J Optim Theory Appl (2007) 133: 77–97
DOI 10.1007/s10957-007-9193-1

D-Gap Functions for Nonsmooth Variational Inequality Problems

K.F. Ng · L.L. Tan

Published online: 20 April 2007

Abstract We study the Clarke generalized gradient of the D-gap functions for the variational inequality problem (VIP) defined by a locally Lipschitz, but not necessarily differentiable, function in an Euclidean space. Using these results, we study the relationship between minimizing sequences and stationary sequences of the D-gap function, regardless of the existence of solutions of (VIP).

Keywords Variational inequality problems · D-gap functions · Minimizing sequences · Stationary sequences

1 Introduction

Throughout this paper let P denote a nonempty closed convex set in an n-dimensional Euclidean space $\Re^n$ and let F be a locally Lipschitz function from $\Re^n$ to itself. We consider VIP(F, P), the variational inequality problem associated with F and P, that is, to find a vector $x^* \in P$ such that

$$F(x^*)^T(x - x^*) \geq 0, \quad \forall x \in P. \tag{1}$$

When P is the nonnegative orthant in $\Re^n$, VIP(F, P) reduces to the nonlinear complementarity problem NCP(F). We refer the reader to the book [1] by Facchinei and Pang for the background information and motivations of the variational inequality problems covering both smooth and nonsmooth functions. In fact nonsmooth variational problems are quite abundant, see [2–4] for recent developments. Below, let us only mention explicitly one of the simplest examples. Consider a Cournot oligopoly

Communicated by M. Fukushima.

K.F. Ng · L.L. Tan (✉)
Department of Mathematics, Chinese University of Hong Kong, Shatin, New Territories, Hong Kong
e-mail: lltan@math.cuhk.edu.hk

problem with two firms with the payoff function $\theta_f : \Re \times \Re \to \Re$ ($f = 1, 2$) being given by

$$\theta_f((q_1, q_2)) \equiv p(Q)q_f - c_f(q_f),$$

where q_f and $c_f(q_f) = w_f q_f$ ($w_f \in \Re$) are respectively the production of firm f and the production cost function, $Q = q_1 + q_2$ is the total production by the two firms, and the market inverse demand function p is given by

$$p(Q) := \begin{cases} (Q-a)^2, & \text{if } 0 \le Q \le c, \\ 2(c-a)Q + (c-a)^2 - 2(c-a)c, & \text{if } Q \ge c \end{cases}$$

with $a > c > 0$. Note that p is once but not twice continuously differentiable, and thus $\nabla_{q_f}\theta_f((q_1, q_2))$ is piecewise smooth for any f. By [3, Proposition 1], a Nash equilibrium (for the above problem) q^N is a solution of a 2-dimensional VIP(F, P) with nonsmooth function

$$F((q_1, q_2)) = -(\nabla_{q_1}\theta_1((q_1, q_2)), \nabla_{q_2}\theta_2((q_1, q_2)))$$

and a production set P in $\Re^2$.

In the last few decades, many methods have been proposed for solving VIP; see [5–25]; among them, one efficient method is based on a merit function see [7]. In this paper, we focus on two special merit functions, namely the regularized gap function f_α [6, 23] and the D-gap function $g_{\alpha\beta}$ [20, 25] respectively defined by

$$f_\alpha(\tau) := \sup_{x \in P} \Psi_\alpha(x, \tau), \quad \tau \in \Re^n, \ \alpha > 0, \tag{2}$$

$$g_{\alpha\beta}(\tau) := f_\alpha(\tau) - f_\beta(\tau), \quad \tau \in \Re^n, \ \beta > \alpha > 0, \tag{3}$$

where

$$\Psi_\alpha(x, \tau) = F(\tau)^T(\tau - x) - \alpha\phi(x, \tau) \tag{4}$$

and $\phi : \Re^n \times \Re^n \to [0, +\infty)$ is a generalized distance function, namely a continuously differentiable function satisfying the following properties:

(C1) $\phi(x, \tau) = 0$ if and only if $x = \tau$;
(C2) the family $\{\phi(\cdot, \tau);\ \tau \in \Re^n\}$ is uniformly strongly convex with modulus $\zeta > 0$:

$$\phi(x'', \tau) - \phi(x', \tau) \ge \nabla_1\phi(x', \tau)^T(x'' - x') + \zeta \parallel x'' - x' \parallel^2, \quad \forall \tau, x', x'' \in \Re^n; \tag{5}$$

(C3) $\{\nabla_1\phi(\cdot, \tau)\}$ is uniformly Lipschitz continuous on $\Re^n$ with modulus $K > 0$:

$$\parallel \nabla_1\phi(x'', \tau) - \nabla_1\phi(x', \tau) \parallel \le K \parallel x'' - x' \parallel, \quad \forall \tau, x', x'' \in \Re^n; \tag{6}$$

(C4) $\nabla_1\phi(x, \tau) = -\nabla_2\phi(x, \tau), \quad \forall x, \tau \in \Re^n$,

where $\nabla_1\phi$ and $\nabla_2\phi$ respectively denote the partial derivatives of ϕ with respect to the first and the second variable of ϕ. For example, the functions ϕ and ψ respectively defined by

$$\phi(x,\tau)=\frac{\|x-\tau\|^2}{2} \tag{7}$$

and

$$\psi(x,\tau)=\frac{\|x-\tau\|^2}{2}+\frac{\sin^2(\tau_1-x_1)}{4},\quad \text{for each } x,\tau\in\Re^n,$$

where τ_1, x_1 denote respectively the first coordinate of τ and x, satisfy (C1)–(C4) with $K=1$ and $\zeta=\frac{1}{2}$ (see [10]) for ϕ and $K=\frac{3}{2}$ and $\zeta=\frac{1}{4}$ for ψ. By (C4), the formulas (6) and (5) can be equivalently written as

$$\|\nabla_2\phi(x',\tau)-\nabla_2\phi(x'',\tau)\|\le K\|x'-x''\|,\quad \forall\tau,x',x''\in\Re^n, \tag{8}$$

and

$$\phi(x'',\tau)\ge\phi(x',\tau)-\nabla_2\phi(x',\tau)^T(x''-x')+\zeta\|x''-x'\|^2,\quad \forall\tau,x',x''\in\Re^n. \tag{9}$$

It is known [23, 25] that x^* solves VIP(F,P) if and only if $f_\alpha(x^*)=0$ and x^* solves the constrained minimization problem

$$\min f_\alpha(\tau),\quad \tau\in P$$

and this is the case if and only if $g_{\alpha\beta}(x^*)=0$ and x^* solves the unconstrained minimization problem

$$\min g_{\alpha\beta}(\tau),\quad \tau\in\Re^n. \tag{10}$$

Moreover, if F is assumed to be continuously differentiable, then f_α and $g_{\alpha\beta}$ are also continuously differentiable; in fact one has [25]

$$\nabla f_\alpha(\tau)=-\nabla F(\tau)(\pi_\alpha(\tau)-\tau)+F(\tau)-\alpha\nabla_2\phi(\pi_\alpha(\tau),\tau) \tag{11}$$

and

$$\nabla g_{\alpha\beta}(\tau)=\nabla F(\tau)(\pi_\beta(\tau)-\pi_\alpha(\tau))+\beta\nabla_2\phi(\pi_\beta(\tau),\tau)-\alpha\nabla_2\phi(\pi_\alpha(\tau),\tau), \tag{12}$$

where $\pi_\alpha(\tau)$ and $\pi_\beta(\tau)$ denote respectively the unique maximizer of $\Psi_\alpha(\cdot,\tau)$ and $\Psi_\beta(\cdot,\tau)$ over P (noting that $\Psi_\gamma(\cdot,\tau)$ $(\gamma>0,\tau\in P)$ is strongly concave, it has a unique maximizer on P [23, 25]). In this smooth situation, Fukushima and Pang provided in [8] conditions under which minimizing sequences and stationary sequences of the D-gap function $g_{\alpha\beta}$ (where ϕ is defined by (7)) can be related. In this paper we provide further conditions to do the same (for a generalized distance function ϕ). Examples are given to illustrate that, even when F is smooth, our results can be applied while the earlier results cannot. Explicitly we prove in Sect. 5 the following results.

Theorem 1.1 *Let $F:\Re^n\to\Re^n$ be locally Lipschitz and let $\beta>\alpha>0$. Let $(\tau_k)\subset\Re^n$ be a stationary sequence (see Definition 2.1(b)) of $g_{\alpha\beta}$ such that*

$$\bigcup_{k=1}^{+\infty}\partial F(\tau_k)\quad\text{and}\quad(g_{\alpha\beta}(\tau_k))\quad\text{are bounded.} \tag{13}$$

Suppose further that either

(i) $\lim_{k\to+\infty}(\pi_\beta(\tau_k)-\pi_\alpha(\tau_k))=0$ *or*
(ii) *For any* $\mu > g_{\inf} := \inf_{\tau\in\Re^n} g_{\alpha\beta}(\tau)$, *there exist a scalar* $r>0$ *and a function* $\delta:(0,+\infty)\to(0,+\infty)$ *with the property*

$\liminf_{t\to t^*}\delta(t)>0 \quad \text{for all } t^*>0$

such that, for all k *and all* $\eta_k\in\partial F(\tau_k)$,

$$(\pi_\beta(\tau_k)-\pi_\alpha(\tau_k))^T\eta_k(\pi_\beta(\tau_k)-\pi_\alpha(\tau_k))\ge\delta([g_{\alpha\beta}(\tau_k)-\mu]_+)\|\pi_\beta(\tau_k)-\pi_\alpha(\tau_k)\|^r,$$

where $[g_{\alpha\beta}(\tau_k)-\mu]_+ := \max\{g_{\alpha\beta}(\tau_k)-\mu,0\}$.

