JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 131, No. 3, pp. 469–483, December 2006 (© 2006)
DOI: 10.1007/s10957-006-9162-0

Modified Combined Relaxation Method for General Monotone Equilibrium Problems in Hilbert Spaces

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Communicated by F. Giannessi

Published Online: 3 November 2006

Abstract. In this paper, we study a class of general monotone equilibrium problems in a real Hilbert space which involves a monotone differentiable bifunction. For such a bifunction, a skew-symmetric type property with respect to the partial gradients is established. We suggest to solve this class of equilibrium problems with the modified combined relaxation method involving an auxiliary procedure. We prove the existence and uniqueness of the solution to the auxiliary variational inequality in the auxiliary procedure. Further, we prove also the weak convergence of the modified combined relaxation method by virtue of the monotonicity and the skew-symmetric type property.

Key Words. General monotone equilibrium problems, modified combined relaxation methods, auxiliary variational inequalities, skew-symmetric type properties.

1. Introduction

Let H be a real Hilbert space whose inner product and norm are denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Let Ω be a nonempty closed convex subset in H and let $f : \Omega \times \Omega \to R$ be a real-valued bifunction such that

$$f(x, x) = 0, \quad \forall x \in \Omega.$$

Then, we can define the equilibrium problem (EP): to find an element $x^* \in \Omega$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in \Omega. \tag{1}$$

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We denote by Ω^* the solution set of this problem. EP represents a rather common and suitable format for many problems arising in economics, mathematical physics, and operations research. Besides, it is closely related with other general problems in nonlinear analysis. For instance, it involves saddle-point problems and variational inequalities in the case when H is finite-dimensional; see e.g., Refs. 1–3 and references therein. A number of methods were designed for solving the EP (1). Most of them are extensions of the corresponding ones for saddle-point problems with convex-concave cost bifunctions; see e.g., Refs. 4–5. On the other hand, several implementable algorithms for concave-convex EP (1) were proposed in Refs. 6–9. In particular, in the case where f is continuously differentiable, the auxiliary procedures were based on an iteration of the gradient projection method, Frank-Wolfe method, and Newton method. In all cases, the corresponding algorithms involve derivative-free line search procedures (see Ref. 9). This approach is called combined relaxation (CR). Recently, Konnov (Ref. 10) considered the EP(1) in the case when H = R^n is a real n-dimensional Euclidean space. He presented a new CR method involving a derivative-free line search procedure for solving the EP (1) in the case where $f(\cdot, y)$ need not be concave. For the reader convenience, we include the Konnov method, where the function g will be defined in Section 2 and $\pi_\Omega(\cdot)$ denotes the projection mapping of H onto Ω.

Konnov's Method (CRM).

Step 0. Initialization. Choose a point $y_0 \in \Omega$ and a sequence of $n \times n$ symmetric matrices $\{A_k\}$ such that

$$\tau' \|p\|^2 \leq \langle A_k p, p\rangle \leq \tau'' \|p\|^2, \quad \forall p \in R^n, 0 < \tau' \leq \tau'' < \infty.$$

Choose numbers $\alpha \in (0, 1), \beta \in (0, 1), \gamma \in (0, 2)$, and $\theta \in (0, 1]$. Set $k := 0$.

Step 1. Auxiliary Procedure. Execute Steps 1a to 1c below.

Step 1a. Determine z_k as the unique solution of the following optimization problem:

$$\min_{x \in \Omega}\{\langle g(y_k), x - y_k\rangle + 0.5\langle A_k(x - y_k), x - y_k\rangle\}.$$

Step 1b. If $z_k = y_k$, stop. Otherwise, determine m as the smallest nonnegative integer such that

$$f(y_k + \beta^m \tilde{\theta}(z_k - y_k), y_k) \geq \alpha\beta^m \tilde{\theta}\langle g(y_k), y_k - z_k\rangle.$$

Step 1c. Set $\theta_k := \beta^m \tilde{\theta}, \quad x_k := y_k + \theta_k(z_k - y_k)$.

Step 2. Main Iteration. Set $g_k := f'_y(x_k, y_k)$. If $g_k = 0$, stop. Otherwise, set

$$y_{k+1} := \pi_\Omega[y_k - \gamma(f(x_k, y_k)/\|g_k\|^2)g_k];$$

set $k := k + 1$ and go to Step 1.

