J Optim Theory Appl (2007) 133: 359–370
DOI 10.1007/s10957-007-9187-z

Weak and Strong Convergence Theorems for a Nonexpansive Mapping and an Equilibrium Problem

A. Tada · W. Takahashi

Published online: 15 May 2007

Abstract In this paper, we introduce two iterative sequences for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in a Hilbert space. Then, we show that one of the sequences converges strongly and the other converges weakly.

Keywords Equilibrium problems · Nonexpansive mappings · Firmly nonexpansive mappings · Weak and strong convergence · Monotonicity

1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.

The equilibrium problem for $f : C \times C \to \mathbb{R}$ is to find $x \in C$ such that $f(x, y) \geq 0$ for all $y \in C$. The set of such solutions is denoted by $EP(f)$.

A mapping S of C into H is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|$$

for all $x, y \in C$; see [1, 2]. We denote the set of fixed points of S by $F(S)$.

Numerous problems in physics, optimization, and economics reduce to finding a solution of the equilibrium problem. Some methods have been proposed to solve the equilibrium problem; see for instance [3–5]. In particular, Combettes and Hirstoaga [6] proposed several methods for solving the equilibrium problem. On the

Communicated by R. Glowinski.

A. Tada · W. Takahashi (✉)
Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Oh-okayama, Meguro, Tokyo, Japan
e-mail: wataru@is.titech.ac.jp

other hand, Mann [7], and Nakajo and Takahashi [8] considered iterative schemes for finding a fixed point of a nonexpansive mapping.

In this paper, motivated by [8, 9], we obtain weak and strong convergence theorems of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the equilibrium problem.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$; let C be a nonempty closed convex subset of H. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x. In a real Hilbert space H, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \le \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection of H onto C. It is also known that $y = P_C(x)$ is equivalent to $\langle x - y, y - z\rangle \ge 0$ for all $z \in C$.

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e. $f(x, y) + f(y, x) \le 0$ for any $x, y \in C$;
(A3) for each $x, y, z \in C$,
$\limsup_{t\downarrow 0} f(tz + (1-t)x, y) \le f(x, y)$;
(A4) $f(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

The following lemma appears implicitly in [3].

Lemma 2.1 *Let C be a nonempty closed convex subset of H, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \textit{for all } y \in C.$$

The following lemma was also given in [6].

Lemma 2.2 *For $r > 0$, $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C \mid f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \textit{ for all } y \in C \right\}$$

for all $x \in H$. Then, the following statements hold:

(i) T_r *is single-valued*;
(ii) T_r *is firmly nonexpansive*, *i.e. for any* $x, y \in H$,
$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;
(iii) $F(T_r) = EP(f)$;
(iv) $EP(f)$ *is closed and convex.*

3 Strong Convergence Theorem

In this section, we show a strong convergence theorem which solves the problem of finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a nonexpansive mapping.

Theorem 3.1 *Let C be a nonempty closed convex subset of H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in H$ and let*

$$u_n \in C \text{ such that } \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \text{for all } y \in C,$$

$$w_n = (1-\alpha_n)x_n + \alpha_n S u_n,$$

$$C_n = \{z \in H \mid \|w_n - z\| \leq \|x_n - z\|\},$$

$$D_n = \{z \in H \mid \langle x_n - z, x - x_n\rangle \geq 0\},$$

$$x_{n+1} = P_{C_n \cap D_n}(x),$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, 1]$ for some $a \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. *Then, $\{x_n\}$ converges strongly to $P_{F(S)\cap EP(f)}(x)$.*

Proof We show first that the sequence $\{x_n\}$ is well defined. Obviously, C_n is closed and D_n is closed and convex for every $n \in \mathbb{N}$. Since $C_n = \{z \in H \mid \|w_n - x_n\|^2 + 2\langle w_n - x_n, x_n - z\rangle \leq 0\}$, C_n is also convex; see [8]. So, $C_n \cap D_n$ is a closed convex subset of H for any $n \in \mathbb{N}$.

