J Optim Theory Appl (2007) 133: 229–240
DOI 10.1007/s10957-007-9174-4

Existence Results for Set-Valued Vector Quasiequilibrium Problems

P.H. Sach · L.A. Tuan

Published online: 24 April 2007

Abstract This paper deals with the set-valued vector quasiequilibrium problem of finding a point (z_0, x_0) of a set $E \times K$ such that $(z_0, x_0) \in B(z_0, x_0) \times A(z_0, x_0)$, and, for all $\eta \in A(z_0, x_0)$,

$$(F(z_0, x_0, \eta), C(z_0, x_0, \eta)) \in \alpha,$$

where α is a subset of $2^Y \times 2^Y$ and $A : E \times K \to 2^K, B : E \times K \to 2^E, F : E \times K \times K \to 2^Y, C : E \times K \times K \to 2^Y$ are set-valued maps, with Y is a topological vector space. Two existence theorems are proven under different assumptions. Correct results of [Hou, S.H., Yu, H., Chen, G.Y.: J. Optim. Theory Appl. **119**, 485–498 (2003)] are obtained from a special case of one of these theorems.

Keywords Vector quasiequilibrium problems · Set-valued maps · Existence theorems · Diagonal quasiconvexity

1 Introduction

The quasi-equilibrium problem is that of finding a point $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in \widehat{B}(x_0) \times \widehat{A}(x_0)$ and

$$\varphi(z_0, x_0, \eta) \geq 0, \quad \forall \eta \in \widehat{A}(x_0), \tag{1}$$

Communicated by H.P. Benson.

The authors are indebted to the referees for valuable remarks.

P.H. Sach (✉)
Institute of Mathematics, Hanoi, Vietnam
e-mail: phsach@math.ac.vn

L.A. Tuan
Ninh Thuan College of Pedagogy, Ninh Thuan, Vietnam

where E (resp. K) is a subset of a topological vector space Z (resp. X), $\widehat{A}: K \to 2^K$ and $\widehat{B}: K \to 2^E$ are set-valued maps, and $\varphi: E \times K \times K \to \mathbb{R}$ (the real line) is a function. Existence results for this problem and its generalizations are obtained in Refs. [2–8] under various assumptions. Recently, new versions of the above quasi-equilibrium problem, where φ is replaced by a set-valued map F with values in a topological vector space Y, are introduced. More precisely, in Ref. [9] condition (1) is replaced by

$$F(z_0, x_0, \eta) \subset F(z_0, x_0, x_0) + C', \quad \forall \eta \in \widehat{A}(x_0),$$

or

$$F(z_0, x_0, x_0) \subset F(z_0, x_0, \eta) + C', \quad \forall \eta \in \widehat{A}(x_0),$$

where $F: E \times K \times K \to 2^Y$ is a set-valued map and $C' \subset Y$ is a convex cone. In Ref. [1] the following problems $(\widehat{\mathrm{P}}_i)$, $i = 1, 2, 3, 4$, are considered:

Problem $(\widehat{\mathrm{P}}_i)$: Find $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in \widehat{B}(x_0) \times \widehat{A}(x_0)$ and

$$\alpha_i(F(z_0, x_0, \eta), \widehat{C}(x_0)), \quad \forall \eta \in \widehat{A}(x_0),$$

where $F: E \times K \times K \to 2^Y$ and $\widehat{C}: K \to 2^Y$ are set-valued maps, α_i is a relation on 2^Y (i.e. α_i is a subset of $2^Y \times 2^Y$) defined by

$$\alpha_1 = \{(a, b) \in 2^Y \times 2^Y : a \not\subset b\},$$
$$\alpha_2 = \{(a, b) \in 2^Y \times 2^Y : a \subset b\},$$
$$\alpha_3 = \{(a, b) \in 2^Y \times 2^Y : a \cap b \neq \emptyset\},$$
$$\alpha_4 = \{(a, b) \in 2^Y \times 2^Y : a \cap b = \emptyset\},$$

and the symbol $\alpha_i(a, b)$ is used to denote that $(a, b) \in \alpha_i$.

Several results for the existence of solutions of each of Problems $(\widehat{\mathrm{P}}_i)$ are established in Theorems 3.1–3.5 of Ref. [1]. Unfortunately, as we will see in Sect. 4, all these results are not true. So, it is clear that to obtain correct results for the above problems we must use assumptions different from or stronger than those of Ref. [1].

In this paper, we study the following general problem:

Problem (P_α): Find a point $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in B(z_0, x_0) \times A(z_0, x_0)$ and

$$\alpha(F(z_0, x_0, \eta), C(z_0, x_0, \eta)), \quad \forall \eta \in A(z_0, x_0),$$

where $A: E \times K \to 2^K$, $B: E \times K \to 2^E$, $F: E \times K \times K \to 2^Y$, $C: E \times K \times K \to 2^Y$ are set-valued maps and α is an arbitrary relation on 2^Y, i.e., a subset of $2^Y \times 2^Y$.

