J Optim Theory Appl (2007) 133: 201–212
DOI 10.1007/s10957-007-9202-4

Gap Functions and Existence of Solutions for a System of Vector Equilibrium Problems

N.J. Huang · J. Li · J.C. Yao

Published online: 18 April 2007

Abstract In this paper, a gap function for a system of vector equilibrium problems is introduced and studied. Some necessary and sufficient conditions for the system of vector equilibrium problems are established. Characterizations of the solutions set for the system of vector equilibrium problems are also derived. Furthermore, some existence results of solutions for the system of vector equilibrium problems are proved.

Keywords Systems of vector equilibrium problems · Gap functions · Convex cones · Point-to-set mappings

1 Introduction

In 1994, Blum and Oettli [1] introduced and studied the following scalar equilibrium problem (in short, EP): given a nonempty set E and a scalar bifunction $f : E \times E \to$

Communicated by X.Q. Yang.

This work was supported by the National Natural Science Foundation of China, the Youth Foundation, Sichuan Education Department of China, the National Natural Science Foundation, Sichuan Education Department of China (2004C018), and a grant from the National Science Council of ROC.

N.J. Huang
Department of Mathematics, Sichuan University, Chengdu, Sichuan, People's Republic of China

J. Li
Department of Mathematics, China West Normal University, Nanchong, Sichuan, People's Republic of China

J.C. Yao (✉)
Department of Applied Mathematics, National Sun Yat-Sen University, Kaohsiung, Taiwan
e-mail: yaojc@math.nsysu.edu.tw

$R=(-\infty,\infty)$, with $f(x,x)\geq 0$ for all $x\in E$, find $x^*\in E$ such that

$$\text{(EP)}\quad f(x^*,y)\geq 0,\quad \forall y\in E.$$

It is well known that EP contains as special cases, for instance, Nash equilibria problems, fixed-point problems, variational inequality and complementarity problems, as well as optimization and control problems. See e.g. [2–7].

On the other hand, the vector variational inequality in a finite-dimensional Euclidean space was introduced first by Giannessi [8] in 1980. This is a generalization of a scalar variational inequality to the vector case by virtue of multicriteria consideration. Since then, many authors have studied vector variational inequalities, vector complementarity problems, and vector equilibrium problems with fixed or moving cones in finite-dimensional Euclidean spaces and infinite-dimensional abstract spaces. See for example [9–21] and the references therein.

Throughout this paper, without other specifications, let I be an index set and for each $i\in I$, let X_i and Y_i be locally convex Hausdorff topological vector spaces. Consider a family of nonempty closed convex subsets $\{K_i\}_{i\in I}$ with K_i in X_i. We denote $X=\prod_{i\in I}X_i$, $Y=\prod_{i\in I}Y_i$ and $K=\prod_{i\in I}K_i$. For each $i\in I$, let $C_i:K\to 2^{Y_i}$ be a point-to-set mapping such that for any $x\in K$, $C_i(x)$ is a pointed, closed and convex cone in Y_i with nonempty interior $\operatorname{int}C_i(x)$. For each $i\in I$, let $e_i:K\to Y_i$ be a vector-valued mapping and for any $x\in K$, $e_i(x)\in\operatorname{int}C_i(x)$. For each $i\in I$, let $f_i:K\times K_i\to Y_i$ be a bifunction. We consider the following system of vector equilibrium problems (in short, SVEP): find $x^*\in K$ such that, for each $i\in I$,

$$\text{(SVEP)}\quad f_i(x^*,y_i)\notin -\operatorname{int}C_i(x^*),\quad \forall y_i\in K_i.$$

We denote by E^S the solutions set of SVEP.

If the index set I is a singleton, then SVEP reduces to the following vector equilibrium problems (in short, VEP): find $x^*\in K$ such that

$$\text{(VEP)}\quad f(x^*,y)\notin -\operatorname{int}C(x^*), forally\in K.$$

We denote by E^0 the solutions set of VEP.

For a suitable choice of the mappings f, C and the spaces X and Y, a number of known classes of vector (scalar) equilibrium problems, vector (scalar) variational inequalities and vector (scalar) complementarity problems can be obtained as special cases of VEP. See e.g. [8, 22–28] and the references therein.

It is well known that gap functions play a crucial role in transforming a variational inequality problem into an optimization problem. Thus powerful optimization solution methods and algorithms can be applied for finding solutions of variational inequalities. In 2002, Yang and Yao [29] introduced the gap functions and established necessary and sufficient conditions for the existence of solutions for vector variational inequalities (in short, VVI) with set-valued mappings. They also investigated the existence of solutions for the generalized VVI with a set-valued mapping by virtue of the existence of a solution of VVI with a single-valued function and a continuous selection theorem. In 2003, the gap function approach was extended to the study for EP by Mastroeni [30]. For some other related works, we refer to [31, 32].