Let $v \in F(S) \cap EP(f)$. From $u_n = T_{r_n} x_n$, we have

$$\|u_n - v\| = \|T_{r_n} x_n - T_{r_n} v\| \leq \|x_n - v\| \tag{1}$$

for every $n \in \mathbb{N}$. From this, we have

$$\begin{aligned}\|w_n - v\| &\leq (1-\alpha_n)\|x_n - v\| + \alpha_n\|Su_n - v\| \\ &\leq (1-\alpha_n)\|x_n - v\| + \alpha_n\|u_n - v\| \\ &\leq (1-\alpha_n)\|x_n - v\| + \alpha_n\|x_n - v\| \\ &= \|x_n - v\|. \end{aligned} \tag{2}$$

So, we have $v \in C_n$; thus,

$$F(S) \cap EP(f) \subset C_n, \tag{3}$$

for every $n \in \mathbb{N}$.

Next we show by induction that $F(S) \cap EP(f) \subset C_n \cap D_n$ for each $n \in \mathbb{N}$. Since $F(S) \cap EP(f) \subset C_1$ and $D_1 = H$, we get

$$F(S) \cap EP(f) \subset C_1 \cap D_1.$$

Suppose that $F(S) \cap EP(f) \subset C_k \cap D_k$ for $k \in \mathbb{N}$. Then, there exists $x_{k+1} \in C_k \cap D_k$ such that

$$x_{k+1} = P_{C_k \cap D_k}(x).$$

Therefore, for each $z \in C_k \cap D_k$, we have

$$\langle x_{k+1} - z, x - x_{k+1} \rangle \geq 0.$$

Since $F(S) \cap EP(f) \subset C_k \cap D_k$, for any $z \in F(S) \cap EP(f)$ we have

$$\langle x_{k+1} - z, x - x_{k+1} \rangle \geq 0,$$

and hence $z \in D_{k+1}$. So, we get

$$F(S) \cap EP(f) \subset D_{k+1}.$$

From this and (3), we have

$$F(S) \cap EP(f) \subset C_{k+1} \cap D_{k+1}.$$

This means that the sequence $\{x_n\}$ is well defined. From Lemma 2.1, the sequence $\{u_n\}$ is also well defined.

Since $F(S) \cap EP(f)$ is a nonempty closed convex subset of H, there exists a unique $z' \in F(S) \cap EP(f)$ such that

$$z' = P_{F(S) \cap EP(f)}(x).$$

From $x_{n+1} = P_{C_n \cap D_n}(x)$, we have

$$\|x_{n+1} - x\| \leq \|z - x\|$$

for all $z \in C_n \cap D_n$. Since $z' \in F(S) \cap EP(f) \subset C_n \cap D_n$, we have

$$\|x_{n+1} - x\| \leq \|z' - x\| \tag{4}$$

for every $n \in \mathbb{N}$. Therefore, $\{x_n\}$ is bounded. From (1) and (2), $\{u_n\}$ and $\{w_n\}$ are also bounded.

Since $x_n = P_{D_n}(x)$ and $x_{n+1} \in D_n$, we have

$$\|x - x_n\| \leq \|x - x_{n+1}\|$$

for every $n \in \mathbb{N}$. Since $\{x_n\}$ is bounded, the sequence $\{\|x - x_n\|\}$ is bounded and nondecreasing. So, there exists $c \in \mathbb{R}$ such that

$$c = \lim_{n\to\infty} \|x - x_n\|.$$

Since $x_n = P_{D_n}(x)$, $x_{n+1} \in D_n$ and $\frac{x_n+x_{n+1}}{2} \in D_n$, from (4) we have

$$\begin{aligned}\|x - x_n\|^2 &\le \left\| x - \frac{x_n + x_{n+1}}{2} \right\|^2 \\ &= \left\| \frac{1}{2}(x - x_n) + \frac{1}{2}(x - x_{n+1}) \right\|^2 \\ &= \frac{1}{2}\|x - x_n\|^2 + \frac{1}{2}\|x - x_{n+1}\|^2 - \frac{1}{4}\|x_n - x_{n+1}\|^2.\end{aligned}$$

So, we get

$$\frac{1}{4}\|x_n - x_{n+1}\|^2 \le \frac{1}{2}\|x - x_{n+1}\|^2 - \frac{1}{2}\|x - x_n\|^2.$$

Since $\lim_{n\to\infty} \|x - x_n\| = c$, we obtain

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{5}$$

From $x_{n+1} \in C_n$, we have

$$\|x_n - w_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \le 2\|x_n - x_{n+1}\|.$$