Obviously, Problem (P_α) includes as special cases all the above Problems $(\widehat{\mathrm{P}}_i)$. We will give two general existence theorems (see Theorems 3.1 and 4.1 below) for Problem (P_α) with different assumptions. In Theorem 3.1 we deal with the case when A is continuous, while in Theorem 4.1 we assume that A has open lower sections (see Sect. 2). To conclude this section, let us observe that, when specializing Theorem 4.1 to the problems considered in Ref. [1], we get correct results.

2 Preliminaries

Let X be a topological space. Each subset of X is a topological space with a topology induced by the given topology of X. In this paper neighborhoods of points of X are assumed to be open. Neighborhoods of $x \in X$ are denoted by $U(x), U_1(x), U_2(x), \ldots$. The symbols $\operatorname{cl} A$ and $\operatorname{int} A$ are used to denote the closure and interior of a subset A of X. If A is a subset of a vector space then $\operatorname{co} A$ denotes the convex hull of A. The empty set is denoted by $\emptyset$.

Let $f : X \to 2^Y$ and $g : X \to 2^Y$ be set-valued maps between the topological spaces X and Y. Then, $f \cap g$ is the set-valued map

$$x \in X \mapsto (f \cap g)(x) = f(x) \cap g(x).$$

We write $f \subset g$ if $f(x) \subset g(x)$ for all $x \in X$. If $\psi : X \to 2^Z$ is a set-valued map between the topological spaces X and Z, then the map $V = f \times \psi : X \to 2^{Y \times Z}$ is defined by $V(x) = f(x) \times \psi(x)$ for all $x \in X$.

We use the continuity properties of set-valued maps in the usual sense of Ref. [10]. Namely, f is upper semicontinuous (usc) if, for any $x \in X$ and any open set $N \supset f(x)$, we have that $N \supset f(x')$ for all x' from some neighborhood $U(x)$ of x. The map f is lower semicontinuous (lsc) if, for any $x \in X$ and any open set N with $f(x) \cap N \neq \emptyset$, we have $f(x') \cap N \neq \emptyset$ for all x' from some neighborhood $U(x)$ of x. The map f is continuous if it is both usc and lsc. If the graph of f, defined by $\operatorname{gr} f$, is a closed (resp. open) set of $X \times Y$, then we say that f has a closed (resp. open) graph. Recall that $\operatorname{gr} f$ is the set of all points $(x, y) \in X \times Y$ such that $y \in f(x)$. We say that f has open lower sections if the inverse map $f^{-1} : Y \to 2^X$, defined by $f^{-1}(y) = \{x \in X : y \in f(x)\}$, is open-valued, i.e., for all $y \in Y f^{-1}(y)$ is open in X. It is known in Ref. [11] that a map having open lower sections must be lsc. It is easy to give examples proving that a continuous map may not have open lower sections. The map f is compact if $f(X)$ is contained in a compact set of Y. The map f is acyclic if it is usc and if, for all $x \in X$, $f(x)$ is nonempty, compact and acyclic. Here a topological space is called acyclic if all of its reduced Čech homology groups over rationals vanish. Observe from Ref. [12] that the Cartesian product of two acyclic sets is acyclic. We say that the map f is closed-valued (resp. open-valued, acyclic-valued, ...) if, for all $x \in X$, $f(x)$ is a closed (resp. open, acyclic, ...) set. A similar definition of a convex-valued map can be introduced if it takes values in a vector space.

We need the following fixed-point theorem (see Ref. [13]).

Theorem 2.1 *Let K be a nonempty convex subset of a locally convex Hausdorff topological vector space X. If $f : K \to 2^K$ is a compact acyclic map, then f has a fixed point, i.e., there exists $x_0 \in K$ such that $x_0 \in f(x_0)$.*

Let β be a relation on 2^Y, i.e., β be a subset of the Cartesian product $2^Y \times 2^Y$. For two sets $a \in 2^Y$ and $b \in 2^Y$, we write $\beta(a,b)$ if and only if $(a,b) \in \beta$. Denote by $\bar{\beta}$ the relation on 2^Y defined by $\bar{\beta} = 2^Y \times 2^Y \setminus \beta$. Then, the symbol $\bar{\beta}(a,b)$ means that $(a,b) \notin \beta$.

Let β be a relation on 2^Y. Let a be a nonempty convex subset of a topological vector space X, let $g : a \times a \to 2^Y$ and $d : a \times a \to 2^Y$ be set-valued maps such that, for all $(x,\eta) \in a \times a$, $g(x,\eta)$ and $d(x,\eta)$ are nonempty.

We say that the pair (g,d) is β-diagonally quasiconvex in η if, for each finite subset $\{x_i,\ i = 1,2,\dots,n\} \subset a$ and each point $x \in \text{co}\{x_i,\ i = 1,2,\dots,n\}$, there exists a point x_i such that $\beta(g(x,x_i), d(x,x_i))$. Observe that this diagonal quasiconvexity property generalizes all notions of diagonal quasiconvexity introduced in Ref. [1]. It is easy to verify that the pair (g,d) is β-diagonally quasiconvex in η if and only if the map

$$\eta \in a \mapsto s_\beta(\eta) := \{x \in a : \beta(g(x,\eta), d(x,\eta))\}$$

is a KKM-map in the sense that

$$\text{co}\{\eta_i,\ i = 1,2,\dots,n\} \subset \bigcup_{i=1}^{n} s_\beta(\eta_i),$$

for each finite set $\{\eta_i,\ i = 1,2,\dots,n\} \subset a$. Therefore, applying the KKM lemma (Ref. [14]) to the map s_β, we can derive the following result.