Inspired and motivated by above research works, in this paper, by virtue of the nonlinear scalarization function introduced by Chen, Yang and Yu [33] a gap function for SVEP is introduced and some necessary and sufficient conditions for SVEP are established. Characterizations of the solutions set for SVEP are also derived. Furthermore, some existence results for SVEP are proved by using the theorem due to Deguire, Tan and Yuan [34].

2 Preliminaries

Let X and Y be two Hausdorff topological vector spaces and let $G : X \to Y$ be a point-to-set mapping. The inverse G^{-1} of G is the point-to-set mapping from the range of G to X defined by

$$x \in G^{-1}(y) \Leftrightarrow y \in G(x).$$

A nonempty subset P of Y is said to be a cone if $\lambda P \subseteq P$ for all $\lambda > 0$. The cone P is called pointed if $P \cap \{-P\} = \{0\}$.

We recall first some definitions which will be needed in the rest of this paper.

Definition 2.1 Let X, Y be two topological vector spaces and let K be a nonempty subset of X. Let $W : K \to 2^Y$ be a point-to-set mapping. The graph of W, denoted by $\mathrm{Gr}(W)$, is

$$\mathrm{Gr}(W) = \{(x, z) \in K \times Y : x \in K, z \in W(x)\}.$$

Definition 2.2 Let X, Y be two topological vector spaces and let K be a nonempty convex subset of X. Let $C : K \to 2^Y$ be a point-to-set mapping such that for any $x \in K$, $C(x)$ is a pointed, closed and convex cone in Y with nonempty interior. Let $f : K \times K \to Y$ be a bifunction. For any given $x \in K$, the vector-valued function $y \to f(x, y)$ is said to be $C(x)$-convex if, for any $y_1, y_2 \in K$ and $t \in [0, 1]$,

$$f(x, ty_1 + (1-t)y_2) \in tf(x, y_1) + (1-t)f(x, y_2) - C(x).$$

Remark 2.1 It is easy to see that for any given $x \in K$, the vector-valued function $y \to f(x, y)$ is said to be $C(x)$-convex if and only if, for any $y_i \in K$ and $t_i \in [0, 1]$ $(i = 1, \ldots, n)$ with $\sum_{i=1}^n t_i = 1$,

$$f\left(x, \sum_{i=1}^n t_i y_i\right) \in \sum_{i=1}^n t_i f(x, y_i) - C(x).$$

Definition 2.3 (See [33]) Let X and Y be two locally convex Hausdorff topological vector spaces, let $C : X \to 2^Y$ be a point-to-set mapping such that, for any $x \in X$, $C(x)$ is a proper, pointed, closed and convex cone in Y with $\mathrm{int}\, C(x) \neq \emptyset$. Let $e : X \to Y$ be a vector-valued mapping such that, for any $x \in X$, $e(x) \in \mathrm{int}\, C(x)$. The nonlinear scalarization function $\xi_e : X \times Y \to R$ is defined as follows:

$$\xi_e(x, y) = \inf\{\lambda \in R : y \in \lambda e(x) - C(x)\}, \quad \forall (x, y) \in X \times Y.$$

If $e(x) = k^0$ for all $x \in X$, then the nonlinear scalarization function ξ_e reduces to the nonlinear scalarization function ξ_{k^0} introduced by Chen and Yang [35]. We observe that the original version of the nonlinear scalarization function was due to Gerstewitz [36].

The following results are very important properties of the nonlinear scalarization function ξ_e.

Lemma 2.1 (See [33]) *Let X and Y be two locally convex Hausdorff topological vector spaces, $C : X \to 2^Y$ a point-to-set mapping such that, for any $x \in X$, $C(x)$ is a proper, pointed, closed and convex cone in Y with $\operatorname{int} C(x) \neq \emptyset$. Let $e : X \to X$ be a vector-valued mapping such that, for any $x \in X$, $e(x) \in \operatorname{int} C(x)$. For each $\lambda \in R$ and $(x, y) \in X \times Y$, we have*

(i) $\xi_e(x, y) < \lambda \Leftrightarrow y \in \lambda e(x) - \operatorname{int} C(x)$;
(ii) $\xi_e(x, y) \leq \lambda \Leftrightarrow y \in \lambda e(x) - C(x)$;
(iii) $\xi_e(x, y) \geq \lambda \Leftrightarrow y \notin \lambda e(x) - \operatorname{int} C(x)$;
(iv) $\xi_e(x, y) > \lambda \Leftrightarrow y \notin \lambda e(x) - C(x)$;
(v) $\xi_e(x, y) = \lambda \Leftrightarrow y \in \lambda e(x) - \partial C(x)$,

where $\partial C(x)$ is the topological boundary of $C(x)$.