Then, (τ_k) *is a minimizing sequence* (*see Definition* 2.1(a)) *of* $g_{\alpha\beta}$.

Theorem 1.2 *Let* $F:\Re^n\to\Re^n$ *be locally Lipschitz and let* (τ_k) *be a minimizing sequence of* $g_{\alpha\beta}$. *Suppose that either*

(i) $g_{\inf}=0$ *and there exists* $\eta_k\in\partial F(\tau_k)$ *for each* k *such that* (η_k) *is bounded or*
(ii) $\cup_{k=1}^{+\infty}\partial F(\tau_k)$ *is bounded and* ∂F *is uniformly upper semicontinuous near* (τ_k) (*see Definition* 2.2).

Then, (τ_k) *is a stationary sequence of* $g_{\alpha\beta}$.

The paper is organized as follows. In Sect. 2 we give the notations used in this paper and present some preliminary results. In Sect. 3, in order to deal with the unconstrained minimization problem such as that in (10), we give an upper estimate of the Clarke generalized gradient of a function of the form $f-g$ where f, g are max-functions. This general result is then applied in Sect. 4 to obtain a formula for the Clarke generalized gradient of $g_{\alpha\beta}$. Using the results obtained in the earlier sections, Theorems 1.1 and 1.2 are proved in Sect. 5.

2 Notations and Preliminary Results

Let $h:\Re^n\to\Re$ be a function and $x\in\Re^n$. We suppose that h is Lipschitz near x. Recall that the Clarke generalized gradient of h at x [26] is defined by

$$\partial h(x) := \{\xi\in\Re^n : \xi^T v\le h^o(x,v),\ \text{for all } v \text{ in } \Re^n\},$$

where $h^o(x,v)$ denotes the generalized directional derivative of h at x along the direction v and is defined by

$$h^o(x,v) := \limsup_{y\to x, t\downarrow 0}\frac{h(y+tv)-h(y)}{t}. \tag{14}$$

It is well known (cf. [26, Proposition 2.12]) that $\partial h(x)\ne\emptyset$ and

$$h^o(x,v)=\max\{\xi^T v:\xi\in\partial h(x)\}\quad \forall v\in\Re^n, \tag{15}$$

that is, $h^o(x,\cdot)$ is the support function of $\partial h(x)$ [26].

Let $G=(f_1(\tau), f_2(\tau), \ldots, f_m(\tau)) : \Re^n \to \Re^m$, where each $f_i : \Re^n \to \Re$ is assumed to be a locally Lipschitz function. Let Ω_G denote the set of points at which G fails to be differentiable and let $\tau \in \Re^n$. The generalized Jacobian of G at τ, denoted by $\partial G(\tau)$, is the convex hull of the set consisting of those $m \times n$ matrices, each of which is the limit of a sequence of the form $\nabla G(\tau_i)$, where $\tau_i \to \tau$ and $\tau_i \notin \Omega_G$. Symbolically, as in [26],

$$\partial G(\tau) := \mathrm{co}\{\lim \nabla G(\tau_i) : \tau_i \to \tau, \tau_i \notin \Omega_G\}. \tag{16}$$

Definition 2.1 Let $\theta : \Re^n \to \Re$ with $\theta_{\inf} = \inf\{\theta(\tau) : \tau \in \Re^n\} \geq -\infty$. A sequence $(\tau_k) \subset \Re^n$ is said to be (a) minimizing if $\lim_{k\to\infty} \theta(\tau_k) = \theta_{\inf}$ and (b) stationary if there exists a sequence (ξ_k) convergent to 0 such that $\xi_k \in \partial\theta(\tau_k)$ for each k.

In general, we use B_X and $\overline{B}_X$ to denote respectively the open and closed unit balls in a normed space X. In the special case when $X = \Re^n$, we denote B_X and $\overline{B}_X$ by B and $\overline{B}$ respectively.

Definition 2.2 (See [8, 27, 28]) Let X and Y be normed spaces and let $T : X \rightrightarrows Y$ be a multifunction. Let (τ_k) be a sequence in X. We say that T is uniformly upper semicontinuous near (τ_k) if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$T(y) \subset T(\tau_k) + \varepsilon \overline{B}_Y,$$

for all large enough k and for all $y \in \tau_k + \delta \overline{B}_X$.

For single-valued functions, one has the terminology "uniformly continuous" in place of "upper uniformly semicontinuous"; that is, $G : X \to Y$ is uniformly continuous near a sequence (τ_k) if and only if, for any $\varepsilon > 0$, there exists $\delta > 0, m$ such that, for all $k \geq m$, $y \in \tau_k + \delta \overline{B}_X$, one has

$$\|G(y) - G(\tau_k)\| \leq \varepsilon.$$

Proposition 2.1 *Let $G : \Re^n \to \Re^n$ be a locally Lipschitz function and let (τ_k) be a sequence in $\Re^n$. Suppose that ∂G is uniformly upper semicontinuous near (τ_k) and $\cup_{k=1}^{+\infty} \partial G(\tau_k)$ is bounded. Then, G is uniformly continuous near the sequence (τ_k).*

Proof By the given assumption, take $M > 0$ such that $\|\xi\| \leq M$ for any k and for any $\xi \in \partial G(\tau_k)$. Let $\varepsilon > 0$. Since ∂G is uniformly upper semicontinuous near (τ_k), there exist $\delta > 0$ with $\delta < \min\{\frac{\varepsilon}{M}, 1\}$ and a natural number m such that

$$\partial G(y) \subset \partial G(\tau_k) + \varepsilon \overline{B}_{\Re^{n\times n}}, \quad \forall y \in \tau_k + \delta \overline{B} \text{ and } k \geq m.$$

Since $\partial G(\tau_k)$ is convex, it follows that

$$\mathrm{co}\{\partial G(z) : z \in [\tau_k, y]\} \subseteq \partial G(\tau_k) + \varepsilon \overline{B}_{\Re^{n\times n}}, \quad \forall y \in \tau_k + \delta \overline{B} \text{ and } k \geq m.$$

Consequently, by the mean-value theorem [11, 12] for locally Lipschitz functions, we have that, for all $k \geq m$ and $y \in \tau_k + \delta \overline{B}$,

$$\begin{aligned} G(y) - G(\tau_k) &\in \mathrm{co}\{\xi^T(y-\tau_k) : \xi \in \partial G(z) \text{ for some } z \in [\tau_k, y]\} \\ &\subseteq \{\xi^T(y-\tau_k) : \xi \in \partial G(\tau_k) + \varepsilon \overline{B}_{\Re^{n\times n}}\} \end{aligned}$$

and so

$$\|G(y) - G(\tau_k)\| \le (M + \varepsilon)\|y - \tau_k\| < 2\varepsilon,$$

by the given property of M. This proves the proposition. □

For the remainder of this section, we assume that $\phi : \Re^n \times \Re^n \to [0, +\infty)$ is a continuously differentiable function satisfying the properties (C1)–(C4) with constants K, ζ. For any $\alpha > 0$, let Ψ_α be defined by (4). Below, we collect some elementary results on VIP which will be useful for the discussion in subsequent sections. In particular, the first assertion in Lemma 2.1 is due to Yamashita, Taji and Fukushima [25], while the other assertions follow immediately from the first assertion and the property (C3). Lemma 2.2 is taken from [10]. Lemma 2.4(i) is known [23, 25] for the special case when F is continuously differentiable. Similarly, Lemma 2.3, Propositions 2.2 and 2.3 extend the corresponding results in [8] (we allow a nonsmooth function F and a generalized distance function $\phi(x, \tau)$ in place of $\frac{\|x-\tau\|^2}{2}$).

Lemma 2.1 *There exists a continuously differentiable function $\psi : \Re^n \to [0, +\infty)$ such that*

$$\phi(x, \tau) = \psi(\tau - x), \quad \forall x, \tau \in \Re^n. \tag{17}$$

Consequently,

$$\nabla_1\phi(-\tau, -x) = -\nabla\psi(\tau - x) = -\nabla_2\phi(x, \tau) \tag{18}$$

and the family $\{\nabla_2\phi(x, \cdot)\}$ is uniformly Lipschitz continuous on $\Re^n$ with modulus K,

$$\|\nabla_2\phi(x, \tau_1) - \nabla_2\phi(x, \tau_2)\| \le K\|\tau_1 - \tau_2\|, \quad \forall \tau_1, \tau_2, x \in \Re^n. \tag{19}$$

Lemma 2.2 *The following statements hold for all $x, \tau \in \Re^n$:*

(i) $\|\nabla_1\phi(x, \tau)\| \le K\|x - \tau\|$;
(ii) $\zeta\|x - \tau\|^2 \le \phi(x, \tau) \le (K - \zeta)\|x - \tau\|^2$;
(iii) $\phi(x, \tau) - \nabla_1\phi(x, \tau)^T(x - \tau) \ge -(K - \zeta)\|x - \tau\|^2$.