Proposition 2.1 *Let a be a nonempty compact convex subset of a topological vector space X and let the pair (g,d) be β-diagonally quasiconvex in η. If for all $\eta \in a$ the above set $s_\beta(\eta)$ is closed in a, then there exists $x \in a$ such that $\beta(g(x,\eta), d(x,\eta))$, $\forall \eta \in a$.*

Proposition 2.2 *The pair (g,d) is β-diagonally quasiconvex in η if and only if $x \notin \text{co}\, t_{\bar{\beta}}(x)$ for all $x \in a$, where*

$$t_{\bar{\beta}}(x) := \{\eta \in a :\ \bar{\beta}(g(x,\eta), d(x,\eta))\}.$$

Proof Obviously, the pair (g,d) is not β-diagonally quasiconvex in η if and only if there exist $\{x_i,\ i = 1,2,\dots,n\} \subset a$ and $x \in \text{co}\{x_i,\ i = 1,2,\dots,n\}$ such that $\bar{\beta}(g(x,x_i), d(x,x_i))$ (i.e., $x_i \in t_{\bar{\beta}}(x)$) for all $i = 1,2,\dots,n$. In other words, the pair (g,d) is not β-diagonally quasiconvex in η if and only if there exists $x \in a$ such that $x \in \text{co}\, t_{\bar{\beta}}(x)$. □

Corollary 2.1 *Let $t_{\bar{\beta}}(x)$ be convex and let $\beta(g(x,x), d(x,x))$ for all $x \in a$. Then, the pair (g,d) is β-diagonally quasiconvex in η.*

Proof If there exists $x \in a$ such that $x \in \text{co}\, t_{\bar{\beta}}(x) = t_{\bar{\beta}}(x)$, then we have $\bar{\beta}(g(x,x), d(x,x))$, a contradiction to the condition $\beta(g(x,x), d(x,x))$ of Corollary 2.1. So, $x \notin \text{co}\, t_{\bar{\beta}}(x)$, for all $x \in a$. It remains to apply Proposition 2.2. □

3 First Existence Result

Throughout this paper, we assume that X, Y and Z are locally convex Hausdorff topological vector spaces and that $E \subset Z$ and $K \subset X$ are nonempty subsets. We also assume that $A : E \times K \to 2^K$, $B : E \times K \to 2^E$, $F : W \to 2^Y$, $C : W \to 2^Y$, $G : W \to 2^Y$ and $D : W \to 2^Y$ are set-valued maps with nonempty values, where $W := E \times K \times K$ is the Cartesian product of the topological spaces E, K and K.

Lemma 3.1 *Let $E \subset Z$ and $K \subset X$ be nonempty sets. Let $A : E \times K \to 2^K$ be a lsc map and let $N : E \times K \to 2^K$ be a closed map. Then, the following map $\varphi : E \times K \to 2^K$ is closed:*

$$(z, \xi) \in E \times K \mapsto \varphi(z, \xi) = \bigcap_{\eta \in A(z,\xi)} N(z, \eta).$$

Proof It suffices to show that the complement of the graph of φ in the topological space $E \times K \times K$ is open. In other words, assuming that

$$\tilde{w} := (\tilde{z}, \tilde{\xi}, \tilde{x}) \notin \operatorname{gr} \varphi := \{(z, \xi, x) \in E \times K \times K : x \in \varphi(z, \xi)\},$$

we can find neighborhoods $U(\tilde{z}), U(\tilde{\xi})$ and $U(\tilde{x})$ such that $w \notin \operatorname{gr} \varphi$ for all $w := (z, \xi, x) \in U(\tilde{z}) \times U(\tilde{\xi}) \times U(\tilde{x})$. Indeed, since $\tilde{w} \notin \operatorname{gr} \varphi$, there exists $\tilde{\eta} \in A(\tilde{z}, \tilde{\xi})$ with $\tilde{x} \notin N(\tilde{z}, \tilde{\eta})$, i.e., $(\tilde{z}, \tilde{\eta}, \tilde{x}) \notin \operatorname{gr} N$. By the closeness of the graph of N there exist neighborhoods $U_1(\tilde{z}), U(\tilde{\eta})$ and $U(\tilde{x})$ such that $(z, \eta, x) \notin \operatorname{gr} N$, i.e., $x \notin N(z, \eta)$, for all $(z, \eta, x) \in U_1(\tilde{z}) \times U(\tilde{\eta}) \times U(\tilde{x})$. Since $\tilde{\eta} \in A(\tilde{z}, \tilde{\xi}) \cap U(\tilde{\eta})$, i.e., $A(\tilde{z}, \tilde{\xi}) \cap U(\tilde{\eta}) \neq \emptyset$, by the lower semicontinuity of A there exist neighborhoods $U_2(\tilde{z})$ and $U(\tilde{\xi})$ such that $A(z, \xi) \cap U(\tilde{\eta}) \neq \emptyset$ for all $z \in U_2(\tilde{z})$ and $\xi \in U(\tilde{\xi})$. Setting $U(\tilde{z}) = U_1(\tilde{z}) \cap U_2(\tilde{z})$, we will prove that $w \notin \operatorname{gr} \varphi$ for all $w = (z, \xi, x) \in U(\tilde{z}) \times U(\tilde{\xi}) \times U(\tilde{x})$. Indeed, since $(z, \xi) \in U_2(\tilde{z}) \times U(\tilde{\xi})$ there exists a point $\eta \in A(z, \xi) \cap U(\tilde{\eta})$. Since $(z, \eta, x) \in U_1(\tilde{z}) \times U(\tilde{\eta}) \times U(\tilde{x})$ we get $x \notin N(z, \eta)$. Thus, given $w = (z, \xi, x) \in U(\tilde{z}) \times U(\tilde{\xi}) \times U(\tilde{x})$ we can find $\eta \in A(z, \xi)$ such that $x \notin N(z, \eta)$. This proves that $w \notin \operatorname{gr} \varphi$, as desired. □