We also need the following lemma.

Lemma 2.2 (See [34]) *Let I be any index set. For each $i \in I$, let K_i be a nonempty convex subset of a Hausdorff topological vector space X_i. Let $G_i : K = \prod_{i \in I} K_i \to 2^{K_i}$ be a point-to-set mapping. Assume that the following conditions hold:*

(i) *For each $i \in I$ and any $x \in K$, $G_i(x)$ is convex.*
(ii) *For each $i \in I$ and any $x \in K$, $x_i \notin G_i(x)$, where x_i is the ith component of x.*
(iii) *For each $i \in I$ and any $y_i \in K_i$, $G_i^{-1}(y_i)$ is open in K.*
(iv) *There exists a nonempty compact subset $D \subseteq K$ and for each $i \in I$, there exists a nonempty compact and convex subset $E_i \subseteq K_i$ such that, $\forall x \in K \backslash D$, $\exists i \in I$ such that $G_i(x) \cap E_i \neq \emptyset$.*

Then, there exists $x^* \in K$ such that, for each $i \in I$, $G_i(x^*) = \emptyset$.

3 Gap Functions for the System of Vector Equilibrium Problems

In this section, we will consider a gap function for SVEP by using the nonlinear scalarization function introduced by Chen, Yang and Yu [33].

Definition 3.1 A function $p : K \to R$ is said to be a gap function for SVEP if it satisfies the following properties:

(i) $p(x) \leq 0$ for all $x \in K$;
(ii) $p(x^*) = 0$ if and only if $x^* \in E^S$.

Let us define two mappings $\phi_0 : K \times I \to R$ and $\phi : K \to R$, respectively, as follows:

$$\phi_0(x,i) = \sup_{y_i \in K_i} \{-\xi_{e_i}(x, f_i(x, y_i))\}$$

and

$$\phi(x) = \inf_{i \in I}\{-\phi_0(x,i)\}. \tag{1}$$

Theorem 3.1 *If for any $x \in K$ and each $i \in I$, $f_i(x, x_i) \in -\partial C_i(x)$, where x_i is the ith component of x, then the function $\phi(x)$ defined by (1) is a gap function for SVEP.*

Proof (i) Since for any $x \in K$ and each $i \in I$, $f_i(x, x_i) \in -\partial C_i(x)$, from Lemma 2.1(v) we obtain

$$\xi_{e_i}(x, f_i(x, x_i)) = 0.$$

It follows that

$$\phi_0(x,i) = \sup_{y_i \in K_i} \{-\xi_{e_i}(x, f_i(x, y_i))\} \geq 0,$$

and so

$$\phi(x) = \inf_{i \in I}\{-\phi_0(x,i)\} \leq 0, \quad \forall x \in K.$$

(ii) If $\phi(x^*) = 0$, then we obtain

$$\inf_{i \in I}\left\{- \sup_{y_i \in K_i} \{-\xi_{e_i}(x^*, f_i(x^*, y_i))\}\right\} = 0.$$

It follows that, for each $i \in I$,

$$- \sup_{y_i \in K_i} \{-\xi_{e_i}(x^*, f_i(x^*, y_i))\} \geq 0,$$

or equivalently,

$$\sup_{y_i \in K_i} \{-\xi_{e_i}(x^*, f_i(x^*, y_i))\} \leq 0,$$

which implies that, for any $y_i \in K_i$,

$$-\xi_{e_i}(x^*, f_i(x^*, y_i)) \leq 0,$$

or equivalently,

$$\xi_{e_i}(x^*, f_i(x^*, y_i)) \geq 0.$$

From Lemma 2.1(iii), we have that $f_i(x^*, y_i) \notin -\operatorname{int} C_i(x^*)$ for all $y_i \in K_i$ from which it follows that $x^* \in E^S$.