Recalling (2), (4) and (3), we have that, for $\beta > \alpha > 0$,

$$\begin{aligned} f_\alpha(\tau) &= \max\{\Psi_\alpha(x, \tau) : x \in P\} \\ &= \Psi_\alpha(\pi_\alpha(\tau), \tau) \\ &= F(\tau)^T(\tau - \pi_\alpha(\tau)) - \alpha\phi(\pi_\alpha(\tau), \tau) \end{aligned} \tag{20}$$

and

$$\begin{aligned} g_{\alpha\beta}(\tau) &= f_\alpha(\tau) - f_\beta(\tau) \\ &= \Psi_\alpha(\pi_\alpha(\tau), \tau) - \Psi_\beta(\pi_\beta(\tau), \tau) \\ &= F(\tau)^T(\pi_\beta(\tau) - \pi_\alpha(\tau)) - \alpha\phi(\pi_\alpha(\tau), \tau) + \beta\phi(\pi_\beta(\tau), \tau). \end{aligned} \tag{21}$$

Lemma 2.3 *Let $\tau \in \Re^n$ and $\beta > \alpha > 0$. Then,*

$$(\beta - \alpha)\phi(\pi_\beta(\tau), \tau) \le g_{\alpha\beta}(\tau) \le (\beta - \alpha)\phi(\pi_\alpha(\tau), \tau) \tag{22}$$

and

$$\zeta(\beta - \alpha)\|\tau - \pi_\beta(\tau)\|^2 \le g_{\alpha\beta}(\tau) \le (K - \zeta)(\beta - \alpha)\|\tau - \pi_\alpha(\tau)\|^2. \tag{23}$$

Proof Clearly (23) follows from (22) and Lemma 2.2(ii). To prove (22), we note by the definitions that

$$\begin{aligned} g_{\alpha\beta}(\tau) &\ge \Psi_\alpha(\pi_\beta(\tau), \tau) - \Psi_\beta(\pi_\beta(\tau), \tau) \\ &= -\alpha\phi(\pi_\beta(\tau), \tau) + \beta\phi(\pi_\beta(\tau), \tau). \end{aligned}$$

Similarly, one can show that

$$g_{\alpha\beta}(\tau) \le (\beta - \alpha)\phi(\pi_\alpha(\tau), \tau).$$

Thus, (22) holds. □

Lemma 2.4 *Let $\beta > \alpha > 0$ and let $\tau, \tau' \in \Re^n$. Then:*

(i) $(\alpha\nabla_1\phi(\pi_\alpha(\tau), \tau) - \beta\nabla_1\phi(\pi_\beta(\tau), \tau))^T(\pi_\alpha(\tau) - \pi_\beta(\tau)) \le 0$;
(ii) $(\alpha\nabla_2\phi(\pi_\alpha(\tau), \tau) - \beta\nabla_2\phi(\pi_\beta(\tau), \tau))^T(\pi_\beta(\tau) - \pi_\alpha(\tau)) \le 0$;
(iii) $\alpha(\nabla_1\phi(\pi_\alpha(\tau), \tau) - \nabla_1\phi(\pi_\alpha(\tau'), \tau'))^T(\pi_\alpha(\tau) - \pi_\alpha(\tau'))$
$\le (F(\tau') - F(\tau))^T(\pi_\alpha(\tau) - \pi_\alpha(\tau'))$.

Proof Since $\pi_\alpha(\tau)$ is the maximizer of $\Psi_\alpha(\cdot, \tau)$ over P, the first-order necessary optimality condition implies that

$$\nabla_1\Psi_\alpha(\pi_\alpha(\tau), \tau)^T(x - \pi_\alpha(\tau)) \le 0, \quad \text{for each } x \in P. \tag{24}$$

Putting $x = \pi_\beta(\tau)$ and recalling (4), we have

$$(-F(\tau) - \alpha\nabla_1\phi(\pi_\alpha(\tau), \tau))^T(\pi_\beta(\tau) - \pi_\alpha(\tau)) \le 0.$$

Similarly,

$$(F(\tau) + \beta\nabla_1\phi(\pi_\beta(\tau), \tau))^T(\pi_\beta(\tau) - \pi_\alpha(\tau)) \le 0.$$

Adding the above two inequalities gives (i). (ii) follows from (i) and (C4). To prove (iii), letting $x = \pi_\alpha(\tau') \in P$ in (24) and recalling (4), we deduce that

$$(-F(\tau) - \alpha\nabla_1\phi(\pi_\alpha(\tau), \tau))^T(\pi_\alpha(\tau') - \pi_\alpha(\tau)) \le 0.$$

Similarly,

$$(F(\tau') + \alpha\nabla_1\phi(\pi_\alpha(\tau'), \tau'))^T(\pi_\alpha(\tau') - \pi_\alpha(\tau)) \le 0.$$

Adding the above two inequalities gives (iii). □

Proposition 2.2 *Let $\beta > \alpha > 0$. Then,*

$$\|\pi_\beta(\tau) - \pi_\alpha(\tau)\| \le \frac{K(\beta-\alpha)}{2\alpha\zeta}\|\tau - \pi_\beta(\tau)\|, \quad \textit{for each } \tau \in \Re^n.$$

Proof By (C2), we have

$$\begin{aligned}&\phi(\pi_\beta(\tau),\tau) - \phi(\pi_\alpha(\tau),\tau)\\ &\quad \ge \nabla_1\phi(\pi_\alpha(\tau),\tau)^T(\pi_\beta(\tau)-\pi_\alpha(\tau)) + \zeta\|\pi_\beta(\tau)-\pi_\alpha(\tau)\|^2\end{aligned}$$

and

$$\phi(\pi_\alpha(\tau),\tau) - \phi(\pi_\beta(\tau),\tau) \ge \nabla_1\phi(\pi_\beta(\tau),\tau)^T(\pi_\alpha(\tau)-\pi_\beta(\tau)) + \zeta\|\pi_\beta(\tau)-\pi_\alpha(\tau)\|^2.$$

Adding the above two inequalities, we obtain

$$0 \ge (\nabla_1\phi(\pi_\alpha(\tau),\tau) - \nabla_1\phi(\pi_\beta(\tau),\tau))^T(\pi_\beta(\tau)-\pi_\alpha(\tau)) + 2\zeta\|\pi_\beta(\tau)-\pi_\alpha(\tau)\|^2. \tag{25}$$

By Lemma 2.4(i) and (25), it follows from the Cauchy-Schwartz inequality that

$$\begin{aligned}2\zeta\|\pi_\beta(\tau)-\pi_\alpha(\tau)\|^2 &\le (\nabla_1\phi(\pi_\alpha(\tau),\tau) - \nabla_1\phi(\pi_\beta(\tau),\tau))^T(\pi_\alpha(\tau)-\pi_\beta(\tau))\\ &\le \frac{\beta-\alpha}{\alpha}\nabla_1\phi(\pi_\beta(\tau),\tau)^T(\pi_\alpha(\tau)-\pi_\beta(\tau))\\ &\le \frac{\beta-\alpha}{\alpha}\|\nabla_1\phi(\pi_\beta(\tau),\tau)\|\|\pi_\alpha(\tau)-\pi_\beta(\tau)\|\end{aligned}$$

and hence

$$\begin{aligned}\|\pi_\beta(\tau)-\pi_\alpha(\tau)\| &\le \frac{\beta-\alpha}{2\alpha\zeta}\|\nabla_1\phi(\pi_\beta(\tau),\tau)\|\\ &\le \frac{\beta-\alpha}{2\alpha\zeta}\cdot K\|\pi_\beta(\tau)-\tau\|,\end{aligned}$$

by Lemma 2.2(i). □

Proposition 2.3 *Let $\alpha > 0$ and let $\tau, \tau' \in \Re^n$. Then,*

$$\|\pi_\alpha(\tau) - \pi_\alpha(\tau')\| \le \frac{1}{2\zeta}K\|\tau-\tau'\| + \frac{1}{2\zeta\alpha}\|F(\tau)-F(\tau')\|. \tag{26}$$

Proof By (C2), we have

$$\begin{aligned}&\phi(\pi_\alpha(\tau'),\tau) - \phi(\pi_\alpha(\tau),\tau)\\ &\quad \ge \nabla_1\phi(\pi_\alpha(\tau),\tau)^T(\pi_\alpha(\tau')-\pi_\alpha(\tau)) + \zeta\|\pi_\alpha(\tau)-\pi_\alpha(\tau')\|^2\end{aligned}$$

and

$$\begin{aligned}&\phi(\pi_\alpha(\tau),\tau) - \phi(\pi_\alpha(\tau'),\tau)\\ &\quad \ge \nabla_1\phi(\pi_\alpha(\tau'),\tau)^T(\pi_\alpha(\tau)-\pi_\alpha(\tau')) + \zeta\|\pi_\alpha(\tau)-\pi_\alpha(\tau')\|^2.\end{aligned}$$