Lemma 3.2 *Let E and K be convex. Let $A : E \times K \to 2^K$ be a compact upper semicontinuous map with closed values, let $B : E \times K \to 2^E$ be a compact acyclic map and $\varphi : E \times K \to 2^K$ be a closed map such that the map $S = \varphi \cap A$ has nonempty acyclic values. Then, for any map $T : E \times K \to 2^K$ such that $S \subset T$, the map $B \times T$ has a fixed point.*

Proof Observe that $f := B \times S$ satisfies all the conditions of Theorem 2.1 with $E \times K$ instead of K. Indeed, by [Ref. [10], Propositions 2 and 7, pp. 71–73] it is usc. Also, it has acyclic values (see the Introduction) and is a compact map (since $f \subset B \times A$ and since both B and A are compact maps). By Theorem 2.1, f has a fixed point. To conclude our proof, it remains to observe that $f \subset B \times T$. □

We will need the following conditions:

Condition (PS): For all $(z,\xi)\in E\times K$ and $x\in A(z,\xi)$,

$$[\forall \eta\in A(z,\xi), \beta(G(z,x,\eta), D(z,x,\eta))]$$
$$\Longrightarrow [\forall \eta\in A(z,\xi), \alpha(F(z,x,\eta), C(z,x,\eta))].$$

Condition (ps): For all $(z,\xi)\in E\times K$, $x\in A(z,\xi)$ and $\eta\in A(z,\xi)$,

$$\beta(G(z,x,\eta), D(z,x,\eta)) \Longrightarrow \alpha(F(z,x,\eta), C(z,x,\eta)).$$

Clearly, condition (ps) $\Longrightarrow$ condition (PS). Condition (ps) is satisfied if, for all $w\in W := E\times K\times K$,

$$\beta(G(w), D(w)) \Longrightarrow \alpha(F(w), C(w)). \tag{2}$$

A special case of (2) with $\beta=\alpha=\alpha_1$ is used in [Ref. [15], Theorem 3.1, condition (iv)] and [Ref. [16], Theorem 3.2, condition (iii)]. For a special case of condition (PS) with $\beta=\alpha=\alpha_1$ and $A(z,\xi)\equiv K$, see e.g. [Ref. [16], Theorem 3.4, condition (iv), and Remark 3.3].

Before formulating the main result of this section, let us introduce the set-valued maps $N_\beta : E\times K\to 2^K$, $L_\alpha : E\times K\to 2^K$, $S_\beta : E\times K\to 2^K$ and $T_\alpha : E\times K\to 2^K$ defined by

$$N_\beta(z,\eta) = \{x\in K : \beta(G(z,x,\eta), D(z,x,\eta))\},$$
$$L_\alpha(z,\eta) = \{x\in K : \alpha(F(z,x,\eta), C(z,x,\eta))\},$$
$$S_\beta(z,\xi) = \{x\in A(z,\xi) : \beta(G(z,x,\eta), D(z,x,\eta)), \forall \eta\in A(z,\xi)\},$$
$$T_\alpha(z,\xi) = \{x\in A(z,\xi) : \alpha(F(z,x,\eta), C(z,x,\eta)), \forall \eta\in A(z,\xi)\}.$$

Theorem 3.1 *Let $E\subset Z$ and $K\subset X$ be nonempty convex sets. Let $A : E\times K\to 2^K$ be a compact continuous map with closed values, let $B : E\times K\to 2^E$ be a compact acyclic map. Let α and β be arbitrary relations on 2^Y. Let (G,D) be a pair of maps satisfying condition (PS). Assume additionally that the map N_β is closed and the map S_β has nonempty acyclic values. Then, there exists a solution of Problem* (P$_\alpha$).

Proof By Lemma 3.1, the map

$$(z,\xi)\in E\times K \mapsto \varphi_\beta(z,\xi) = \bigcap_{\eta\in A(z,\xi)} N_\beta(z,\eta)$$

is closed. On the other hand, $S_\beta(z,\xi)=\varphi_\beta(z,\xi)\cap A(z,\xi)$ and, by condition (PS), $S_\beta(z,\xi)\subset T_\alpha(z,\xi)$ for all $(z,\xi)\in E\times K$. Applying Lemma 3.2 proves that $B\times T_\alpha$ has a fixed point which is exactly a solution of Problem (P$_\alpha$). □

The following corollary is derived from Theorem 3.1 with $\beta=\alpha$ and $(G,D)=(F,C)$.