Conversely, if $x^* \in E^S$, then for each $i \in I$,

$$f_i(x^*, y_i) \notin -\operatorname{int} C_i(x^*), \quad \forall y_i \in K_i.$$

From Lemma 2.1(iii), we have

$$\xi_{e_i}(x^*, f_i(x^*, y_i)) \geq 0,$$

or equivalently,

$$-\xi_{e_i}(x^*, f_i(x^*, y_i)) \leq 0,$$

for any $y_i \in K_i$. Then,

$$\phi_0(x^*, i) = \sup_{y_i \in K_i} \{-\xi_{e_i}(x^*, f_i(x^*, y_i))\} \leq 0.$$

Thus, we have

$$-\phi_0(x^*, i) \geq 0$$

and therefore,

$$\phi(x^*) = \inf_{i \in I}\{-\phi_0(x^*, i)\} \geq 0. \tag{2}$$

Now, (i) and (2) imply

$$\phi(x^*) = 0.$$

This completes the proof. □

Corollary 3.1 *If for any $x \in K$ and each $i \in I$, $f_i(x, x_i) \in -\partial C_i(x)$, where x_i is the ith component of x, then $\{x^* : \phi(x^*) = 0\} = E^S$, where the function $\phi(x)$ is defined by* (1).

If the index set I is a singleton, then the mapping $\phi : K \to R$ defined by (1) reduces to the following:

$$\phi(x) = -\sup_{y \in K}\{-\xi_e(x, f(x, y))\}. \tag{3}$$

Consequently, we have the following result from Theorem 3.1.

Corollary 3.2 *If for any $x \in K$, $f(x, x) \in -\partial C(x)$, then the function $\phi(x)$ defined by* (3) *is a gap function for VEP and $\{x^* : \phi(x^*) = 0\} = E^0$.*

We observe that gap functions have been also discussed with a different point of view in [5, Sect. 6].

4 Existence of Solutions for the System of Vector Equilibrium Problems

In order to derive existence theorems for SVEP, we show first the following results.

Proposition 4.1 *If for each $i \in I$, we define $F_i : K_i \to 2^K$ such that*

$$F_i(y_i) = \{x^* \in K : \xi_{e_i}(x^*, f_i(x^*, y_i)) \geq 0\}, \quad \forall y_i \in K_i,$$

then

$$\bigcap_{i \in I} \bigcap_{y_i \in K_i} F_i(y_i) = E^S.$$

Proof Let $x^* \in \bigcap_{i\in I} \bigcap_{y_i \in_i} F_i(y_i)$. Then for each $i \in I$, $x^* \in \bigcap_{y_i \in K_i} F_i(y_i)$. That is, $x^* \in K$ and for each $i \in I$, $\xi_{e_i}(x^*, f_i(x^*, y_i)) \geq 0$ for all $y_i \in K_i$. Hence by Lemma 2.1(iii), we have

$$f_i(x^*, y_i) \notin -\operatorname{int} C_i(x^*), \quad \forall y_i \in K_i.$$

That is, $x^* \in E^S$.

Conversely, suppose that $x^* \in E^S$. Then, $x^* \in K$ and for each $i \in I$,

$$f_i(x^*, y_i) \notin -\operatorname{int} C_i(x^*), \quad \forall y_i \in K_i.$$

Again by Lemma 2.1(iii), we obtain

$$\xi_{e_i}(x^*, f_i(x^*, y_i)) \geq 0,$$

i.e., $x^* \in F_i(y_i)$ for all $y_i \in K_i$. Hence,

$$x^* \in \bigcap_{i \in I} \bigcap_{y_i \in K_i} F_i(y_i).$$

This completes the proof. □

Remark 4.1 If the index set I is a singleton in Proposition 4.1, then we can obtain the characterization of the solutions set E^0 of VEP via the nonlinear scalarization function.

Proposition 4.2 *Let the following assumptions hold*:

(i) *For each $i \in I$ and any $y_i \in K_i$, the vector-valued function, $x \mapsto f_i(x, y_i)$ is continuous.*
(ii) *For each $i \in I$ the point-to-set mapping $W_i : K \to 2^{Y_i}$ has closed graph in $K \times Y_i$ where $W_i(x) = Y_i \setminus (-\operatorname{int} C_i(x))$, $\forall x \in K$.*

Then, for each $i \in I$ and any given $y_i \in K_i$, $F_i(y_i)$ is either closed or empty, where the mapping $F_i : K_i \to 2^K$ is defined by

$$F_i(y_i) = \{x^* \in K : \xi_{e_i}(x^*, f_i(x^*, y_i)) \geq 0\}, \quad \forall y_i \in K_i.$$