Adding these two inequalities and making use of Lemma 2.4(iii), we have

$$
\begin{aligned}
0 &\geq (\nabla_1\phi(\pi_\alpha(\tau),\tau) - \nabla_1\phi(\pi_\alpha(\tau'),\tau))^T(\pi_\alpha(\tau') - \pi_\alpha(\tau)) + 2\zeta\|\pi_\alpha(\tau') - \pi_\alpha(\tau)\|^2 \\
&\geq \frac{1}{\alpha}(F(\tau') - F(\tau))^T(\pi_\alpha(\tau') - \pi_\alpha(\tau)) + 2\zeta\|\pi_\alpha(\tau') - \pi_\alpha(\tau)\|^2 \\
&\quad + (\nabla_1\phi(\pi_\alpha(\tau'),\tau') - \nabla_1\phi(\pi_\alpha(\tau'),\tau))^T(\pi_\alpha(\tau') - \pi_\alpha(\tau)). \qquad (27)
\end{aligned}
$$

Consequently, by (C4), (19) and the Cauchy-Schwartz inequality, we have

$$
\begin{aligned}
0 \geq &-\frac{1}{\alpha}\|F(\tau') - F(\tau)\|\|\pi_\alpha(\tau') - \pi_\alpha(\tau)\| - K\|\tau - \tau'\|\|\pi_\alpha(\tau') - \pi_\alpha(\tau)\| \\
&+ 2\zeta\|\pi_\alpha(\tau') - \pi_\alpha(\tau)\|^2.
\end{aligned}
$$

Thus, (26) is seen to hold. □

Remark 2.1 Since F is locally Lipschitz, Proposition 2.3 implies that for all $\beta > \alpha > 0$, π_α and π_β are also locally Lipschitz, and it follows from (20), (21) that f_α, f_β and $g_{\alpha\beta}$ are all locally Lipschitz.

3 Directional Derivative of the Difference Functions of Max Functions

In this section, we follow the approach of Clarke in [26, Theorem 2.8.6], to provide an upper estimate for the Clarke generalized directional derivative $Q^o(\cdot,\cdot)$ of the function Q, which is defined by

$$Q(\tau) = G(\tau) - H(\tau), \quad \forall \tau \in \Re^n \qquad (28)$$

with

$$G(\tau) = \max_{i\in I} G_i(\tau) \qquad (29)$$

and

$$H(\tau) = \max_{j\in J} H_j(\tau), \qquad (30)$$

where I and J are two index sets and each of G_i, H_j is a real-valued function on $\Re^n$. We make the following blanket assumptions throughout this section.

Assumption 3.1 (i) For each $\tau \in \Re^n$, we assume that the active index subset $I(\tau)$ for τ in (29) is nonempty, that is,

$$I(\tau) := \{i \in I : G_i(\tau) = G(\tau)\} \neq \emptyset. \qquad (31)$$

Similarly, we assume that, regarding (30),

$$J(\tau) := \{j \in J : H_j(\tau) = H(\tau)\} \neq \emptyset. \qquad (32)$$

(ii) For each $\tau \in \Re^n$, there exist positive real numbers δ_τ and L_τ such that, for each $z \in \tau + \delta_\tau B$, each $i \in I(z)$ and each $j \in J(z)$, $G_i - H_j$ is Lipschitz on $\tau + \delta_\tau B$ with modulus L_τ, that is,

$$|(G_i(\tau') - H_j(\tau')) - (G_i(\tau'') - H_j(\tau''))| \leq L_\tau \|\tau' - \tau''\|, \quad \forall \tau', \tau'' \in (\tau + \delta_\tau B). \tag{33}$$

Remark 3.1 (a) By (33), Q is also locally Lipschitz and

$$|Q(\tau_1) - Q(\tau_2)| \leq L_\tau \|\tau_1 - \tau_2\|, \quad \forall \tau_1, \tau_2 \in \tau + \delta_\tau B. \tag{34}$$

In fact, if $\tau_1, \tau_2 \in (\tau + \delta_\tau B)$, $i \in I(\tau_1)$ and $j \in J(\tau_2)$, then

$$\begin{aligned} Q(\tau_1) - Q(\tau_2) &= (G_i(\tau_1) - H(\tau_1)) - (G(\tau_2) - H_j(\tau_2)) \\ &\leq (G_i(\tau_1) - H_j(\tau_1)) - (G_i(\tau_2) - H_j(\tau_2)) \\ &\leq L_\tau \|\tau_1 - \tau_2\|, \end{aligned}$$

and thus (31) holds by interchanging the roles of τ_1 and τ_2.

(b) By (a) and the Rademacher theorem, Q and also each $G_i - H_j$ are differentiable almost everywhere. Note that, if $z \in \tau + \delta_\tau B$ and if Q (resp. $G_i - H_j$) is differentiable at z, then

$$\|\nabla Q(z)\| \leq L_\tau \quad (\text{resp. } \|\nabla (G_i - H_j)(z)\| \leq L_\tau). \tag{35}$$

Proposition 3.1 *Let $\tau, v \in \Re^n$ and let S be a set of measure zero in $\Re^n$. Consider the subset of $\Re$ defined by*

$$D = \Big\{ \lim_{k \to +\infty} \nabla (G_{i_k} - H_{j_k})(\tau_k) : \tau_k, z_k, z'_k \to \tau \text{ with } \tau_k \notin S, i_k \in I(z_k), \\ j_k \in J(z'_k) \text{ for each } k \Big\}.$$

Then, the following assertions hold:

(i) *D is nonempty and compact.*
(ii) *A real number r belongs to D if and only if for any $\varepsilon > 0$ there exist $\tau^\varepsilon \in (\tau + \varepsilon B) \setminus S$, $z^\varepsilon, z'^\varepsilon \in \tau + \varepsilon B$, $t^\varepsilon \in I(z^\varepsilon)$, and $t'^\varepsilon \in J(z'^\varepsilon)$ such that $G_{t^\varepsilon} - H_{t'^\varepsilon}$ is differentiable at τ^ε and*

$$|r - \nabla (G_{t^\varepsilon} - H_{t'^\varepsilon})(\tau^\varepsilon)| < \varepsilon.$$

(iii) *The generalized directional derivative of Q at τ along the direction v satisfies the inequality*

$$Q^o(\tau, v) \leq \max\{\xi^T v : \xi \in D\}. \tag{36}$$

(iv) $\emptyset \neq \partial Q(\tau) \subset \mathrm{co}\, D.$ (37)

Proof Take sequences $(z_k), (z'_k) \to \tau$. By (i) and (ii) of Assumption 3.1 and making use of the Rademacher theorem, there exist sequences (i_k), (j_k) and (τ_k) such that $(\tau_k) \to \tau$ and, for each k, $i_k \in I(z_k)$, $j_k \in J(z'_k)$, $\tau_k \in (\tau + \delta_\tau B) \setminus S$ and

$\nabla(G_{i_k} - H_{j_k})(\tau_k)$ exists. Together with (35), it follows that D is a nonempty bounded set. The verification for (ii) is routine and it follows that D is closed. Therefore, D is compact and hence the set $\{\xi^T v : \xi \in D\}$ has a maximal element. Note that co D and $\partial Q(\tau)$ are nonempty convex compact sets in $\Re^n$ and $Q^o(\tau, \cdot)$ is the support function of $\partial Q(\tau)$ (see [26, Proposition 2.1.2]) for each τ. Hence, (37) follows from (36) by [26, Proposition 2.1.4]. To prove (36), we may assume $\|v\| = 1$ and denote m for $\max\{\xi^T v : \xi \in D\}$. Then for any $\varepsilon > 0$, the definitions of m and D imply that there exists $\delta > 0$ such that if $x, z, z' \in \tau + 2\delta B$, $i \in I(z)$ and $j \in J(z')$ satisfy

$$x \notin S, \quad \nabla(G_i - H_j)(x) \text{ exists,}$$

then one has

$$(\nabla(G_i - H_j)(x))^T v < m + \varepsilon. \tag{38}$$

We assume without loss of generality that $\delta < \frac{\delta_\tau}{2}$. Let s be any number in $(0, \delta)$. We will show that

$$\frac{1}{s}([G(x+sv) - H(x+sv)] - [G(x) - H(x)]) < m + \varepsilon, \quad \forall x \in \tau + \delta B. \tag{39}$$

Let $x \in \tau + \delta B$. By Assumption 3.1(i), there exist $i \in I(x + sv)$ and $j \in J(x)$. Let Ω_{ij} be the set of points in $\tau + \delta B$ at which $G_i - H_j$ fails to be differentiable. The Rademacher theorem shows that Ω_{ij} is a set of measure zero. Let y be any point in $\tau + \delta B$ such that the line-segment $[y, y + sv]$ meets $S \cup \Omega_{ij}$ in a set of one-dimensional measure zero. (Note that almost all $y \in \tau + \delta B$ have this property since $S \cup \Omega_{ij}$ has measure zero.) With the help of (38), we obtain

$$\begin{aligned}
&[G_i(y+sv) - H_j(y+sv)] - [G_i(y) - H_j(y)] \\
&\quad = \int_0^s (\nabla(G_i - H_j)(y + \mu v))^T v \, d\mu \\
&\quad < s(m + \varepsilon).
\end{aligned} \tag{40}$$

Since $G_i - H_j$ is continuous on $\tau + \delta B$, this inequality must in fact hold for all y in $\tau + \delta B$. Taking $y = x$, we have

$$\begin{aligned}
&[G(x+sv) - H(x+sv)] - [G(x) - H(x)] \\
&\quad \le [G_i(x+sv) - H_j(x+sv)] - [G_i(x) - H_j(x)] \\
&\quad < s(m + \varepsilon).
\end{aligned}$$