Corollary 3.1 *Let E, K, A and B be as in Theorem 3.1. Assume additionally that the map L_α is closed and the map T_α has nonempty acyclic values. Then, there exists a solution of Problem* (P_α).

Theorem 3.2 *Let $E \subset Z$ and $K \subset X$ be nonempty convex sets. Let $A : E \times K \to 2^K$ be a compact continuous map with closed convex values and let $B : E \times K \to 2^E$ be a compact acyclic map. Let α and β be arbitrary relations on 2^Y. Let (G, D) be a pair of maps satisfying condition (PS). Assume additionally that the map N_β is closed and has convex values and that, for each $z \in E$, the pair $(G(z,\cdot,\cdot), D(z,\cdot,\cdot))$ is β-diagonally quasiconvex in the variable η. Then, there exists a solution of Problem* (P_α).

Proof By Theorem 3.1, it suffices to verify that the map S_β has nonempty acyclic values. From the closeness of the map N_β, it follows that, for all $(z,\eta) \in E \times K$, the set $N_\beta(z,\eta)$ is closed in K. On the other hand, $A(z,\xi) \subset K$. So, for all $(z,\xi,\eta) \in E \times K \times K$ the set $N_\beta(z,\eta) \cap A(z,\xi)$ is closed in $A(z,\xi)$. Now, for fixed $(z,\xi) \in E \times K$, let us set $a = A(z,\xi)$, $g = G(z,\cdot,\cdot)$ and $d = D(z,\cdot,\cdot)$. Then, the set $s_\beta(\eta)$ mentioned in Proposition 2.1 is closed in a since in our case $s_\beta(\eta) = N_\beta(z,\eta) \cap A(z,\xi)$. Therefore, because of the validity of all the requirements of Proposition 2.1, we claim that, for all $(z,\xi) \in E \times K$, $S_\beta(z,\xi)$ is nonempty. To prove that $S_\beta(z,\xi)$ is acyclic it suffices to show that it is convex. Indeed, let us rewrite $S_\beta(z,\xi)$ as

$$S_\beta(z,\xi) = \bigcap_{\eta \in A(z,\xi)} [N_\beta(z,\eta) \cap A(z,\xi)].$$

Since $N_\beta(z,\eta) \cap A(z,\xi)$ is convex and since the intersection of a family of convex sets is convex we conclude that $S_\beta(z,\xi)$ is convex, as required. □

Corollary 3.2 *Let E, K, A and B be as in Theorem 3.2. Assume additionally that the map L_α is closed and has convex values and that, for each $z \in E$, the pair $(F(z,\cdot,\cdot), C(z,\cdot,\cdot))$ is α-diagonally quasiconvex in the variable η. Then, there exists a solution of Problem* (P_α).

Remark 3.1 If E and K are compact sets then the upper semicontinuity of the maps A and B used in Theorems 3.1 and 3.2 implies that the set

$$M = \{(z,x) \in E \times K : (z,x) \in B(z,x) \times A(z,x)\} \tag{3}$$

is closed in $E \times K$. This property will be assumed to be satisfied in Sect. 4, but instead of the continuity of A we will require that A has open lower sections. When $B(z,x) \equiv E$ and $A(z,x) \equiv A(x)$ then the closeness of M is equivalent to the closeness of the set $\{x \in K : x \in A(x)\}$. This requirement is introduced in Ref. [15].

4 Second Existence Result

This section is devoted to existence theorems where the map A satisfies assumptions different from those of Sect. 3. Namely, the continuity of set-valued map A will be replaced by the requirement that A has open lower sections and it is such that the set M of fixed points of the set-valued map $B \times A$ is closed in $E \times K$. Also, condition (PS) will be replaced by the stronger condition (ps).

Before formulating the main result of this section (Theorem 4.1), let us consider some lemmas.

Lemma 4.1 *Let m' be a subset of a topological vector space X, and let $\varphi : m' \to 2^{M'}$ and $\psi : m' \to 2^{M'}$ be set-valued maps with open lower sections such that $[\mathrm{co}\,\varphi(x)] \cap \psi(x) \neq \emptyset$ for all $x \in m$, where M' is a convex set and $m \subset m'$ is a subset closed in m'. Then, the map $\sigma : m' \to 2^{M'}$, defined by*

$$\sigma(x) = \begin{cases} [\mathrm{co}\,\varphi(x)] \cap \psi(x), & \text{if } x \in m, \\ \psi(x), & \text{if } x \in m' \setminus m, \end{cases}$$

has open lower sections.

Proof Let $\tilde{v} \in M'$ be an arbitrary point. Since $\phi = \mathrm{co}\,\varphi$ has open lower sections (see Ref. [11]) and since $m' \setminus m$ is open in m', both the sets $U_1 := \phi^{-1}(\tilde{v}) \cap \psi^{-1}(\tilde{v})$ and $U_2 := (m' \setminus m) \cap \psi^{-1}(\tilde{v})$ are open in m'. On the other hand, it is obvious that $U_i \subset \sigma^{-1}(\tilde{v}),\ i = 1, 2$.