Proof For each $i \in I$ and any given $y_i \in K_i$ if $F_i(y_i) \neq \emptyset$, then let $\{u_\alpha\} \subset F_i(y_i)$ be a net such that $u_\alpha \to u^*$. Then, $u_\alpha \in K$ and

$$\xi_{e_i}(u_\alpha, f_i(u_\alpha, y_i)) \geq 0.$$

Since K is closed, $u^* \in K$. By Lemma 2.1(iii), it follows that

$$f_i(u_\alpha, y_i) \notin -\operatorname{int} C_i(u_\alpha),$$

that is,

$$(u_\alpha, f_i(u_\alpha, y_i)) \in \operatorname{Gr}(W_i).$$

From assumption (i), we have $(u_n\alpha, f_i(u_\alpha, y_i)) \to (u^*, f_i(u^*, y_i))$. Since for each $i \in I$, $W_i(x) = Y_i \backslash (-\operatorname{int} C_i(x))$ for all $x \in K$ and since W_i has a closed graph in $K \times Y_i$, we obtain

$$(u^*, f_i(u^*, y_i)) \in \operatorname{Gr}(W_i),$$

that is,

$$f_i(u^*, y_i) \notin -\operatorname{int} C_i(u^*).$$

Again by Lemma 2.1(iii), we obtain

$$\xi_{e_i}(u^*, f_i(u^*, y_i)) \geq 0.$$

Hence $u^* \in F_i(y_i)$. Consequently, $F_i(y_i)$ is closed. This completes the proof. □

Let the index set I be a singleton in Proposition 4.2. Then, we have the following result.

Corollary 4.1 *Let the following assumptions hold*:

(i) *For any $y \in K$, the vector-valued function $x \mapsto f(x, y)$ is continuous.*
(ii) *The point-to-set mapping $W : K \to 2^Y$ has closed graph in $K \times Y$, where $W(x) = Y \backslash (-\operatorname{int} C(x))$, $\forall x \in K$.*

Then, for any given $y \in K$, $F(y)$ is either closed or empty, where the mapping $F : K \to 2^K$ is defined by

$$F(y) = \{x^* \in K : \xi_e(x^*, f(x^*, y)) \geq 0\}, \quad \forall y \in K.$$

We remark that taking into account Lemma 2.1 (in the single case), the closedness of the set $F(y)$ in Corollary 4.1 already appeared in [15, 37, 38].

We state and prove now the main result of this section.

Theorem 4.1 *Let the following assumptions hold*:

(i) *For each $i \in I$ and any $x \in K$, $\xi_{e_i}(x, f_i(x, x_i)) \geq 0$ where x_i is the ith component of x.*
(ii) *For each $i \in I$ and any $x \in K$, the vector-valued function $y_i \to f_i(x, y_i)$ is $C_i(x)$-convex.*
(iii) *For each $i \in I$ and any $y_i \in K_i$, the vector-valued function $x \mapsto f_i(x, y_i)$ is continuous.*
(iv) *For each $i \in I$, the point-to-set mapping $W_i : K \to 2^{Y_i}$ has a closed graph in $K \times Y_i$ where $W_i(x) = Y_i \backslash (-\operatorname{int} C_i(x))$, $\forall x \in K$.*

(v) *There exists a nonempty compact subset $D \subseteq K$ and, for each $i \in I$, there exists a nonempty compact and convex subset $E_i \subseteq K_i$ such that, for each $x \in K \backslash D$, there exist $i \in I$ and $\overline{y_i} \in E_i$ such that*

$$\xi_{e_i}(x, f_i(x, \overline{y_i})) < 0.$$

Then,

$$\bigcap_{i \in I} \bigcap_{y_i \in K_i} F_i(y_i) \neq \emptyset,$$

that is, $E^S \neq \emptyset$, where the mapping $F_i : K_i \to 2^K$ is defined by

$$F_i(y_i) = \{x^* \in K : \xi_{e_i}(x^*, f_i(x^*, y_i)) \geq 0\}, \quad \forall y_i \in K_i.$$

Furthermore, E^S is compact.

Proof For each given $i \in I$, we define a point-to-set mapping $G_i : K \to 2^{K_i}$ by

$$G_i(x) = \{y_i \in K_i : \xi_{e_i}(x, f_i(x, y_i)) < 0\}, \quad \forall x \in K.$$

Then, for each $i \in I$ and any $x \in K$, $G_i(x)$ is convex. To show this, for each given $i \in I$ and any given $x \in K$, let $y_{i_1}, y_{i_2} \in G_i(x)$ and $\lambda \in (0, 1)$. Then, we obtain

$$\xi_{e_i}(x, f_i(x, y_{i_j})) < 0, \quad j = 1, 2.$$

For each $i \in I$ and any $x \in K$, since the vector-valued function $y_i \mapsto f_i(x, y_i)$ is C_i-convex,

$$f_i(x, \lambda y_{i_1} + (1-\lambda) y_{i_2}) \in \lambda f_i(x, y_{i_1}) + (1-\lambda) f_i(x, y_{i_2}) - C_i(x).$$