This proves (39). Passing to the limits as $x \to \tau$ and $s \downarrow 0$, (39) implies that

$$\begin{aligned}
Q^o(\tau, v) &= \limsup_{x \to \tau, s \downarrow 0} \frac{Q(x+sv) - Q(x)}{s} \\
&= \limsup_{x \to \tau, s \downarrow 0} \frac{[G(x+sv) - H(x+sv)] - [G(x) - H(x)]}{s} \\
&\le m + \varepsilon.
\end{aligned}$$

Then, (36) holds as $\varepsilon > 0$ is arbitrary. □

4 D-Gap Function

For the remainder of this paper, let $F, P, \alpha, \beta, \phi, \Psi_\alpha, f_\alpha, g_{\alpha\beta}, \pi_\alpha$ and π_β be as in Sect. 1. In this section, we study the Clarke generalized gradient of $g_{\alpha\beta}$. In particular,

$$g_{\alpha\beta}(\tau) = \max_{x\in P} \Psi_\alpha(x,\tau) - \max_{x\in P} \Psi_\beta(x,\tau),$$

the maxima being attained exactly at points $x = \pi_\alpha(\tau)$ and $x' = \pi_\beta(\tau)$ respectively. Together with the following Lemma 4.1, $(f_\alpha, f_\beta, \Psi_\alpha(x,\cdot), \Psi_\beta(x,\cdot), x \in P, x \in P)$ satisfies Assumption 3.1 stated for $(G, H, G_i, H_j, i \in I, j \in J)$; hence, by Proposition 3.1,

$$g^o_{\alpha\beta}(\tau) \le \max\{\xi^T v : \xi \in D(\tau)\}, \qquad \emptyset \ne \partial g_{\alpha\beta}(\tau) \subset \operatorname{co} D(\tau), \tag{41}$$

where

$$D(\tau) = \Big\{ \lim_{i\to+\infty} \nabla_2(\Psi_\alpha(\pi_\alpha(z_k),\tau_k) - \Psi_\beta(\pi_\beta(z'_k),\tau_k)) : \tau_k, z_k, z'_k \to \tau \text{ and } \tau_k \notin \Omega_F \Big\}, \tag{42}$$

$\nabla_2(\Psi_\alpha(\pi_\alpha(z_k),\tau_k) - \Psi_\beta(\pi_\beta(z'_k),\tau_k))$ denotes the derivative of the function $\Psi_\alpha(\pi_\alpha(z_k),\cdot) - \Psi_\beta(\pi_\beta(z'_k),\cdot)$ at τ_k and Ω_F denotes the set of all points at which F fails to be differentiable (Ω_F is of measure zero by the Rademacher theorem). We will show in Theorem 4.1 that the equality in (41) holds.

Lemma 4.1 *Let $\tau \in \Re^n$. Then, there exist $\delta_\tau, L_\tau > 0$ such that each function in the family*

$$\{\Psi_\alpha(\pi_\alpha(z),\cdot) - \Psi_\beta(\pi_\beta(z),\cdot) : z \in \tau + \delta_\tau B\}$$

is Lipschitz on $\tau + \delta_\tau B$ with modulus L_τ; that is

$$|[\Psi_\alpha(\pi_\alpha(z),\tau') - \Psi_\beta(\pi_\beta(z),\tau')] - [\Psi_\alpha(\pi_\alpha(z),\tau'') - \Psi_\beta(\pi_\beta(z),\tau'')]| \le L_\tau \|\tau' - \tau''\| \tag{43}$$

for any $z, \tau', \tau'' \in \tau + \delta_\tau B$. Consequently, $g_{\alpha\beta}$ is also Lipschitz with modulus L_τ on $\tau + \delta_\tau B$.

Proof Let $M_\tau > 0$ be a Lipschitz constant for F on $\tau + \delta_\tau B$ for some $\delta_\tau > 0$. Note that there exists a constant $C_1 > 0$ such that $\|F(\tau')\| \le C_1$ for all $\tau' \in \tau + \delta_\tau B$ (e.g., take $C_1 := \|F(\tau)\| + M_\tau\delta_\tau$). Similarly, by (26), there exists a constant $C_2 > 0$ such that

$$\|\pi_\alpha(\tau')\|, \|\pi_\beta(\tau')\| \le C_2, \quad \forall \tau' \in \tau + \delta_\tau B.$$

Now, in view of (4), we write, for all $z, \tau' \in \tau + \delta_\tau B$,

$$\Psi_\alpha(\pi_\alpha(z),\tau') - \Psi_\beta(\pi_\beta(z),\tau') := \Psi_1(\pi_\alpha(z),\pi_\beta(z),\tau') + \Psi_2(\pi_\alpha(z),\pi_\beta(z),\tau'),$$

where

$$\Psi_1(\pi_\alpha(z), \pi_\beta(z), \tau') = F(\tau')^T(\pi_\beta(z) - \pi_\alpha(z))$$

and

$$\Psi_2(\pi_\alpha(z), \pi_\beta(z), \tau') = -\alpha\phi(\pi_\alpha(z), \tau') + \beta\phi(\pi_\beta(z), \tau').$$

Then,

$$|\Psi_1(\pi_\alpha(z), \pi_\beta(z), \tau') - \Psi_1(\pi_\alpha(z), \pi_\beta(z), \tau'')| \le 2C_2 M_\tau \|\tau' - \tau''\|.$$

Moreover, since ϕ is continuously differentiable, there exists $C_3 > 0$ such that, for all $z, \tau' \in \tau + \delta_\tau B$,

$$\| -\alpha\nabla_2\phi(\pi_\alpha(z), \tau') + \beta\nabla_2\phi(\pi_\beta(z), \tau')\| \le C_3;$$

hence by the mean-value theorem,

$$\begin{aligned}&|\Psi_2(\pi_\alpha(z), \pi_\beta(z), \tau') - \Psi_2(\pi_\alpha(z), \pi_\beta(z), \tau'')| \le C_3\|\tau' - \tau''\|,\\ &\quad \forall z, \tau', \tau'' \in \tau + \delta_\tau B.\end{aligned}$$

Thus, (43) holds with δ_τ and $L_\tau := 2C_2M_\tau + C_3 > 0$. The rest of the lemma follows from Remark 3.1(a). □

The set $\partial F(\tau)z$ used in the following theorem is of course to be understood as $\{\xi z : \xi \in \partial F(\tau)\}$.

Theorem 4.1 *Let $\beta > \alpha > 0$. Let $\tau \in \Re^n$ and let $D(\tau)$ denote the set defined by* (42). *Then,*

$$\partial g_{\alpha\beta}(\tau) = \text{co}\, D(\tau) = \partial F(\tau)(\pi_\beta(\tau) - \pi_\alpha(\tau)) - \alpha\nabla_2\phi(\pi_\alpha(\tau), \tau) + \beta\nabla_2\phi(\pi_\beta(\tau), \tau) \tag{44}$$

and

$$g^o_{\alpha\beta}(\tau) = \max\{\langle\xi,\ v\rangle : v \in D(\tau)\} = \max\{\langle\xi,\ v\rangle : v \in \text{co}\, D(\tau)\}. \tag{45}$$

Proof The second equality of (45) is trivial. Moreover, by [26, Proposition 2.1.4], the first equality in (44) follows from (45). Further, by (4), we note that

$$\nabla_2\Psi_\gamma(z, \tau') = \nabla F(\tau')(\tau' - z) + F(\tau') - \gamma\nabla_2\phi(z, \tau'), \quad \forall\gamma > 0,$$

for each $\tau' \in \Re^n \backslash \Omega_F$ and $z \in \Re^n$. Thus,

$$\begin{aligned}D(\tau) = \Big\{&\lim_{k\to+\infty} \nabla F(\tau_k)(\pi_\beta(z'_k) - \pi_\alpha(z_k)) - \alpha\nabla_2\phi(\pi_\alpha(z_k), \tau_k)\\ &+ \beta\nabla_2\phi(\pi_\beta(z'_k), \tau_k) : \tau_k, z_k, z'_k \to \tau, \tau_k \notin \Omega_F\Big\}.\end{aligned}$$

Making use of the facts that ϕ is continuously differentiable, that F, π_α, π_β are continuous and that $\{\nabla F(z) : z \in \tau + \delta_\tau B\}$ is bounded, one has

$$\begin{aligned} D(\tau) &= \left\{ \lim_{k\to+\infty} \nabla F(\tau_k)(\pi_\beta(\tau) - \pi_\alpha(\tau)) : \tau_k \to \tau, \tau_k \notin \Omega_F \right\} - \alpha \nabla_2 \phi(\pi_\alpha(\tau), \tau) \\ &\quad + \beta \nabla_2 \phi(\pi_\beta(\tau), \tau) \\ &= \left\{ \left(\lim_{k\to+\infty} \nabla F(\tau_k) \right) (\pi_\beta(\tau) - \pi_\alpha(\tau)) : \tau_k \to \tau, \tau_k \notin \Omega_F \right\} - \alpha \nabla_2 \phi(\pi_\alpha(\tau), \tau) \\ &\quad + \beta \nabla_2 \phi(\pi_\beta(\tau), \tau). \end{aligned} \tag{46}$$