By (5), we obtain

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{6}$$

For $v \in F(S) \cap EP(f)$, we have, from Lemma 2.2,

$$\begin{aligned}\|u_n - v\|^2 &= \|T_{r_n}x_n - T_{r_n}v\|^2 \le \langle T_{r_n}x_n - T_{r_n}v, x_n - v\rangle \\ &= \langle u_n - v, x_n - v\rangle = \frac{1}{2}\left\{\|u_n - v\|^2 + \|x_n - v\|^2 - \|x_n - u_n\|^2\right\};\end{aligned}$$

hence,

$$\|u_n - v\|^2 \le \|x_n - v\|^2 - \|x_n - u_n\|^2.$$

Therefore, by the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}\|w_n - v\|^2 &\le (1-\alpha_n)\|x_n - v\|^2 + \alpha_n\|Su_n - v\|^2 \\ &\le (1-\alpha_n)\|x_n - v\|^2 + \alpha_n\|u_n - v\|^2 \\ &\le (1-\alpha_n)\|x_n - v\|^2 + \alpha_n\left\{\|x_n - v\|^2 - \|x_n - u_n\|^2\right\} \\ &= \|x_n - v\|^2 - \alpha_n\|x_n - u_n\|^2.\end{aligned}$$

Since $\{\alpha_n\} \subset [a, 1]$, we get

$$\begin{aligned} a\|x_n - u_n\|^2 \le \alpha_n \|x_n - u_n\|^2 &\le \|x_n - v\|^2 - \|w_n - v\|^2 \\ &\le \|x_n - w_n\|\{\|x_n - v\| + \|w_n - v\|\}. \end{aligned}$$

From this and (6), we obtain

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{7}$$

Since $\liminf_{n\to\infty} r_n > 0$, we get

$$\lim_{n\to\infty} \left\| \frac{x_n - u_n}{r_n} \right\| = \lim_{n\to\infty} \frac{1}{r_n} \|x_n - u_n\| = 0. \tag{8}$$

Since $\alpha_n S u_n = w_n - (1 - \alpha_n) x_n$, we have

$$\begin{aligned} a\|u_n - Su_n\| \le \alpha_n \|Su_n - u_n\| &= \|w_n - (1 - \alpha_n)x_n - \alpha_n u_n\| \\ &\le (1 - \alpha_n)\|u_n - x_n\| + \|w_n - u_n\| \\ &\le \|u_n - x_n\| + \|w_n - x_n\| + \|x_n - u_n\| \\ &= 2\|x_n - u_n\| + \|x_n - w_n\|. \end{aligned}$$

From (6) and (7), we obtain also

$$\lim_{n\to\infty} \|u_n - Su_n\| = 0. \tag{9}$$

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$. From (7), we obtain also that $u_{n_i} \rightharpoonup w$. Since $\{u_{n_i}\} \subset C$ and C is closed and convex, we obtain $w \in C$.

We shall show $w \in EP(f)$. By $u_n = T_{r_n} x_n$, we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \text{for all } y \in C.$$

From the monotonicity of f, we get

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge f(y, u_n), \quad \text{for all } y \in C;$$

hence,

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \ge f(y, u_{n_i}), \quad \text{for all } y \in C.$$

From (8) and condition (A4), we have

$$0 \ge f(y, w), \quad \text{for all } y \in C.$$

For t with $0 < t \le 1$ and $y \in C$, let $y_t = ty + (1 - t)w$. Since $y \in C$ and $w \in C$, we obtain $y_t \in C$ and hence $f(y_t, w) \le 0$. So, we have

$$0 = f(y_t, y_t) \le tf(y_t, y) + (1 - t)f(y_t, w) \le tf(y_t, y).$$

Dividing by t, we get

$$f(y_t, y) \geq 0, \quad \text{for all } y \in C.$$

Letting $t \downarrow 0$ and from (A3), we get

$$f(w, y) \geq 0, \quad \text{for all } y \in C.$$

Therefore, we obtain $w \in EP(f)$.