To prove that $\sigma^{-1}(\tilde{v})$ is open in m', it suffices to show that, for any point $\tilde{x} \in \sigma^{-1}(\tilde{v})$, there exists a set U such that U is open in m' and $\tilde{x} \in U \subset \sigma^{-1}(\tilde{v})$. Obviously, $U = U_1$ (resp. $U = U_2$) has this property if $\tilde{x} \in m$ (resp. $\tilde{x} \in m' \setminus m$). The proof of Lemma 4.1 is thus complete. □

Remark 4.1 Lemma 4.1 can be established by using the following formulas (see Ref. [17]), valid for all $\tilde{v} \in M'$:

$$\sigma^{-1}(\tilde{v}) = [\psi^{-1}(\tilde{v}) \cap \phi^{-1}(\tilde{v})] \cup [(m' \setminus m) \cap \psi^{-1}(\tilde{v})].$$

Lemma 4.2 *Let α and β be relations on 2^Y. Let $E \subset Z$ and $K \subset X$ be compact convex sets. Let $A : E \times K \to 2^K$ be a map with convex values and open lower sections, and let $B : E \times K \to 2^E$ be a compact acyclic map such that the set M (see (3)) is closed in $E \times K$. Let $L : E \times K \to 2^K$ and $N : E \times K \to 2^K$ be such that, for all $(z, x) \in M$,*

$$A(z, x) \cap L(z, x) \subset A(z, x) \cap N(z, x). \tag{4}$$

Assume additionally that N has open lower sections and that, for all $(z, x) \in M$,

$$x \notin \mathrm{co}\, N(z, x). \tag{5}$$

Then, there exists a point $(z_0, x_0) \in M$ such that $A(z_0, x_0) \cap L(z_0, x_0) = \emptyset$.

Proof Assume to the contrary that, for all $(z, x) \in M$,

$$A(z, x) \cap L(z, x) \neq \emptyset,$$

which by (4) implies that

$$A(z,x) \cap N(z,x) \neq \emptyset.$$

Setting $Q := \operatorname{co} N$, we derive from Lemma 4.1 that the map $H : E \times K \to 2^K$, defined by

$$H(z,x) = \begin{cases} Q(z,x) \cap A(z,x), & \text{if } (z,x) \in M, \\ A(z,x), & \text{if } (z,x) \in E \times K \setminus M, \end{cases}$$

has open lower sections. Observing that $E \times K$ is a compact Hausdorff topological space and H has nonempty convex values, we claim from Theorem 8.1.3 of [Ref. [18], p. 97] that H has a continuous selection, i.e., there exists a continuous single-valued map $h : E \times K \to K$ such that $h(z,x) \in H(z,x)$ for all $(z,x) \in E \times K$. Now, let us construct a set-valued map $\psi : E \times K \to 2^{E \times K}$ by setting $\psi(z,x) = B(z,x) \times \{h(z,x)\}$. Observe from Theorem 2.1 that ψ has a fixed point, i.e., there exists a point $(z_0, x_0) \in E \times K$ such that $(z_0, x_0) \in \psi(z_0, x_0)$. Since $\psi \subset B \times A$ it follows that (z_0, x_0) is also a fixed point of $B \times A$, i.e., $(z_0, x_0) \in M$. Since $(z_0, x_0) \in M$ and (z_0, x_0) is a fixed point of ψ, we have

$$(z_0, x_0) \in B(z_0, x_0) \times H(z_0, x_0),$$

which implies that

$$x_0 \in [\operatorname{co} N(z_0, x_0)] \cap A(z_0, x_0) \subset \operatorname{co} N(z_0, x_0),$$

a contradiction to (5). □

Remark 4.2 If in the proof of Lemma 4.2 we set $H \equiv A$, then the continuous selection h must be such that $h(z,x) \in A(z,x)$. This implies that a fixed point of $\psi = B \times h$ is also a fixed point of $B \times A$, i.e., $M \neq \emptyset$. This result is obtained under the assumption that $A : E \times K \to 2^K$ is a map with convex values and open lower sections, and that $B : E \times K \to 2^E$ is a compact acyclic map. Thus, the set M appearing in Lemma 4.2 must be nonempty.

Consider now the set-valued maps $N'_{\bar\beta} : E \times K \to 2^K$ and $L'_{\bar\alpha} : E \times K \to 2^K$ defined by

$$N'_{\bar\beta}(z,x) = \{\eta \in K : \bar\beta(G(z,x,\eta), D(z,x,\eta))\},$$

$$L'_{\bar\alpha}(z,x) = \{\eta \in K : \bar\alpha(F(z,x,\eta), C(z,x,\eta))\}.$$

Theorem 4.1 *Let $E \subset Z$ and $K \subset X$ be nonempty compact convex sets. Let $A : E \times K \to 2^K$ be a map with convex values and open lower sections, and let $B : E \times K \to 2^E$ be a compact acyclic map such that the set M (see (3)) is closed in $E \times K$. Let α and β be arbitrary relations on 2^Y. Let (G, D) be a pair of maps satisfying condition (ps). Assume additionally that $N'_{\bar\beta}$ has open lower sections and that, for all $(z,x) \in M$,*

$$x \notin \operatorname{co} N'_{\bar\beta}(z,x). \tag{6}$$

Then, there exists a solution of Problem (P_α).