Note that there are a number of rules of choosing the matrix sequence $\{A_k\}$ satisfying the inequality in Step 0. In particular, the simplest choice $A_k \equiv I$ corresponds to z_k being the projection of $y_k - g(y_k)$ onto Ω. In the case where the feasible set Ω is defined by affine constraints, the optimization problem in Step 1a is a convex quadratic programming problem which can be solved by standard finite algorithms; see Refs. 11–14.

Furthermore, by virtue of the monotonicity of f and the skew-symmetric type property of f with respect to its partial gradients, Konnov (Ref. 10) proved that the sequence $\{y_k\}$ generated by the above method converges to a solution of the EP (1).

In this paper, the general monotone equilibrium problem involving a monotone differentiable bifunction in a real Hilbert space is considered. For such a bifunction, a skew-symmetric type property with respect to the partial gradients is established. Motivated and inspired by the Konnov method (CRM), we suggest and propose the modified combined relaxation method involving an auxiliary procedure for solving the equilibrium problem. Moreover, we prove the existence and uniqueness of the solution to the auxiliary variational inequality in the auxiliary procedure. Further, we prove also the weak convergence of the modified combined relaxation method by virtue of the monotonicity and the skew-symmetric type property.

2. Preliminary Considerations

First, we give definitions and some basic properties for bifunctions.

Definition 2.1. A bifunction $f : \Omega \times \Omega \to R$ is said to be:

(i) an equilibrium bifunction if $f(x, x) = 0, \forall x \in \Omega$;
(ii) monotone if $f(x, y) + f(y, x) \leq 0, \forall x, y \in \Omega$.

The blanket assumptions of this paper are the following:

(A1) Ω is a nonempty convex and closed subset of H, $\Omega \subset \Lambda$, where Λ is an open convex subset of H;
(A2) $f : \Lambda \times \Lambda R$ is a differentiable monotone equilibrium bifunction such that $f(x, \cdot)$ is convex for each $x \in \Lambda$;
(A3) the EP (1) is solvable.

In the rest of this paper, we define $g, \tilde{g} : \Lambda \to H$ as follows:

$$g(x) = f'_y(x, y)|_{y=x}, \quad \forall x \in \Lambda, \tag{2}$$

$$\hat{g}(y) = f'_x(x, y)|_{x=y}, \quad \forall y \in \Lambda. \tag{3}$$

Motivated and inspired by the Konnov CR method, we intend to give a modified CR method for solving the EP(1) in a real Hilbert space H under Assumptions (A1)–(A3). To this end, we need first to make use of the following property of monotone equilibrium bifunctions.

Proposition 2.1. See Ref. 10, p. 331 in the case where H is finite-dimensional. Let assumptions (A1)–(A2) be fulfilled. Then, for all $z \in \Lambda$, we have

$$f_x'(x, z)|_{x=z} = -f_y'(z, y)|_{y=z}. \tag{4}$$

Proof. Fix $z \in \Lambda$. By the convexity of $f(z, \cdot)$, we have

$$f(z, w) - f(z, z) \geq \langle f_y'(z, y)|_{y=z}, w - z\rangle, \quad \forall w \in \Lambda.$$

Since f is monotone, it follows that

$$-f(w, z) + f(z, z) \geq \langle f_y'(z, y)|_{y=z}, w - z\rangle, \quad \forall w \in \Lambda.$$

For brevity, set

$$\mu(m) = -f(w, z).$$

Note that Λ is open. Thus, whenever $t > 0$ is small enough, we deduce that, for each $u \in H$,

$$(\mu(z + tu) - \mu(z))/t \geq \langle g(z), tu\rangle/t = \langle g(z), u\rangle,$$

where g is defined by (2). From the differentiability of μ, it follows that, for each $u \in H$,

$$\langle \mu'(z), u\rangle = \lim_{t\to 0^+} [\mu(z + tu) - \mu(z)]/t \geq \langle g(z), u\rangle.$$

Setting $u = g(z) - \mu'(z)$ in this inequality gives

$$\|g(z) - \mu'(z)\|^2 \leq 0.$$

This implies that $\mu'(z) = g(z)$; i.e., (4) is valid as desired. □

Next, we need also to make use of an extension of the well-known Minty lemma.