For each $i \in I$ and any $x \in K$, let the vector-valued function $y_i \mapsto \xi_{e_i}(x, y_i)$ be positively homogeneous, monotone and subadditive (see Propositions 2.4 and 2.5 in [33]). It follows that

$$\begin{aligned}
\xi_{e_i}(x, f_i(x, \lambda y_{i_1} + (1-\lambda) y_{i_2})) &\leq \xi_{e_i}(x, \lambda f_i(x, y_{i_1}) + (1-\lambda) f_i(x, y_{i_2})) \\
&\leq \xi_{e_i}(x, \lambda f_i(x, y_{i_1})) + \xi_{e_i}(x, (1-\lambda) f_i(x, y_{i_2})) \\
&= \lambda \xi_{e_i}(x, f_i(x, y_{i_1})) + (1-\lambda) \xi_{e_i}(x, f_i(x, y_{i_2})) \\
&< 0,
\end{aligned}$$

that is, $\lambda y_{i_1} + (1-\lambda) y_{i_2} \in G_i(x)$, and so $G_i(x)$ is convex. Considering conditions (i), (iii) and (iv), from Proposition 4.2, we have that, for each $i \in I$ and any $y_i \in K_i$, the set

$$\begin{aligned}
K \backslash [G_i^{-1}(y_i)] &= F_i(y_i) \\
&= \{x \in K : \xi_{e_i}(x, f_i(x, y_i)) \geq 0\}
\end{aligned}$$

is closed in K and so $G_i^{-1}(y_i)$ is open in K. By condition (i), it follows that, for each $i \in I$ and any $x \in K$, $x_i \notin G_i(x)$. From condition (v), we have that there exists a nonempty compact subset $D \subseteq K$ and for each $i \in I$, there exists a nonempty compact and convex subset $E_i \subseteq K_i$, such that for each $x \in K \backslash D$, there exists $i \in I$ such that $G_i(x) \cap E_i \neq \emptyset$. Thus by Lemma 2.2, there exists $x^* \in K$ such that for each $i \in I$, $G_i(x^*) = \emptyset$, that is, for each $i \in I$, $x^* \in \bigcap_{y_i \in K_i} F_i(y_i)$. Hence, $x^* \in \bigcap_{i \in I} \bigcap_{y_i \in K_i} F_i(y_i) = E^S$. Furthermore, from condition (v), we obtain that every point outside D is not a solution of SVEP, i.e., $E^S \subseteq D$ and so E^S is compact. The proof is complete. □

We note that condition (v) is related to condition (∗) in [38]. It is worth noting that, in Theorem 4.1, there is no generalized monotonicity assumption, which was imposed in [15, 37, 38]. The reason is that, in [15, 37, 38], the KKM theory approach was employed, which is not the case in this paper.

Remark 4.2 Theorem 4.1 extended Theorem 2.2 in [39] for a moving cone. But the proof of Theorem 4.1 is different from the proof of Theorem 2.2 in [39].

Example 4.1 Let $X_1 = X_2 = Y_1 = Y_2 = R$ and let $K_1 = K_2 = [0, 2]$. Then, $X = Y = R^2$ and $K = [0, 2] \times [0, 2]$. Let $C_1(x) = C_2(x) = [0, +\infty)$, $e_1(x) = x_1x_2 + 1$ and $e_2(x) = x_1x_2 + 2$ for all $x = (x_1, x_2) \in K$. Then for any $x \in K$, $e_i(x) \in \operatorname{int} C_i(x)$, $i = 1, 2$. For $i = 1, 2$, we define $f_i : K \times K_i \to Y_i$ by

$$f_1(x, y_1) = x_2 + 2y_1 - 2x_1, \quad \forall (x, y_1) \in K \times K_1,$$

and

$$f_2(x, y_2) = x_1 + 2y_2 - 2x_2, \quad \forall (x, y_2) \in K \times K_2,$$

where $x = (x_1, x_2) \in K$. Then it is easy to verify that conditions (i)–(iv) of Theorem 4.1 hold. Furthermore, if set $D = [0, 1] \times [0, 1]$, $E_1 = E_2 = [0, 2]$, then we can prove that condition (v) of Theorem 4.1 holds. By Lemma 2.1(iii), we have