We next show that $w \in F(S)$. Assume $w \notin F(S)$. Then, from the Opial theorem [10] and (9), we have

$$\begin{aligned}\liminf_{i\to\infty} \|u_{n_i} - w\| &< \liminf_{i\to\infty} \|u_{n_i} - Sw\| \\ &\leq \liminf_{i\to\infty}\{\|u_{n_i} - Su_{n_i}\| + \|Su_{n_i} - Sw\|\} \\ &\leq \liminf_{i\to\infty} \|u_{n_i} - w\|.\end{aligned}$$

This is a contradiction. So, we get $w \in F(S)$. Therefore, we obtain $w \in F(S) \cap EP(f)$.

From $z' = P_{F(S)\cap EP(f)}(x)$ and (4), we have

$$\begin{aligned}\|x - z'\| \leq \|x - w\| &\leq \liminf_{i\to\infty} \|x - x_{n_i}\| \\ &\leq \limsup_{i\to\infty} \|x - x_{n_i}\| \leq \|x - z'\|;\end{aligned}$$

hence,

$$\lim_{i\to\infty} \|x - x_{n_i}\| = \|x - w\| = \|x - z'\|.$$

Since H is a Hilbert space, we obtain

$$x_{n_i} \to w = z'.$$

Since $z' = P_{F(S)\cap EP(f)}(x)$, we can conclude that

$$x_n \to P_{F(S)\cap EP(f)}(x). \qquad \square$$

As direct consequences of Theorem 3.1, we can obtain two corollaries.

Corollary 3.1 *Let C be a nonempty closed convex subset of H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *such that $EP(f) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in H$ and let*

$$\begin{aligned}&u_n \in C \text{ such that } \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \text{for all } y \in C, \\ &C_n = \{z \in H \mid \|u_n - z\| \leq \|x_n - z\|\}, \\ &D_n = \{z \in H \mid \langle x_n - z, x - x_n\rangle \geq 0\}, \\ &x_{n+1} = P_{C_n \cap D_n}(x),\end{aligned}$$

for every $n \in \mathbb{N}$, *where* $\{r_n\} \subset (0, \infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$. *Then*, $\{x_n\}$ *converges strongly to* $P_{EP(f)}(x)$.

Proof Putting $S = I$ and $\alpha_n = 1$ in Theorem 3.1, we obtain Corollary 3.1. □

Corollary 3.2 *Let* C *be a nonempty closed convex subset of* H *and let* S *be a nonexpansive mapping of* C *into* H *such that* $F(S) \neq \emptyset$. *Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_1 = x \in H$ *and let*

$$\begin{aligned} &u_n \in C \text{ such that } \quad \langle y - u_n, u_n - x_n\rangle \geq 0, \quad \text{for all } y \in C,\\ &w_n = (1-\alpha_n)x_n + \alpha_n S u_n,\\ &C_n = \{z \in H \mid \|w_n - z\| \leq \|x_n - z\|\},\\ &D_n = \{z \in H \mid \langle x_n - z, x - x_n\rangle \geq 0\},\\ &x_{n+1} = P_{C_n \cap D_n}(x), \end{aligned}$$

for every $n \in \mathbb{N}$, *where* $\{\alpha_n\} \subset [a, 1]$ *for some* $a \in (0, 1)$. *Then*, $\{x_n\}$ *converges strongly to* $P_{F(S)}(x)$.

Proof Putting $f(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ in Theorem 3.1, we obtain Corollary 3.2. □

4 Weak Convergence Theorem

In this section, we consider a weak convergence theorem motivated by [9]. To prove it, we need two results which were used in [9].

Lemma 4.1 *Let* $\{\alpha_n\}$ *be a sequence of real numbers such that* $0 < a \leq \alpha_n \leq b < 1$ *for all* $n \in \mathbb{N}$. *Let* $\{v_n\}$ *and* $\{w_n\}$ *be sequences of* H *such that*

$$\begin{aligned} &\limsup_{n\to\infty} \|v_n\| \leq c, \qquad \limsup_{n\to\infty} \|w_n\| \leq c,\\ &\lim_{n\to\infty} \|\alpha_n v_n + (1-\alpha_n)w_n\| = c, \quad \text{for some } c > 0. \end{aligned}$$

Then, $\lim_{n\to\infty} \|v_n - w_n\| = 0$.

Lemma 4.2 *Let* C *be a nonempty closed convex subset of* H. *Let* $\{x_n\}$ *be a sequence in* H. *Suppose that, for all* $y \in C$,

$$\|x_{n+1} - y\| \leq \|x_n - y\|,$$

for every $n \in \mathbb{N}$. *Then*, $\{P_C(x_n)\}$ *converges strongly to some* $z \in C$.

