = \{\bar{x}_n\}$ (i.e., each $\bar{x}_n$ is the unique solution corresponding to the Pareto optimal value $f(\bar{x}_n)$).*

Corollary 3.2 *Let $f \in C_m(X)$ and $x^* \in S(f)$. If $f^{-1}(f(x^*)) = \{x^*\}$, i.e., the Pareto optimal value $f(x^*)$ corresponds to the unique efficient solution x^*, then x^* is essential.*

Remark 3.1 The following example shows that the assumption $f^{-1}(f(x^*)) = \{x^*\}$ is not necessary for x^* being essential.

Example 3.3 Let $X = [0, 2]$. Define $f : X \mapsto R^2$ by

$$f(x) = \begin{cases} (x-1, 1-x), & x \in [0,1], \\ (0,0), & x \in [1,2]. \end{cases}$$

Then

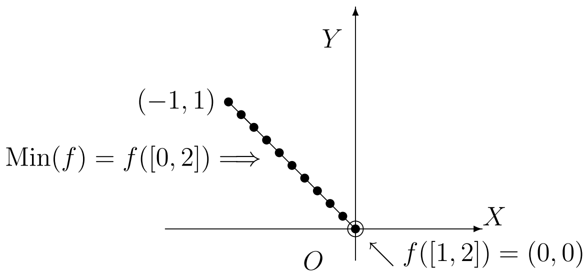
$$\text{Min}(f) = \{(-y, y)\colon\ y \in [0,1]\}.$$

$$S(f) = [0, 2].$$

In this case, $f(x) \equiv (0, 0)$ and $x \in [1, 2]$. But it is clear that for this function f, $x = 1$ is essential according to Corollary 3.1 (see Fig. 1).

Corollary 3.3 *Let $f \in C_m(X)$. If f is strongly quasiconvex on X, then for the function f, all its efficient solutions are essential, i.e., the efficient solution mapping S is l.s.c. at f.*

Fig. 1 $f^{-1}(f(x^*)) = \{x^*\}$ not necessary for essential solutions



Proof It is immediate from Corollary 3.2. In fact, each optimal valued corresponds to the unique efficient solution. If not, let $x_1^* \neq x_2^*$ and $f(x_1^*) = f(x_2^*)$ is a Pareto optimal value. Observe that f is strongly quasiconvex. Then, f_i satisfies

$$f_i(tx_1 + (1-t)x_2) < \max\{f_i(x_1), f_i(x_2)\} < f_i(x_1) = f_i(x_2),$$
$$\forall i = 1, \dots, m, t \in (0, 1).$$

This is a contradiction with that $f(x_1^*) = f(x_2^*)$ is a Pareto optimal value. □

Corollary 3.4 *Let $f \in C_m(X)$. Then, the efficient solution mapping S is l.s.c. at f if f is injective, i.e., $f(x) \neq f(x')$ whenever $x \neq x'$.*

Combining Theorem 2.2 and Theorem 3.2, we obtain the following results.

Theorem 3.4 *There exists a dense G_δ subset Q of $C_m(X)$ with the properties: for each $f \in Q$, there is at least one efficient solution $x^* \in T(w, f) \subset S(f)$ such that, for any open neighborhood $U(x^*)$ of x^*, there exists $\bar{x} \in S(f)$ satisfying $\bar{x} \in U(x^*)$ and $f^{-1}(f(\bar{x})) \cap [X \setminus U(x^*)] = \emptyset$.*

Proof By Theorem 2.2, there exists a dense G_δ subset Q of $C_m(X)$ such that, for each $f \in Q$, there is at least one essential solution $x^* \in T(w, f) \subset S(f)$. Hence x^* has the desired properties mentioned in this theorem and the proof is complete. □

As a special case, we deduce the following results for scalar optimization problems (see [7]).

Corollary 3.5 *For each fixed $w \in R_+^m$, there exists a dense G_δ subset Q of $C_m(X)$ such that $x^* \in \mathrm{T}(f, w)$ is the unique additive weight solution subject to w for each $f \in Q$.*

Proof For each fixed $w = (w_1, \dots, w_m) \in R_+^m$, by Lemma 2.4, there exists a dense G_δ subset Q of $C_m(X)$ such that $\mathrm{T}(\cdot, w)$ is l.s.c. at each $f \in Q$. Then, for each $f \in Q$, every solution in $\mathrm{T}(f, w)$ is an essential additive weight solution. Now, we prove that $\mathrm{T}(f, w)$ is an one point set. If not, let $x_1^*, x_2^* \in \mathrm{T}(f, w)$ and $x_1^* \neq x_2^*$. Then,

$$F_{f,w}(x_1^*) = F_{f,w}(x_2^*) = \min_{x \in X} F_{f,w}(x).$$

Since X is compact, choose open neighborhoods $U(x_1^*)$ and $V(x_1^*)$ of x_1^* such that $x_2^* \notin U(x_1^*)$ and $\overline{V(x_1^*)} \subset U(x_1^*)$.