Proof Condition (ps) yields (4) with $L = L'_{\bar{\alpha}}$ and $N = N'_{\bar{\beta}}$. Applying Lemma 4.2 proves that there exists a point $(z_0, x_0) \in M$ such that the intersection of $A(z_0, x_0)$ and $L'_{\bar{\alpha}}(z_0, x_0)$ is empty. This proves that (z_0, x_0) is a solution of (P_α). □

Remark 4.3 Condition (6) holds if for all $z \in E$ the pair $(G(z,\cdot,\cdot), D(z,\cdot,\cdot))$ is β-diagonally quasiconvex in the variable η (see Proposition 2.2).

Corollary 4.1 *Let all the assumptions of Theorem 4.1 be satisfied, except for condition (6). Assume additionally that:*

(i) *For all* $(z, x) \in E \times K$, $N'_{\bar{\beta}}(z, x)$ *is convex.*
(ii) *For all* $(z, x) \in E \times K$, $\bar{\beta}(G(z, x, x), D(z, x, x))$.

Then, there exists a solution of Problem (P_α).

Proof By Corollary 2.1, for all $z \in E$, the pair $(G(z,\cdot,\cdot), D(z,\cdot,\cdot))$ is β-diagonally quasiconvex in η. Corollary 4.1 is thus a consequence of Theorem 4.1 and Remark 4.3. □

The following result is derived from Theorem 4.1 with $\beta = \alpha$ and $(G, D) = (F, C)$.

Corollary 4.2 *Let* E, K, A *and* B *be as in Theorem 4.1. Assume additionally that* $L'_{\bar{\alpha}}$ *has open lower sections and that, for all* $(z, x) \in M$, $x \notin \operatorname{co} L'_{\bar{\alpha}}(z, x)$. *Then, there exists a solution of Problem* (P_α).

From Corollary 4.2 and Remark 4.3 we obtain the following result.

Corollary 4.3 *Let* E, K, A *and* B *be as in Theorem 4.1. If* $L'_{\bar{\alpha}}$ *has open lower sections and if, for all* $z \in E$, *the pair* $(F(z,\cdot,\cdot), C(z,\cdot,\cdot))$ *is* α*-diagonally quasiconvex in the variable* η, *then there exists a solution of Problem* (P_α).

Remark 4.4 Corollary 4.3 fails to hold if the closeness assumption of M is not satisfied. As an example illustrating this remark, let us take the following example from Ref. [19].

Example 4.1 Consider Problem (P_{α_1}), where $X = Y = Z = \mathbb{R}$, $E = K = [0, 1] \subset \mathbb{R}$, $B(z, x) \equiv \{1\}$, $C(z, x, \eta) \equiv -\operatorname{int}\mathbb{R}_+$ (the negative half-line), $F(z, x, \eta) = \{z(x - \eta)\} \subset \mathbb{R}$ for all $z, x, \eta \in [0, 1]$ and

$$A(z, x) = \begin{cases} [0, 1], & \text{if } x \in [0, 1), \\ \{0\}, & \text{if } x = 1. \end{cases}$$

In this example there does not exist a solution of Problem (P_{α_1}), though all the assumptions of Corollary 4.3, except the closeness of the set M, are satisfied.

Remark 4.5 Theorems 3.1 and 3.3 of Ref. [1] give sufficient conditions for the existence of solutions of Problems $(\widehat{\mathrm{P}}_1)$ and $(\widehat{\mathrm{P}}_4)$ (see the Introduction). Unfortunately, although all conditions of each of these theorems are satisfied in Example 4.1, it is easy to see that in this example both Problems $(\widehat{\mathrm{P}}_1)$ and $(\widehat{\mathrm{P}}_4)$ have no solution. Hence, both Theorems 3.1 and 3.3 of Ref. [1] are incorrect. From Corollary 4.2, we see that the absence of the closeness of the set M is a reason for this incorrectness.

Remark 4.6 If in Example 4.1 we replace condition $C(z,x,\eta) \equiv -\operatorname{int}\mathbb{R}_+$ by the condition $C(z,x,\eta) \equiv \mathbb{R}_+$, then we obtain a counterexample proving that Theorems 3.4 and 3.5 of Ref. [1], which give conditions for the existence of solutions of Problems $(\widehat{\mathrm{P}}_3)$ and $(\widehat{\mathrm{P}}_2)$, are incorrect. This incorrectness disappears if we add the assumption of the closeness of M to each of these theorems.

The following result gives the existence of solutions of Problem (P_α) without the compactness and convexity assumptions for E and K.

Theorem 4.2 *Let all the assumptions of Theorem 4.1 be satisfied, except for the compactness and convexity of the sets E and K. Assume additionally that there exist nonempty compact convex sets $E_1 \subset E$, $K_1 \subset K$ and a nonempty set $K_2 \subset K_1$ such that:*

(i) $A(E_1 \times K_2) \subset K_1$.
(ii) *For all $(z,x) \in E_1 \times K_1$, the set $A(z,x) \cap K_1$ is nonempty and the set $B(z,x) \cap E_1$ is nonempty and acyclic.*
(iii) *For all $(z,x) \in E_1 \times (K_1 \setminus K_2)$ with $z \in B(z,x)$, there exists $\eta \in A(z,x) \cap K_1$ such that $\bar{\alpha}(F(z,x,\eta), C(z,x,\eta))$.*

Then, there exists a solution (z_0,x_0) of Problem (P_α) with $(z_0,x_0) \in E_1 \times K_2$.