Together with (16), it follows that the second equality in (44) holds. Thus, it remains to prove (45). In view of (41), it suffices to show that, for all $v \in \Re^n$,

$$\max\{\xi^T v : \xi \in D(\tau)\} \le g^o_{\alpha\beta}(\tau, v). \tag{47}$$

To do this, let $\xi = \lim_{k\to+\infty}(\nabla F(\tau_k)(\pi_\beta(\tau) - \pi_\alpha(\tau))) - \alpha\nabla_2\phi(\pi_\alpha(\tau), \tau) + \beta\nabla_2\phi(\pi_\beta(\tau), \tau)$ for some sequence $(\tau_k) \subset \Re^n \setminus \Omega_F$ with $(\tau_k) \to \tau$. Then,

$$\begin{aligned} \xi^T v &= \lim_{k\to+\infty} (\nabla F(\tau_k)(\pi_\beta(\tau) - \pi_\alpha(\tau)))^T v - \alpha \nabla_2 \phi(\pi_\alpha(\tau), \tau)^T v \\ &\quad + \beta \nabla_2 \phi(\pi_\beta(\tau), \tau)^T v \\ &= \lim_{k\to+\infty} (\pi_\beta(\tau) - \pi_\alpha(\tau))^T (\nabla F(\tau_k)^T v) - \alpha \nabla_2 \phi(\pi_\alpha(\tau), \tau)^T v \\ &\quad + \beta \nabla_2 \phi(\pi_\beta(\tau), \tau)^T v. \end{aligned}$$

Note that

$$\lim_{k\to+\infty} \nabla F(\tau_k)^T v = \lim_{k\to+\infty} \lim_{t\downarrow 0} \frac{F(\tau_k + tv) - F(\tau_k)}{t}.$$

Consequently, there exist a subsequence (τ_{k_i}) of (τ_k) and a sequence $(t_i) \downarrow 0$ such that

$$\lim_{k\to+\infty} \nabla F(\tau_k)^T v = \lim_{i\to+\infty} \frac{F(\tau_{k_i} + t_i v) - F(\tau_{k_i})}{t_i}.$$

For simplicity of notations, we henceforth assume that the above (τ_{k_i}) is (τ_k) itself, i.e.,

$$\lim_{k\to+\infty} \nabla F(\tau_k)^T v = \lim_{k\to+\infty} \frac{F(\tau_k + t_k v) - F(\tau_k)}{t_k}. \tag{48}$$

We note that

$$\begin{aligned} &g_{\alpha\beta}(\tau_k + t_k v) - g_{\alpha\beta}(\tau_k) \\ &\quad = f_\alpha(\tau_k + t_k v) - f_\beta(\tau_k + t_k v) - f_\alpha(\tau_k) + f_\beta(\tau_k) \\ &\quad = \Psi_\alpha(\pi_\alpha(\tau_k + t_k v), \tau_k + t_k v) - \Psi_\beta(\pi_\beta(\tau_k + t_k v), \tau_k + t_k v) - \Psi_\alpha(\pi_\alpha(\tau_k), \tau_k) \\ &\qquad + \Psi_\beta(\pi_\beta(\tau_k), \tau_k) \end{aligned}$$

$$
\begin{aligned}
&\geq \Psi_\alpha(\pi_\alpha(\tau_k), \tau_k + t_k v) - \Psi_\beta(\pi_\beta(\tau_k + t_k v), \tau_k + t_k v) - \Psi_\alpha(\pi_\alpha(\tau_k), \tau_k) \\
&\quad + \Psi_\beta(\pi_\beta(\tau_k + t_k v), \tau_k) \\
&= (F(\tau_k + t_k v))^T(\tau_k + t_k v - \pi_\alpha(\tau_k)) - \alpha\phi(\pi_\alpha(\tau_k), \tau_k + t_k v) \\
&\quad - (F(\tau_k + t_k v))^T(\tau_k + t_k v - \pi_\beta(\tau_k + t_k v)) + \beta\phi(\pi_\beta(\tau_k + t_k v), \tau_k + t_k v) \\
&\quad - F(\tau_k)^T(\tau_k - \pi_\alpha(\tau_k)) + \alpha\phi(\pi_\alpha(\tau_k), \tau_k) \\
&\quad + F(\tau_k)^T(\tau_k - \pi_\beta(\tau_k + t_k v)) - \beta\phi(\pi_\beta(\tau_k + t_k v), \tau_k) \\
&= (F(\tau_k + t_k v) - F(\tau_k))^T(\pi_\beta(\tau_k + t_k v) - \pi_\alpha(\tau_k)) \\
&\quad - \alpha(\phi(\pi_\alpha(\tau_k), \tau_k + t_k v) - \phi(\pi_\alpha(\tau_k), \tau_k)) \\
&\quad + \beta(\phi(\pi_\beta(\tau_k + t_k v), \tau_k + t_k v) - \phi(\pi_\beta(\tau_k + t_k v), \tau_k)).
\end{aligned}
$$

Moreover, for any k, we apply the mean-value theorem to the differentiable function ϕ to find some $s_k, s_k' \in [0, t_k]$ such that

$$
\frac{\phi(\pi_\alpha(\tau_k), \tau_k + t_k v) - \phi(\pi_\alpha(\tau_k), \tau_k)}{t_k} = \nabla_2\phi(\pi_\alpha(\tau_k), \tau_k + s_k v)^T v \tag{49}
$$

and

$$
\begin{aligned}
&\frac{\phi(\pi_\beta(\tau_k + t_k v), \tau_k + t_k v) - \phi(\pi_\beta(\tau_k + t_k v), \tau_k)}{t_k} \\
&\qquad = \nabla_2\phi(\pi_\beta(\tau_k + t_k v), \tau_k + s_k' v)^T v.
\end{aligned} \tag{50}
$$

Note that the limits (when $k \to +\infty$) in (49) and (50) are $\nabla_2\phi(\pi_\alpha(\tau), \tau)^T v$ and $\nabla_2\phi(\pi_\beta(\tau), \tau)^T v$ respectively. Hence, by (48),

$$
\begin{aligned}
&\limsup_{k\to+\infty} \frac{g_{\alpha\beta}(\tau_k + t_k v) - g_{\alpha\beta}(\tau_k)}{t_k} \\
&\quad \geq \lim_{k\to+\infty} \left(\frac{F(\tau_k + t_k v) - F(\tau_k)}{t_k}\right)^T (\pi_\beta(\tau_k + t_k v) - \pi_\alpha(\tau_k)) \\
&\qquad - \alpha\nabla_2\phi(\pi_\alpha(\tau), \tau)^T v + \beta\nabla_2\phi(\pi_\beta(\tau), \tau)^T v \\
&\quad = \lim_{k\to+\infty} (\nabla F(\tau_k)^T v)^T(\pi_\beta(\tau) - \pi_\alpha(\tau)) - \alpha\nabla_2\phi(\pi_\alpha(\tau), \tau)^T v \\
&\qquad + \beta\nabla_2\phi(\pi_\beta(\tau), \tau)^T v \\
&\quad = \lim_{k\to+\infty} (\pi_\beta(\tau) - \pi_\alpha(\tau))^T(\nabla F(\tau_k)^T v) - \alpha\nabla_2\phi(\pi_\alpha(\tau), \tau)^T v \\
&\qquad + \beta\nabla_2\phi(\pi_\beta(\tau), \tau)^T v \\
&\quad = \xi^T v.
\end{aligned} \tag{51}
$$

Consequently, it follows that

$$\begin{aligned} g^o_{\alpha\beta}(\tau, v) &= \limsup_{\tau'\to\tau, t\downarrow 0} \frac{g_{\alpha\beta}(\tau' + tv) - g_{\alpha\beta}(\tau')}{t} \\ &\geq \limsup_{k\to\infty} \frac{g_{\alpha\beta}(\tau_k + t_k v) - g_{\alpha\beta}(\tau_k)}{t_k} \\ &= \xi^T v \end{aligned}$$

and so, by (46), we see that (47) holds. □

Remark 4.1 In the special case when ϕ is defined by (7) and $\beta = \frac{1}{\alpha}$, Theorem 4.1 is due to Xu [24] where his argument relies on the concrete representation of π_α by virtue of the Euclidean projection to P. This is of course no longer the case when ϕ is only abstractly given.

5 Minimizing and Stationary Sequences of the D-Gap Function

This section is devoted to study relations between minimizing sequences and stationary sequences of $g_{\alpha\beta}$; in particular Theorems 1.1 and 1.2 stated earlier in the Introduction will be proved in this section. Let Δ be the set of all positive-valued functions δ defined on $(0, +\infty)$ with the property that

$$\liminf_{t\to t^*} \delta(t) > 0, \quad \text{for all } t^* > 0; \tag{52}$$

that is, set Δ consists of all positive-valued functions defined on $(0, +\infty)$ with the property that

$$\left[\lim_{k\to+\infty} t_k = t_* > 0\right] \Rightarrow \liminf_{k\to+\infty} \delta(t_k) > 0$$

(see [7]). For example, all positive-valued continuous functions defined on $(0, +\infty)$ belong to Δ.