Proposition 2.2. See the Minty Lemma in Ref. 1, Section 10.1. Suppose that assumptions (A1)–(A2) are fulfilled. If we denote by Ω^d the solution set to the problem of finding an element $y^* \in \Omega$ such that

$$f(x, y^*) \leq 0, \forall x \in \Omega, \tag{5}$$

then $\Omega^* = \Omega^d$.

Proof. At first, suppose that the EP(1) has a solution $x^* \in \Omega$. Then, we have

$$f(x^*, y) \geq 0, \quad \forall y \in \Omega.$$

Since f is monotone, it follows that

$$-f(y, x^*) \geq 0, \quad \forall y \in \Omega;$$

i.e.,

$$f(y, x^*) \leq 0, \quad \forall y \in \Omega.$$

This implies that $x^* \in \Omega$ is a solution of problem (5).

Conversely, suppose that problem (5) has a solution $x^* \in \Omega$. Then, we have

$$f(y, x^*) \leq 0, \quad \forall y \in \Omega.$$

Hence, for each $y \in \Omega$ and $t \in (0, 1)$, we get

$$[f(x^* + t(y - x^*), x^*) - f(x^*, x^*)]/t \leq 0.$$

Thus, taking the limit as $t \to 0^+$, we obtain

$$\langle \tilde{g}(x^*), y - x^* \rangle \leq 0, \quad \forall y \in \Omega,$$

where $\tilde{g}$ is defined by (3). By using (4), we have

$$\langle -g(x^*), y - x^* \rangle \leq 0, \quad \forall y \in \Omega;$$

i.e.,

$$\langle f'_y(x^*, y)|_{y=x^*}, y - x^* \rangle \geq 0, \quad \forall y \in \Omega.$$

According to (A2), $f(x, \cdot)$ is convex for each $x \in \Lambda$. Hence, for each $y \in \Omega$, we derive

$$\begin{aligned} f(x^*, y) &= f(x^*, y) - f(x^*, x^*) \\ &\geq \langle f'_y(x^*, y)|_{y=x^*}, y - x^* \rangle \geq 0. \end{aligned}$$

This shows that $x^* \in \Omega$ is a solution of the EP (1). □

Remark 2.1.

(i) Property (4) can be treated as the skew-symmetricity of monotone equilibrium bifunctions with respect to their partial gradients.
(ii) If f is a concave-convex equilibrium bifunction, the property (4) was established in the case when H is finite-dimensional; see Ref. 5.
(iii) From the proof of Proposition 2.2, it follows that the subdifferential of $-f(\cdot, y)$ is nonempty at y, but $f(\cdot, y)$ need not be concave in general.

We recall now the well-known optimality condition for the EP(1), which is intended to be used for establishing convergence results of the modified CR method in the next section.

Proposition 2.3. See Ref. 5, p. 238 in the case where H is finite-dimensional. Suppose that (A1)–(A2) are fulfilled. Then, the EP (1) is equivalent to the following VI: find $x^* \in \Omega$ such that

$$\langle g(x^*), y - x^* \rangle \geq 0, \quad \forall y \in \Omega, \tag{6}$$

where g is defined by (2).

Proof. Let $x^* \in \Omega$ be a solution of the EP(1). Then, we have

$$f(x^*, y) \geq 0, \quad \forall y \in \Omega.$$

Hence, for each $y \in \Omega$ and $t \in (0, 1)$, we obtain

$$[f(x^*, x^* + t(y - x^*)) - f(x^*, x^*)]/t \geq 0.$$

Taking the limit as $t \to 0^+$, we obtain (6). Conversely, let $x^* \in \Omega$ satisfy (6). Since $f(x, \cdot)$ is convex for each $x \in \Lambda$, we have that, for each $y \in \Omega$,

$$f(x^*, y) = f(x^*, y) - f(x^*, x^*) \geq \langle g(x^*), y - x^* \rangle \geq 0.$$

This implies that the EP (1) has a solution $x^* \in \Omega$. □

Remark 2.2. If the cost mapping g in (6) enjoys certain (generalized) monotonicity properties, one can suggest various algorithms to solve the VI (6) in R^n; see e.g., Ref. 15 and the references therein.

The proof of the following lemma is quite straightforward; hence, it is omitted.

Lemma 2.1. Let $A : \Omega \to H$ be sequentially continuous from the weak topology to the strong topology. Then, the function $g : \Omega \to R$, defined as $g(x) = \langle Ax, y - x \rangle$ for each fixed $y \in \Omega$, is weakly continuous.