Now, we show the following weak convergence theorem which solves the problem of finding a common element of the set of solutions for an equilibrium problem and the set of fixed points of a nonexpansive mapping.

Theorem 4.1 *Let C be a nonempty closed convex subset of H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in H$ and let*

$$u_n \in C \text{ such that } \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \text{for all } y \in C,$$
$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) S u_n,$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then, $\{x_n\}$ converges weakly to $w \in F(S) \cap EP(f)$, where $w = \lim_{n\to\infty} P_{F(S)\cap EP(f)}(x_n)$.

Proof By Lemma 2.1, $\{u_n\}$ and $\{x_n\}$ are well defined. Let $v \in F(S) \cap EP(f)$ and $u_n = T_{r_n} x_n$. For any $n \in \mathbb{N}$, we have

$$\|u_n - v\| = \|T_{r_n} x_n - T_{r_n} v\| \leq \|x_n - v\|.$$

Therefore, we have

$$\begin{aligned}\|x_{n+1} - v\| &\leq \alpha_n \|x_n - v\| + (1-\alpha_n)\|Su_n - v\| \\ &\leq \alpha_n \|x_n - v\| + (1-\alpha_n)\|u_n - v\| \\ &\leq \alpha_n \|x_n - v\| + (1-\alpha_n)\|x_n - v\| = \|x_n - v\|. \end{aligned} \tag{10}$$

So, there exists $c \in \mathbb{R}$ such that

$$c = \lim_{n\to\infty} \|x_n - v\|. \tag{11}$$

Hence, $\{x_n\}$ and $\{u_n\}$ are bounded.

Next, for $v \in F(S) \cap EP(f)$, as in the proof of Theorem 3.1, we get

$$\|u_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - u_n\|^2.$$

Therefore, we have

$$\begin{aligned}\|x_{n+1} - v\|^2 &\leq \alpha_n \|x_n - v\|^2 + (1-\alpha_n)\|Su_n - v\|^2 \\ &\leq \alpha_n \|x_n - v\|^2 + (1-\alpha_n)\|u_n - v\|^2 \\ &\leq \alpha_n \|x_n - v\|^2 + (1-\alpha_n)\big(\|x_n - v\|^2 - \|x_n - u_n\|^2\big) \\ &= \|x_n - v\|^2 - (1-\alpha_n)\|x_n - u_n\|^2 \\ &\leq \|x_n - v\|^2 - (1-b)\|x_n - u_n\|^2. \end{aligned}$$

So, we obtain

$$(1-b)\|x_n - u_n\|^2 \leq \|x_n - v\|^2 - \|x_{n+1} - v\|^2.$$

From (11), we get

$$\lim_{n\to\infty} \|x_n - u_n\| = 0.$$

Since $\liminf_{n\to\infty} r_n > 0$, we obtain

$$\lim_{n\to\infty} \left\| \frac{x_n - u_n}{r_n} \right\| = 0.$$

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to w. From $\|x_n - u_n\| \to 0$, we also have that $u_{n_i} \rightharpoonup w$. From $\{u_{n_i}\} \subset C$, we have $w \in C$.

Now, let us show $w \in F(S) \cap EP(f)$. First, as in the proof of Theorem 3.1, we can show $w \in EP(f)$. Let us show that $w \in F(S)$.

Let $v \in F(S) \cap EP(f)$. Since $\|Su_n - v\| \le \|u_n - v\| \le \|x_n - v\|$, from (11), we have

$$\limsup_{n\to\infty} \|Su_n - v\| \le c.$$

Further, we have

$$\lim_{n\to\infty} \|\alpha_n(x_n - v) + (1-\alpha_n)(Su_n - v)\| = \lim_{n\to\infty} \|x_{n+1} - v\| = c.$$