By the Urysohn lemma, we can construct continuous function β such that $\beta(x) = 0$ if $x \in \overline{V(x^*)}$ and $\beta(x) = 1$ if $x \in X \setminus U(x^*)$ and $0 \leq \beta(x) \leq 1$. For each $n \in N$, let $\vec{\mathbf{n}} = (1/n, \dots, 1/n)$. Define $f^n \in C_m(X)$ by

$$f^n(x) = f(x) - \beta(x)\vec{\mathbf{n}}, \quad \forall x \in X.$$

Then, $f^n \to f$. Note that

$$F_{(f^n,w)}(x_2^*) = \sum_{i=1}^{m} w_i f_i^n(x_2^*) = \sum_{i=1}^{m} w_i f_i(x_2^*) - \frac{1}{n} = F_{(f,w)}(x_2^*) - \frac{1}{n}.$$

Take an arbitrary $x \in V(x_1^*)$. It follows that

$$F_{f^n,w}(x) = \sum_{i=1}^{m} w_i f_i^n(x) = \sum_{i=1}^{m} w_i f_i(x) = F_{f,w}(x).$$

Thus,

$$F_{f^n,w}(x) = F_{f,w}(x) \geq F_{f,w}(x_2^*) > F_{f,w}(x_2^*) - \frac{1}{n} = F_{f^n,w}(x_2^*), \quad \forall x \in V(x_1^*).$$

Consequently, $x \notin \mathrm{T}(f^n, w)$, $\forall x \in V(x_1^*)$. Hence, $V(x_1^*) \cap \mathrm{T}(f^n, w) = \emptyset$, which contradicts the fact that x_1^* is an essential additive weight solution. The proof is complete. □

Corollary 3.5 shows that, in the sense of Baire categories, most problems in the form of (VP) have a unique additive weight solution.

Corollary 3.6 *There exists a dense G_δ subset Q of $C_1(X)$ such that, for each $f \in Q$, the scalar optimization problem f has a unique optimal solution in X.*

Proof By Theorem 3.4, there exists some dense G_δ subset Q of $C_1(X)$ such that each $f \in Q$ has at least one essential efficient solution (optimal solution). Hence there is some $x^* \in S(f)$ such that for any open neighborhood $U(x^*)$ there exists $\bar{x} \in S(f)$ satisfying $\bar{x} \in U(x^*)$ and $f^{-1}(f(\bar{x})) \cap [X \setminus U(x^*)] = \emptyset$. We can observe that in the special case of real-valued function $f^{-1}(f(\bar{x})) \cap [X \setminus U(x^*)] = \emptyset$ for any open neighborhood $U(x^*)$ if and only if x^* is the unique optimal solution of f. The proof is complete. □

Corollary 3.6 shows that in the sense of Baire categories, most scalar optimization problems have a unique optimal solution.

4 Conclusions

In this paper, we have proved some stability results for efficient solutions of the problem

$$\text{(VP)} \quad \min f(x), \quad \text{s.t.} \quad x \in X,$$

where $f : X \mapsto R^m$ is a continuous function over the compact subset of a metric space X. Here, stability is intended as semicontinuity property of the set-valued mapping $S(f)$, which associates to f. The problem is nontrivial as, in general, $S(f)$ is neither upper semicontinuous nor lower semicontinuous (see Example 2.2).

In Sect. 2, we prove the partly upper semicontinuity of $S(f)$, i.e., the existence of an u.s.c. set-valued mapping, whose values are subsets of $S(f)$ for each f. Using this result, along with the linear scalarization (whose solutions are named "additive weight solutions"), we prove that most problems in the form of (VP) (in the Baire category) have at least one essential solution.

In Sect. 3, we formulate some necessary and sufficient conditions for efficient solutions to be essential and also characterize essential solution x^* by $f^{-1}(x^*) = \{x^*\}$. Especially, we delineate the generic uniqueness of the additive weight solution of (VP) and the solution of scalar optimization.

Acknowledgements This work is supported by NSFC, Grant 10561003. The authors express their gratitude to the editor and referees for many useful comments and suggestions.

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