$$\bigcap_{y_1 \in K_1} F_1(y_1) = \{x = (x_1, x_2) \in K : x_2 + 2y_1 - 2x_1 \geq 0, \ \forall y_1 \in K_1\}$$

$$= \left\{(x_1, x_2) : 0 \leq x_2 \leq 2, 0 \leq x_1 \leq \frac{x_2}{2}\right\},$$

and

$$\bigcap_{y_2 \in K_2} F_2(y_2) = \{x = (x_1, x_2) \in K : x_1 + 2y_2 - 2x_2 \geq 0, \ \forall y_2 \in K_2\}$$

$$= \left\{(x_1, x_2) : 0 \leq x_1 \leq 2, 0 \leq x_2 \leq \frac{x_1}{2}\right\},$$

and so

$$\bigcap_{i=1}^{2} \bigcap_{y_i \in K_i} F_i(y_i) = E^S = \{(0, 0)\}.$$

If the index set I is a singleton in Theorem 4.1, then we have the following result.

Corollary 4.2 *Let the following assumptions hold*:

(i) *For any* $x \in K$, $\xi_e(x, f(x,x)) \geq 0$.
(ii) *For any* $x \in K$, *the vector-valued function* $y \to f(x,y)$ *is* $C(x)$*-convex.*
(iii) *For any* $y \in K$, *the vector-valued function* $x \mapsto f(x,y)$ *is continuous.*
(iv) *The point-to-set mapping* $W : K \to 2^Y$ *has a closed graph in* $K \times Y$, *where* $W(x) = Y \backslash (-\operatorname{int} C(x))$, $\forall x \in K$.
(v) *There exists a nonempty compact and convex subset* $D \subseteq K$, *such that* $\forall x \in K \backslash D$, $\exists \bar{y} \in D$ *such that*

$$\xi_e(x, f(x, \overline{y})) < 0.$$

Then,

$$\bigcap_{y \in K} F(y) \neq \emptyset,$$

that is, $E^0 \neq \emptyset$, *where the mapping* $F : K \to 2^K$ *is defined by*

$$F(y) = \{x^* \in K : \xi_e(x^*, f(x^*, y)) \geq 0\}, \quad \forall y \in K.$$

Furthermore, E^0 *is compact.*

References

1. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)
2. Schaible, S., Yao, J.C., Zeng, L.C.: A proximal method for pseudomonotone type variational-like inequalities. Taiwan. J. Math. **10**, 497–513 (2006)
3. Zeng, L.C., Lin, L.J., Yao, J.C.: Auxiliary problem method for mixed variational-like inequalities. Taiwan. J. Math. **10**, 515–529 (2006)
4. Chadli, O., Wong, N.C., Yao, J.C.: Equilibrium problems with applications to eigenvalue problems. J. Optim. Theory Appl. **117**, 245–266 (2003)
5. Flores-Bazán, F.: Existence theory for finite-dimensional pseudomonotone equilibrium problems. Acta Appl. Math. **77**, 249–297 (2003)
6. Giannessi, F. (ed.): Vector Variational Inequalities and Vector Equilibrium Problems. Kluwer Academic, Dordrecht (2000)
7. Isac, G., Bulavski, V.A., Kalashnikov, V.V.: Complementarity, Equilibrium, Efficiency and Economics. Kluwer Academic, Dordrecht (2002)
8. Giannessi, F.: Theorem of alternative, quadratic programs, and complementarity problems. In: Cottle, R.W., Giannessi, F., Lions, J.L. (eds.) Variational Inequality and Complementarity Problems, pp. 151–186. Wiley, Chichester (1980)
9. Ansari, Q.H., Schaible, S., Yao, J.C.: The system of generalized vector equilibrium problems with applications. J. Glob. Optim. **22**, 3–16 (2002)
10. Ansari, Q.H., Yao, J.C.: An existence result for generalized vector equilibrium problem. Appl. Math. Lett. **12**, 53–56 (1999)
11. Bianchi, M., Hadjisavvas, N., Schaible, S.: Vector equilibrium problems with generalized monotone bifunctions. J. Optim. Theory Appl. **92**, 527–542 (1997)
12. Ding, X.P., Yao, J.C., Lin, L.J.: Solutions of system of generalized vector quasi-equilibrium problems in locally G-convex uniform spaces. J. Math. Anal. Appl. **298**, 398–410 (2004)
13. Fang, Y.P., Huang, N.J.: Existence results for systems of strongly implicit vector variational inequalities. Acta Math. Hung. **103**, 265–277 (2004)
14. Fang, Y.P., Huang, N.J.: Vector equilibrium type problems with $(S)_+$-conditions. Optimization **53**, 269–279 (2004)