Proof of Theorem 1.1 In view of (13), it is sufficient to show that any cluster point g^* of $(g_{\alpha\beta}(\tau_k))$ equals $g_{\inf} := \inf_{\tau\in\Re^n} g_{\alpha\beta}(\tau)$. For simplicity of notations, let us assume that g^* is the limit of the whole sequence $(g_{\alpha\beta}(\tau_k))$. Moreover, since $(g_{\alpha\beta}(\tau_k))$ is assumed bounded, $(\tau_k - \pi_\beta(\tau_k))$ and $(\pi_\beta(\tau_k) - \pi_\alpha(\tau_k))$ are bounded by Lemma 2.3 and Proposition 2.2. By the stationarity assumption, there exists a sequence (ξ_k) with each $\xi_k \in \partial g_{\alpha\beta}(\tau_k)$ such that

$$\lim_{k\to+\infty} \xi_k = 0. \tag{53}$$

In view of Theorem 4.1, each ξ_k can be represented as

$$\begin{aligned} \xi_k &= \tau_k^*(\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)) + \beta\nabla_2\phi(\pi_\beta(\tau_k), \tau_k) - \alpha\nabla_2\phi(\pi_\alpha(\tau_k), \tau_k) \\ &= \tau_k^*(\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)) + (\beta - \alpha)\nabla_2\phi(\pi_\beta(\tau_k), \tau_k), + A_k, \end{aligned} \tag{54}$$

where each $\tau_k^* \in \partial F(\tau_k)$ and

$$A_k := \alpha(\nabla_2\phi(\pi_\beta(\tau_k), \tau_k) - \nabla_2\phi(\pi_\alpha(\tau_k), \tau_k)). \tag{55}$$

Then, by (13) and an earlier remark,

$$(\tau_k^*), (\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)) \quad \text{and} \quad (\tau_k - \pi_\alpha(\tau_k)) \quad \text{are bounded.} \tag{56}$$

Let us assume that (i) is satisfied. We will prove the desired result by establishing a stronger claim that

$$\lim_{k\to+\infty} g_{\alpha\beta}(\tau_k) = g_{\inf} = 0. \tag{57}$$

Indeed, by (55), (8) and (i),

$$\|A_k\| \le \alpha K \|\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)\| \to 0. \tag{58}$$

Consequently, by (53), (54) and (56), (i) implies that $\nabla_2\phi(\pi_\beta(\tau_k), \tau_k) \to 0$ and hence that $\nabla_2\phi(\pi_\alpha(\tau_k), \tau_k) \to 0$ by (55) and (58). Recalling (C1) and (9) (with τ, x'', x' in place of τ_k, τ_k and $\pi_\alpha(\tau_k)$ respectively), we have

$$\begin{aligned} 0 &\le \phi(\pi_\alpha(\tau_k), \tau_k) \le \phi(\pi_\alpha(\tau_k), \tau_k) + \zeta\|\tau_k - \pi_\alpha(\tau_k)\|^2 \\ &\le \nabla_2\phi(\pi_\alpha(\tau_k), \tau_k)^T(\tau_k - \pi_\alpha(\tau_k)) \to 0 \end{aligned}$$

and it follows from (22) that (57) holds. It remains to consider the case when (ii) is assumed. We claim that, if $\lim_{k\to+\infty} g_{\alpha\beta}(\tau_k) = g^* > g_{\inf}$, then (ii)⇒(i) (and hence a contradiction wrt the first part of our proof). Granting this, the proof of the theorem will easily be completed by making use of the assumption (13) as we did at the beginning of the proof. To prove our claim, let μ be a scalar satisfying $g^* > \mu > g_{\inf}$. Then, by (ii), there exist a scalar $r > 0$ and a function $\delta \in \Delta$ such that, for all k,

$$\begin{aligned} 0 &\le \delta([g_{\alpha\beta}(\tau_k) - \mu]_+)\|\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)\|^r \\ &\le (\pi_\beta(\tau_k) - \pi_\alpha(\tau_k))^T \tau_k^*(\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)). \end{aligned} \tag{59}$$

Since $\delta \in \Delta$ and $\lim_{k\to+\infty}[g_{\alpha\beta}(\tau_k) - \mu]_+ = g^* - \mu > 0$, one has

$$\liminf_{k\to+\infty} \delta([g_{\alpha\beta}(\tau_k) - \mu]_+) > 0. \tag{60}$$

Thus, $\|(\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)\|^r \to 0$ provided that

$$\limsup_{k\to+\infty} (\pi_\beta(\tau_k) - \pi_\alpha(\tau_k))^T \tau_k^*(\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)) \le 0. \tag{61}$$

To verify (61), we take the inner product with $\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)$ on both sides in the first equality in (54), make use of Lemma 2.4(ii) and we deduce that

$$(\pi_\beta(\tau_k) - \pi_\alpha(\tau_k))^T \tau_k^*(\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)) \le \xi_k^T(\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)) \to 0, \tag{62}$$

thanks to (53) and (56). Therefore, (61) is true and this completes the proof. □

Remark 5.1 The condition (ii) in Theorem 1.1 is automatically satisfied if F is strongly monotone. In fact, if F is strongly monotone with modulus $\lambda > 0$, then $u^T \nabla F(\tau) u \geq \lambda \|u\|^2$ for all $u, \tau \in \Re^n$ provided that $\nabla F(\tau)$ exists. And so, by the definition of $\partial F(\cdot)$, (ii) holds with $\delta \equiv \lambda$ and $r = 2$. Fukushima and Pang [8] gave an example of smooth F which shows that (ii) does not imply the monotonicity of F (see [1, p. 154]) for the definitions).

Proof of Theorem 1.2 Suppose that (i) is assumed. Then, by the assumptions, $\lim_{k\to+\infty} g_{\alpha\beta}(\tau_k) = g_{\inf} = 0$. Replace τ by τ_k in (23), pass to the limits, and we deduce that

$$\lim_{k\to+\infty} (\tau_k - \pi_\beta(\tau_k)) = 0. \tag{63}$$

Then, by Proposition 2.2,

$$\lim_{k\to+\infty} (\pi_\alpha(\tau_k) - \pi_\beta(\tau_k)) = 0 \quad \text{and} \quad \lim_{k\to+\infty} (\tau_k - \pi_\alpha(\tau_k)) = 0. \tag{64}$$

Since (η_k) is bounded, it follows that

$$\lim_{k\to+\infty} \eta_k (\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)) = 0. \tag{65}$$

On the other hand, by (63), (64), Lemma 2.2(i) and (C4), one has

$$\lim_{k\to+\infty} \nabla_2 \phi(\pi_\alpha(\tau_k), \tau_k) = \lim_{k\to+\infty} \nabla_2 \phi(\pi_\beta(\tau_k), \tau_k) = 0.$$

It follows from (44) and (65) that (τ_k) is a stationary sequence of $g_{\alpha\beta}$. Next, we consider the case when (ii) is assumed. By [27, Lemma 3.4], it is sufficient to show that $\partial g_{\alpha\beta}$ is uniformly upper semicontinuous near (τ_k). By the assumptions, Proposition 2.1 implies that F is uniformly continuous near (τ_k) and it follows from Proposition 2.3 that π_α is uniformly continuous near (τ_k). Consequently, $\nabla_2 \phi(\pi_\alpha(\cdot), \cdot)$ is also uniformly continuous, because of the following estimates:

$$\begin{aligned}
&\|\nabla_2 \phi(\pi_\alpha(\tau_k), \tau_k) - \nabla_2 \phi(\pi_\alpha(\tau'), \tau')\| \\
&\quad \leq \|\nabla_2 \phi(\pi_\alpha(\tau_k), \tau_k) - \nabla_2 \phi(\pi_\alpha(\tau'), \tau_k)\| + \|\nabla_2 \phi(\pi_\alpha(\tau'), \tau_k) - \nabla_2 \phi(\pi_\alpha(\tau'), \tau')\| \\
&\quad \leq K \|\pi_\alpha(\tau_k) - \pi_\alpha(\tau')\| + K \|\tau_k - \tau'\|,
\end{aligned}$$

thanks to (8) and (19). The same assertion is of course also valid for β in place of α. Hence, by (44), the uniform upper semicontinuity of $\partial g_{\alpha\beta}$ will follow if one can prove the corresponding property for $\partial F(\cdot)(\pi_\beta(\cdot) - \pi_\alpha(\cdot))$. For the latter, let us take $M > 0$ to be the radius of a ball containing the bounded set $\cup_{k=1}^{+\infty} \partial F(\tau_k)$, and let $\varepsilon \in (0, 1)$ be given. Then by the uniform upper semicontinuity of ∂F and of $\pi_\beta - \pi_\alpha$, there exist $\delta > 0$ and a natural number m such that

$$\partial F(\tau') \subset \partial F(\tau_k) + \varepsilon \overline{B}_{\Re^{n\times n}} \subset (M+1) \overline{B}_{\Re^{n\times n}}, \tag{66}$$

$$(\pi_\beta(\tau') - \pi_\alpha(\tau')) \subset (\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)) + \varepsilon \overline{B}, \tag{67}$$

whenever $k \geq m$ and $\|\tau' - \tau_k\| \leq \delta$. By Lemma 2.3, Proposition 2.2 and the boundedness property of $(g_{\alpha\beta}(\tau_k))$, there exists $M' > 0$ such that $\|\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)\| \leq M'$ for all k. It follows from (66) and (67) that

$$\begin{aligned}
&\partial F(\tau')(\pi_\beta(\tau') - \pi_\alpha(\tau')) \\
&\quad \subset (\partial F(\tau_k) + \varepsilon \overline{B}_{\Re^{n\times n}})((\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)) + \varepsilon \overline{B}) \\
&\quad \subset \partial F(\tau_k)(\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)) + (M + M' + 1)\varepsilon \overline{B},
\end{aligned}$$

whenever $k \geq m$ and $\|\tau' - \tau_k\| \leq \delta$. Thus, $\partial F(\cdot)(\pi_\beta(\cdot) - \pi_\alpha(\cdot))$ is uniformly upper semicontinuous near (τ_k). □

Remark 5.2 In the special case when ϕ is the one defined by (7) and when F is assumed to be continuously differentiable, Theorem 1.2(ii) and Theorem 1.1(ii) are due to Fukushima and Pang [8, Theorem 4.1].