By Lemma 4.1, we obtain

$$\lim_{n\to\infty} \|Su_n - x_n\| = 0.$$

Also, we have

$$\|Su_n - u_n\| \le \|Su_n - x_n\| + \|x_n - u_n\|.$$

Therefore, we get

$$\lim_{n\to\infty} \|Su_n - u_n\| = 0.$$

From this and $u_{n_i} \rightharpoonup w$, we obtain $w \in F(S)$. Then, $w \in F(S) \cap EP(f)$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w'$. Then, we have

$$w' \in F(S) \cap EP(f).$$

If $w \neq w'$, from the Opial theorem [10] we get

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - w\| &= \liminf_{i\to\infty} \|x_{n_i} - w\| < \liminf_{i\to\infty} \|x_{n_i} - w'\| \\
&= \lim_{n\to\infty} \|x_n - w'\| = \liminf_{j\to\infty} \|x_{n_j} - w'\| \\
&< \liminf_{j\to\infty} \|x_{n_j} - w\| = \lim_{n\to\infty} \|x_n - w\|.
\end{aligned}$$

This is a contradiction. So, we have $w = w'$. This implies that

$$x_n \rightharpoonup w \in F(S) \cap EP(f).$$

Let $z_n = P_{F(S)\cap EP(f)}(x_n)$. Since $w \in F(S)\cap EP(f)$, we have

$$\langle x_n - z_n, z_n - w\rangle \geq 0.$$

Using (10) and Lemma 4.2, we have that $\{z_n\}$ converges strongly to some $w_0 \in F(S)\cap EP(f)$. Since $\{x_n\}$ converges weakly to w, we have

$$\langle w - w_0, w_0 - w\rangle \geq 0.$$

Therefore, we obtain

$$w = w_0 = \lim_{n\to\infty} P_{F(S)\cap EP(f)}(x_n). \qquad \square$$

As direct consequences of Theorem 4.1, we can obtain two corollaries.

Corollary 4.1 *Let C be a nonempty closed convex subset of H. Let f be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying* (A1)–(A4) *such that $EP(f)\neq\emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in H$; let*

$$u_n \in C \text{ such that } \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \text{for all } y \in C,$$
$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)u_n,$$

for every $n\in\mathbb{N}$, where $\{\alpha_n\}\subset[a,b]$ for some $a,b\in(0,1)$ and $\{r_n\}\subset(0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then, $\{x_n\}$ converges weakly to $w\in EP(f)$, where $w = \lim_{n\to\infty} P_{EP(f)}(x_n)$.

Proof Putting $S = I$ in Theorem 4.1, we obtain Corollary 4.1. $\square$

Corollary 4.2 *Let C be a nonempty closed convex subset of H and let S be a nonexpansive mapping of C into H such that $F(S)\neq\emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x\in H$; let*

$$u_n \in C \text{ such that } \quad \langle y - u_n, u_n - x_n\rangle \geq 0, \quad \text{for all } y\in C,$$
$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Su_n,$$

for every $n\in\mathbb{N}$, where $\{\alpha_n\}\subset[a,b]$ for some $a,b\in(0,1)$. Then, $\{x_n\}$ converges weakly to $w\in F(S)$, where $w = \lim_{n\to\infty} P_{F(S)}(x_n)$.

Proof Putting $f(x,y) = 0$ for all $x,y\in C$ and $r_n = 1$ in Theorem 4.1, we obtain Corollary 4.2. $\square$

References

1. Takahashi, W.: Convex Analysis and Approximation of Fixed Points. Yokohama Publishers, Yokohama (2000)
2. Takahashi, W.: Nonlinear Functional Analysis. Yokohama Publishers, Yokohama (2000)

3. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123–145 (1994)
4. Flam, S.D., Antipin, A.S.: Equilibrium programming using proximal-like algorithms. Math. Program. **78**, 29–41 (1997)
5. Moudafi, A., Thera, M.: Proximal and dynamical approaches to equilibrium problems. In: Lecture Notes in Economics and Mathematical Systems, vol. 477, pp. 187–201. Springer, New York (1999)
6. Combettes, P.L., Hirstoaga, S.A.: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**, 117–136 (2005)
7. Mann, W.R.: Mean value methods in iteration. Proc. Am. Math. Soc. **4**, 506–510 (1953)
8. Nakajo, K., Takahashi, W.: Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups. J. Math. Anal. Appl. **279**, 372–379 (2003)
9. Takahashi, W., Toyoda, M.: Weak convergence theorems for nonexpansive mappings and monotone mappings. J. Optim. Theory Appl. **113**, 417–428 (2003)
10. Opial, Z.: Weak convergence of the sequence of successive approximation for nonexpansive mappings. Bull. Am. Math. Soc. **73**, 591–597 (1967)