15. Hadjisavvas, N., Schaible, S.: From scalar to vector equilibrium problems in the quasimonotone case. J. Optim. Theory Appl. **96**, 297–309 (1998)
16. Huang, N.J., Gao, C.J.: Some generalized vector variational inequalities and complementarity problems for multivalued mappings. Appl. Math. Lett. **16**, 1003–1010 (2003)
17. Huang, N.J., Li, J., Thompson, H.B.: Implicit vector equilibrium problems with applications. Math. Comput. Model. **37**, 1343–1356 (2003)
18. Li, J., Huang, N.J., Kim, J.K.: On implicit vector equilibrium problems. J. Math. Anal. Appl. **283**, 501–512 (2003)
19. Li, S.J., Teo, K.L., Yang, X.Q.: Generalized vector quasiequilibrium problems. Math. Methods Oper. Res. **61**, 385–397 (2005)
20. Ansari, Q.H., Scahible, S., Yao, J.C.: Vector quasivariational inequalities over product spaces. J. Global Optim. **32**, 437–449 (2005)
21. Chadli, O., Yang, X.Q., Yao, J.C.: On generalized vector prevariational and prequasivariational inequalities. J. Math. Anal. Appl. **295**, 392–403 (2004)
22. Giannessi, F. (ed.): Vector Variational Inequalities and Vector Equilibria. Mathematical Theories. Kluwer Academic, Dordrecht (2000)
23. Tan, N.X., Tinh, P.N.: On the existence of equilibrium points of vector functions. Numer. Funct. Anal. Optim. **19**, 141–156 (1998)
24. Ansari, Q.H., Oettli, W., Schläger, D.: A generalization of vector equilibria. Math. Methods Oper. Res. **46**, 147–152 (1997)
25. Fu, J.: Simultaneous vector variational inequalities and vector implicit complementarity problem. J. Optim. Theory Appl. **93**, 141–151 (1997)
26. Lee, G.M., Kim, D.S., Lee, B.S.: On noncooperative vector equilibrium. Indian J. Pure Appl. Math. **27**, 735–739 (1996)
27. Zeng, L.C., Wong, N.C., Yao, J.C.: Convergence of hybrid steepest-descent methods for generalized variational inequalities. Acta Math. Sinica, Engl. Ser. **22**, 1–12 (2006)
28. Konnov, I.V., Schaible, S., Yao, J.C.: Combined relaxation method for mixed equilibrium problems. J. Optim. Theory Appl. **126**, 309–322 (2005)
29. Yang, X.Q., Yao, J.C.: Gap functions and existence of solutions to set-valued vector variational inequalities. J. Optim. Theory Appl. **115**, 407–417 (2002)
30. Mastroeni, G.: Gap functions for equilibrium problems. J. Glob. Optim. **27**, 411–426 (2003)
31. Li, J., He, Z.Q.: Gap functions and existence of solutions to generalized vector variational inequalities. Appl. Math. Lett. **18**, 989–1000 (2005)
32. Yang, X.Q.: On the gap functions of prevariational inequalities. J. Optim. Theory Appl. **116**, 437–452 (2003)
33. Chen, G.Y., Yang, X.Q., Yu, H.: A nonlinear scalarization function and generalized quasi-vector equilibrium problems. J. Glob. Optim. **32**, 451–466 (2005)
34. Deguire, P., Tan, K.K., Yuan, G.X.Z.: The study of maximal elements, fixed points for L_S majorized mappings, and their applications to minimax and variational inequalities in the product topological spaces. Nonlinear Anal. Theory, Methods, Appl. **37**, 933–951 (1999)
35. Chen, G.Y., Yang, X.Q.: Characterizations of variable domination structures via nonlinear scalarization. J. Optim. Theory Appl. **112**, 97–110 (2002)
36. Gerstewitz, C.T.: Nonconvex duality in vector optimization. Wissensch. Z., Technische Hochschule Leuna-Mersebung **25**, 357–364 (1983) (in German)
37. Oettli, W.: A remark on vector-valued equilibria and generalized monotonicity. Acta Math. Vietnam. **22**, 213–221 (1997)
38. Flores-Bazán, F., Flores-Bazán, F.: Vector equilibrium problems under asymptotic analysis. J. Glob. Optim. **26**, 141–166 (2003)
39. Ansari, Q.H., Schaible, S., Yao, J.C.: System of vector equilibrium problems and its applications. J. Optim. Theory Appl. **107**, 547–557 (2000)