Below, we provide an example, where Theorem 1.1(i) is applicable but not Theorem 1.1(ii). Similarly, there are examples for which (i) in Theorem 1.2 is applicable but not (ii).

Example 5.1 We consider a 2-dimensional smooth NCP with the following function:

$$F(\tau^1, \tau^2) \equiv \left(-\tau^1, \frac{1}{(\tau^1 + \tau^2 - 1)^2 + 1} - 1\right)^T, \quad \forall (\tau^1, \tau^2) \in \Re^2.$$

Let $\tau' = (1, 0)^T$ and $\tau = (1, 1)^T$. Then,

$$(F(\tau') - F(\tau))^T(\tau' - \tau) = \left((-1, 0) - \left(-1, -\frac{1}{2}\right)\right)\begin{pmatrix}0\\-1\end{pmatrix} = -\frac{1}{2} < 0.$$

Thus F is not monotone. Consider the sequence (τ_k) with

$$\tau_k = (\tau_k^1, \tau_k^2)^T \equiv \left(\frac{1}{k}, 1 - \frac{1}{k}\right)^T, \quad \text{for all } k.$$

Now let $\beta > \alpha > 0$ and let ϕ be defined by (7). Then, for any $\gamma > 0$ and any natural number k, one has

$$\pi_\gamma(\tau_k) = \operatorname*{Proj}_{\Re^2_+}\left(\tau_k - \frac{F(\tau_k)}{\gamma}\right) = \left(\frac{1}{k}\left(1 + \frac{1}{\gamma}\right), 1 - \frac{1}{k}\right)^T.$$

Thus, $\lim_{k\to+\infty}(\pi_\alpha(\tau_k) - \pi_\beta(\tau_k)) = \lim_{k\to+\infty}(\frac{1}{k}(\frac{1}{\alpha} - \frac{1}{\beta}), 0)^T = (0, 0)^T$ and the condition (i) in Theorem 1.1 is satisfied. On the other hand, $\nabla F(\tau_k) = \left(\begin{smallmatrix}-1 & 0\\ 0 & 0\end{smallmatrix}\right)$ and

$$\begin{aligned}
&(\pi_\beta(\tau_k) - \pi_\alpha(\tau_k))^T \nabla F(\tau_k)(\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)) \\
&\quad = \left(\frac{1}{k}\left(\frac{1}{\alpha} - \frac{1}{\beta}\right) 0\right)\begin{pmatrix}-1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{k}(\frac{1}{\alpha} - \frac{1}{\beta})\\ 0\end{pmatrix}
\end{aligned}$$

$$= \left(-\frac{1}{k}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right) 0\right) \begin{pmatrix} \frac{1}{k}(\frac{1}{\alpha}-\frac{1}{\beta}) \\ 0 \end{pmatrix}$$
$$= -\frac{1}{k^2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2 < 0.$$

Therefore, the condition (ii) in Theorem 1.1 is not satisfied. Moreover, $\pi_\gamma(\tau_k) - \tau_k = (\frac{1}{\gamma k}, 0)^T \to (0,0)^T$, for any $\gamma > 0$. Hence,

$$\nabla g_{\alpha\beta}(\tau_k) = \nabla F(\tau_k)(\pi_\beta(\tau_k) - \pi_\alpha(\tau_k)) - \alpha(\tau_k - \pi_\alpha(\tau_k)) + \beta(\tau_k - \pi_\beta(\tau_k)) \to (0,0)$$

as $k \to +\infty$. Therefore, (τ_k) is a stationary sequence of $g_{\alpha\beta}$, and so by Theorem 1.1, it is also a minimizing sequence of $g_{\alpha\beta}$.

Acknowledgements We are indebted to Professors Y.R. He, J.S. Pang and D. Ralph for helpful suggestions. We also thank the referee for careful reading and valuable comments.

References

1. Facchinei, F., Pang, J.S.: Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer, New York (2003)
2. Pang, J.S., Fukushima, M.: Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games. Comput. Manag. Sci. **1**, 21–56 (2005)
3. Harrington, J.E., Hobbs, B.F., Pang, J.S., Liu, A., Roch, G.: Collusive game solutions via optimization. Math. Program. **104**, 407–435 (2005)
4. Pang, J.S., Hobbs, B.F.: Nash-Cournot equilibria in electric power markets with piecewise linear demand functions and joint constraints. Oper. Res. (to appear)
5. Cottle, R.W.: Nonlinear programs with positively bounded Jacobians. SIAM J. Appl. Math. **14**, 147–158 (1966)
6. Fukushima, M.: Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems. Math. Program. **53**, 99–110 (1992)
7. Fukushima, M.: Merit functions for variational inequality and complementarity problems. In: DiPillo, G., Giannessi, F. (eds.) Nonlinear Optimization and Applications, pp. 155–170. Plenum, New York (1996)
8. Fukushima, M., Pang, J.S.: Minimizing and stationary sequences of merit functions for complementarity problems and variational inequalities. In: Complementarity and Variational Problems, Baltimore, MD, 1995, pp. 91–104. SIAM, Philadelphia (1997)
9. Harker, P.T., Pang, J.S.: Finite dimensional variational inequality and nonlinear complementary problems: a survey of theory, algorithms and applications. Math. Program. **48**, 161–220 (1990)
10. Huang, L.R., Ng, K.F.: Equivalent optimization formulations and error bounds for variational inequality problems. J. Optim. Theory Appl. **125**, 299–314 (2005)
11. Jeyakumar, V., Luc, D.T.: Approximate Jacobian matrices for nonsmooth continuous maps and C^1-optimization. SIAM J. Control Optim. **36**, 1815–1832 (1998)
12. Jiang, H.Y., Qi, L.Q.: Local uniqueness and convergence of iterative methods for nonsmooth variational inequalities. J. Math. Anal. Appl. **196**, 314–331 (1995)
13. King, A.J., Rokafellar, R.T.: Sensitivity analysis for nonsmooth generalized equations. Math. Program. **55**, 193–202 (1992)
14. Kyparisis, J.: Uniqueness and differentiability of solutions of parametric nonlinear complementarity problems. Math. Program. **36**, 105–113 (1986)
15. Luc, D.T.: Existence results for densely pseudomonotone variational inequalities. J. Math. Anal. Appl. **254**, 291–308 (2001)
16. Luc, D.T.: The Frechet approximate Jacobian and local uniqueness in variational inequalities. J. Math. Anal. Appl. **268**, 629–646 (2002)
17. Mordukhovich, B.S.: Stability theory for parametric generalized equations and variational inequalities via nonsmooth analysis. Trans. Am. Math. Soc. **343**, 609–658 (1994)

18. Ng, K.F., Tan, L.L.: Error bounds for nonsmooth variational inequality problems. Math. Program. **110**, 405–429 (2007)
19. Pang, J.S.: Complementarity problems. In: Horst, R., Pardalos, P.M. (eds.) Handbook on Global Optimization, Kluwer Academic, Boston (1994)
20. Peng, J.M.: Equivalence of variational inequality problems to unconstrained minimization. Math. Program. **78**, 347–355 (1997)
21. Robinson, S.M.: Generalized equations. In: Bachem, A., Grötschel, M., Korte, B. (eds.) Mathematical Programming: The State of the Art, pp. 346–367. Springer, Berlin (1983)
22. Stampacchia, G.: Formes bilinaires coercitives sur les ensembles convexes. C.R. Acad. Sci. **258**, 4413–4416 (1964)
23. Wu, J.H., Florian, M., Marcotte, P.: A general descent framework for the monotone variational inequality problem. Math. Program. **61**, 281–300 (1993)
24. Xu, H.F.: Regularized gap functions and D-Gap functions for nonsmooth variational inequalities. Appl. Optim. **47**, 153–176 (2001)
25. Yamashita, N., Taji, K., Fukushima, M.: Unconstrained optimization reformulations of variational inequality problems. J. Optim. Theory Appl. **92**, 439–456 (1997)
26. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley, New York (1983)
27. Huang, L.R., Ng, K.F., Penot, J.P.: On minimizing and critical sequences in nonsmooth optimization. SIAM J. Optim. **10**, 999–1019 (2000)
28. Chou, C.C., Ng, K.F., Pang, J.S.: Minimizing and stationary sequences of constrained optimization problems. SIAM J. Control Optim. **36**, 1908–1936 (1998)