For each $D \subseteq H$, we denote by co(D) the convex hull of D. A point-to-set mapping $G : H \to 2^H$ is called a KKM mapping if, for every finite subset $\{u_1, u_2, \ldots, u_n\}$ of H,

$$\text{co}(\{u_1, u_2, \cdots, u_n\}) \subseteq \bigcup_{i=1}^{n} G(u_i).$$

Lemma 2.2. See Ref. 16. Let K be an arbitrary nonempty subset in a Hausdorff topological vector space E and let $G : K \to 2^E$ be a KKM mapping. If $G(x)$ is closed for all $x \in K$ and is compact for at least one $x \in K$, then $\bigcap_{x \in K} G(x) \neq \emptyset$.

Lemma 2.3. See Ref. 17. Let $\{a_k\}$ and $\{b_k\}$ be two sequences of nonnegative real numbers satisfying the inequality

$$a_{k+1} \leq a_k + b_k, \quad \forall k \geq 0.$$

If $\sum_{k=0}^{\infty} b_k < \infty$, then $\lim_{k \to \infty} a_k$ exists.

In the sequel, we use the following notation. For a given sequence $\{x_k\}$, $\omega_w(x_k)$ denotes the weak ω-limit set of $\{x_k\}$; that is,

$$\omega_w(x_k) := \{x \in H : w-\lim_{i \to \infty} x_{k_i} = x \text{ for some subsequence } \{x_{k_i}\} \text{ of } \{x_k\}, k_i \uparrow \infty\},$$

where w-$\lim_{i \to \infty} x_{k_i} = x$ means the weak convergence of $\{x_{k_i}\}$ to x; i.e., $x_{k_i} \to x$ weakly.

3. Modified Combined Relaxation Method and Its Convergence

The modified combined relaxation method (Modified CRM, for short) for solving the EP(1) under Assumptions (A1)–(A3) can be described as follows.

Algorithm 3.1. Modified CRM.

Step 0. Initialization. Choose a point $y_0 \in \Omega$ and a sequence of mappings $A_k : \Omega \to H, k = 0, 1, \ldots$ such that, for all $x, y \in \Omega$,

$$\langle A_k x - A_k y, x - y \rangle \geq \tau' \|x - y\|^2, \tag{7a}$$

$$\|A_k x - A_k y\| \leq \tau'' \|x - y\|, \tag{7b}$$

with $0 < \tau' \leq \tau'' < \infty$. Choose numbers $\alpha \in (0, 1), \beta \in (0, 1), \gamma \in (0, 2)$, and $\tilde{\theta} \in (0, 1]$. Set $k := 0$.

Step 1. Auxiliary Procedure. See Steps 1a to 1c below.

Step 1a. Determine $z_k \in \Omega$ as the unique solution of the following VI:

$$\langle g(y_k) + 0.5(A_k Z_k - A_k y_k), x - z_k \rangle \geq 0, \quad \forall x \in \Omega. \tag{8}$$

Step 1b. If $z_k = y_k$, stop. Otherwise, determine m as the smallest nonnegative integer such that

$$f(y_k + \beta^m \tilde{\theta}(z_k - y_k), y_k) \geq \alpha \beta^m \tilde{\theta} \langle g(y_k), y_k - z_k \rangle. \tag{9}$$

Step 1c. Set $\theta_k := \beta^m \tilde{\theta}$, $x_k := y_k + \theta_k (z_k - y_k)$.

Step 2. Main Iteration. Set $g_k := f'_y(x_k, y_k)$. If $g_k = 0$, stop. Otherwise, choose two relaxation parameters $\alpha_k, \beta_k \in [0, 1]$, with $\alpha_k + \beta_k \leq 1$, and compute the $(n+1)$th iterate

$$\begin{aligned} y_{k+1} := &(1 - \alpha_k - \beta_k) y_k \\ &+ \alpha_k \pi_\Omega [y_k - \gamma (f(x_k, y_k)/\|g_k\|^2) g_k] + \beta_k e_k, \end{aligned} \tag{10}$$

where $\{e_k\}$ is an error sequence in Ω introduced to take into account possible inexact computation.