Proof Let us consider the maps

$$
\begin{aligned}
(z,x) \in E_1 \times K_1 &\mapsto A_1(z,x) := A(z,x) \cap K_1,\\
(z,x) \in E_1 \times K_1 &\mapsto B_1(z,x) := B(z,x) \cap E_1,\\
(z,x) \in E_1 \times K_1 &\mapsto L'_{1\bar{\alpha}}(z,x) := L'_{\bar{\alpha}}(z,x) \cap K_1,\\
(z,x) \in E_1 \times K_1 &\mapsto N'_{1\bar{\beta}}(z,x) := N'_{\bar{\beta}}(z,x) \cap K_1.
\end{aligned}
$$

Since all the assumptions of Theorem 4.1 with $E_1, K_1, A_1, B_1, L'_{1\bar{\alpha}}$ and $N'_{1\bar{\beta}}$ instead of $E, K, A, B, L'_{\bar{\alpha}}$ and $N'_{\bar{\beta}}$ are satisfied, we can find

$$(z_0,x_0) \in M_1 := \{(z,x) \in E_1 \times K_1 : (z,x) \in B_1(z,x) \times A_1(z,x)\}$$

such that

$$\alpha(F(z_0,x_0,\eta), C(z_0,x_0,\eta)), \quad \forall \eta \in A_1(z_0,x_0). \tag{7}$$

Since $z_0 \in B_1(z_0,x_0)$, we derive from (iii) that condition (7) cannot be satisfied if $x_0 \in K_1 \setminus K_2$. Therefore, $x_0 \in K_2$, which together with (i) yields $A_1(z_0,x_0) = A(z_0,x_0) \cap K_1 = A(z_0,x_0)$. From this and (7), we conclude that (z_0,x_0) is a solution of Problem (P_α). □

References

1. Hou, S.H., Yu, H., Chen, G.Y.: On vector quasiequilibrium problems with set-valued maps. J. Optim. Theory Appl. **119**, 485–498 (2003)
2. Chan, D., Pang, J.S.: The generalized quasivariational inequality problems. Math. Oper. Res. **7**, 211–222 (1982)
3. Chang, S., Lee, B.S., Wu, X., Cho, Y.J., Lee, G.M.: On the generalized quasivariational inequality problems. J. Math. Anal. Appl. **203**, 686–711 (1996)
4. Chen, G.Y., Li, S.J.: Existence of solutions for a generalized quasivector variational inequality. J. Optim. Theory Appl. **90**, 321–334 (1996)
5. Cubiotti, P.: Generalized quasi-variational inequalities in infinite-dimensional normed spaces. J. Optim. Theory Appl. **92**, 457–475 (1997)
6. Lin, L.J., Park, S.: On some generalized quasiequilibrium problems. J. Math. Anal. Appl. **224**, 167–181 (1998)
7. Tian, G.: Generalized quasi-variational-like inequality problem. Math. Oper. Res. **18**, 752–764 (1993)
8. Yu, S.J., Yao, J.C.: On the generalized nonlinear variational inequalities and quasivariational inequalities. J. Math. Anal. Appl. **198**, 178–193 (1996)
9. Tan, N.X.: On the existence of solutions of quasivariational inclusion problems. J. Optim. Theory Appl. **123**, 619–638 (2004)
10. Aubin, J.P.: Mathematical Methods of Game and Economic Theory. North-Holland, Amsterdam (1979)
11. Yannelis, N.C., Prabhakar, N.D.: Existence of maximal element and equilibria in linear topological spaces. J. Math. Econ. **12**, 233–245 (1983)
12. Massey, W.S.: Singular Homology Theory. Springer, New York (1970)
13. Park, S.: Some coincidence theorems on acyclic multifunctions and applications to KKM theory. In: Tan, K.-K. (ed.) Fixed Point Theory and Applications, pp. 248–277. World Scientific, Singapore (1992)
14. Fan, K.: A generalization of Tychonoff's fixed point theorem. Math. Ann. **142**, 305–310 (1961)
15. Ansari, Q.H., Flores-Bazan, F.: Generalized vector quasiequilibrium problems with applications. J. Math. Anal. Appl. **277**, 246–256 (2003)
16. Lin, L.J., Wu, Z.T., Kassay, G.: Existence of equilibria for multivalued mappings and its application to vector equilibria. J. Optim. Theory Appl. **114**, 189–208 (2002)
17. Ding, X.P.: Existence of solutions for quasiequilibrium problem in noncompact topological spaces. Comput. Math. Appl. **39**, 13–21 (2000)
18. Klein, E., Thompson, A.C.: Theory of Correspondences. Wiley, New York (1984)
19. Sach, P.H., Tuan, L.A.: Generalizations of vector quasivariational inclusion problems with set-valued maps (2007, submitted)