Remark 3.1. If $H = R^n$, if A_k is an $n \times n$ symmetric matrix and if the feasible set Ω is defined by affine constraints, then clearly (8) is a convex quadratic programming problem which can be solved by standard finite algorithms; see Refs. 11–14. We note also that condition (17) in Ref. 10 implies condition (7). This observation follows from the fact that, if we let $A_k = (a^k_{ij})$, then condition (17) in Ref. 10 inplies that $|a^k_{ij}| \leq \tau''$, $\forall_i$, $j = 1, \ldots, n$ and $\forall k = 1, 2, \ldots$.

In order to ensure the existence of solutions to the subproblem (8), we need the following assumption:

(A4) For each $k \geq 0$, $A_k : \Omega \to H$ is sequentially continuous form the weak topology to the strong topology.

First, we give some properties of the subproblem (8) and investigate the termination criteria.

Lemma 3.1. Let Assumptions (A1), (A2), (A4) hold. Then:

(i) Problem (8) has a unique solution.

(ii) For each $k \geq 0$, the solution z_k of problem (8) is a solution of the following optimization problem:

$$\min_{x \in \Omega} \{\langle g(y_k), x - y_k \rangle + 0.5 \langle A_k x - A_k y_k, x - y_k \rangle\}. \tag{11}$$

(iii) If $z_k = y_k$, then $y_k \in \Omega^*$.

(iv) $f(x_k, y_k) \geq 0$.

(v) If $g_k = 0$, then $x_k \in \Omega^*$.

Proof.

(i) Existence of Solutions of Problem (8). For the sake of simplicity, we write (8) as follows: find $\bar{x} \in \Omega$ such that

$$\langle g(y_k) + 0.5(A_k\bar{x} - A_k y_k), y - \bar{x}\rangle \geq 0, \quad \forall y \in \Omega.$$

For each fixed $k \geq 0$ and each $y \in \Omega$, we define

$$G(y) = \{x \in \Omega : \langle g(y_k) + 0.5(A_k x - A_k y_k), y - x\rangle \geq 0\}.$$

Note that, since $y \in G(y)$, $G(y)$ is nonempty for each $y \in \Omega$. Now, we claim that G is a KKM mapping. Indeed, suppose that there exists a finite subset $\{u_1, u_2, \cdots, u_n\}$ of Ω and that $\alpha_i \geq 0, \forall i = 1, 2, \cdots, n$, with $\sum_{i=1}^n \alpha_i = 1$ such that $\hat{x} = \sum_{i=1}^n \alpha_i u_i \notin G(u_i), \forall i = 1, 2, \cdots, n$. Then, we have

$$\begin{aligned} 0 &= \langle g(y_k) + 0.5(A_k\hat{x} - A_k y_k), \hat{x} - \hat{x}\rangle \\ &= \sum_{i=1}^n \alpha_i \langle g(y_k) + 0.5(A_k\hat{x} - A_k y_k), u_i - \hat{x}\rangle < 0, \end{aligned}$$

which is a contradiction. Hence, G is a KKM mapping.

In view of Assumption (A4) and Lemma 2.1, we can see readily that $G(y)$ is a weakly closed subset of Ω for each $y \in \Omega$. Moreover, from (7), we know that $G(y)$ is bounded and hence weakly compact for every point $y \in \Omega$. Hence, by Lemma 2.2, we have $\bigcap_{y\in\Omega} G(y) \neq \emptyset$, which clearly implies that there exists at least one solution to problem (8).

Uniqueness of Solutions of Problem (8). Let x_1 and x_2 be two solutions of problem (8). Then,

$$\langle g(y_k) + 0.5(A_k x_1 - A_k y_k), y - x_1\rangle \geq 0, \tag{12}$$

$$\langle g(y_k) + 0.5(A_k x_2 - A_k y_k), y - x_2\rangle \geq 0, \tag{13}$$

for all $y \in \Omega$. Taking $y = x_2$ in (12), $y = x_1$ in (13) and adding these inequalities, we get

$$\begin{aligned} &\langle g(y_k) + 0.5(A_k x_1 - A_k y_k), x_2 - x_1\rangle + \langle g(y_k) \\ &+ 0.5(A_k x_2 - A_k y_k), x_1 - x_2\rangle \geq 0, \end{aligned}$$

implying that

$$\langle A_k x_1 - A_k x_2, x_1 - x_2\rangle \leq 0.$$

According to (7) in Algorithm 3.1, we get

$$\tau' \|x_1 - x_2\|^2 \leq \langle A_k x_1 - A_k x_2, x_1 - x_2\rangle \leq 0;$$

therefore, $x_1 = x_2$ since $\tau' > 0$. Hence, the solution of problem (8) is unique.

(ii) The conclusion follows from the fact that VI (8) is a necessary and sufficient condition for the convex optimization problem (11); see e.g. Ref. 11.

(iii) Suppose that $z_k = y_k$. Then, it follows from (8) that

$$\langle g(y_k), x - y_k \rangle \geq 0, \quad \forall x \in \Omega.$$

By using Proposition 2.3, we derive $y_k \in \Omega^*$.

(iv) Taking $x = y_k$ in (8), we obtain

$$\langle g(y_k) + 0.5(A_k z_k - A_k y_k), y_k - z_k \rangle \geq 0;$$

hence, using (7), we have

$$\begin{aligned} \langle g(y_k), z_k - y_k \rangle &\leq -0.5\langle A_k z_k - A_k y_k, z_k - y_k \rangle \\ &\leq -0.5\tau' \|z_k - y_k\|^2 \leq 0. \end{aligned} \tag{14}$$

Utilizing (9) now yields $f(x_k, y_k) \geq 0$.

(v) Suppose that

$$g_k = f_y'(x_k, y_k) = 0.$$

By the convexity of $f(x_k, \cdot)$, we have that

$$f(x_k, x) - f(x_k, y_k) \geq \langle g_k, x - y_k \rangle = 0, \quad \forall x \in \Omega. \tag{15}$$

Setting $x = x_k$ in this inequality yields $f(x_k, y_k) \leq 0$. On account of (iv), we obtain

$$f(x_k, y_k) = 0,$$

which together with (15) implies that x_k solves the EP (1). The proof is complete. □

Therefore, the modified CRM can terminate only with a solution. For this reason, in what follows, we suppose that it generates an infinite sequence $\{y_k\}$. To obtain convergence, we need the following additional assumptions.

(A5) For each $y \in \Omega$, the gradient of the function $f(\cdot, y)$ is locally Lipschitz continuous.

(A6) $g : \Omega \to H$ is sequentially continuous from the weak topology to the strong topology where g is defined by (2).

Lemma 3.2. Let Assumptions (A1), (A2), (A4), (A5) hold. Then:

(i) It holds that

$$\langle g(y_k), z_k - y_k \rangle \leq -0.5\tau' \|z_k - y_k\|^2. \tag{16}$$

(ii) It holds that $\theta_k > 0$ and that, if $\{y_k\}$ is bounded, $\theta_k \geq \theta' > 0$, for $k = 0, 1, \ldots$.

(iii) It holds that

$$\langle g_k, y_k - y^* \rangle \geq f(x_y, y_k) \tag{17}$$

and that

$$\|y_{k+1} - y^*\|^2 \leq \|y_k - y^*\|^2 - \alpha_k \gamma(2-\gamma)[f(x_k, y_k)/\|g_k\|]^2$$
$$+ \beta_k \|e_k - y^*\|^2, \tag{18}$$

for all $y^* \in \Omega^*$.

Proof.

(i) It is easy to see that inequality (16) follows from (14).

(ii) For brevity, set

$$\hat{g}(x) = f'_x(x, y_k).$$

Taking any $\theta \in (0, \tilde{\theta})$, we have that, for some $\xi \in [0, 1]$,

$$\begin{aligned} f(y_k + \theta(z_k - y_k), y_k) &= f(y_k, y_k) + \theta \langle \hat{g}(y_k + \xi\theta(z_k - y_k)), z_k - y_k \rangle \\ &= \theta \langle \hat{g}(y_k), z_k - y_k \rangle + \theta \langle \hat{g}(y_k + \xi\theta(z_k - y_k)) \\ &\quad - \hat{g}(y_k), z_k - y_k \rangle \\ &\geq \theta \langle \tilde{g}(y_k), z_k - y_k \rangle - L_k(\theta \|z_k - y_k\|)^2, \end{aligned}$$

where L_k is the Lipschitz constant for $\hat{g}$ on the segment $[z_k, y_k]$. Taking into account (16) and (4), we obtain

$$\begin{aligned} f(y_k + \theta(z_k - y_k), y_k) &\geq -\theta \langle g(y_k), z_k - y_k \rangle - L_k(\theta \|z_k - y_k\|)^2 \\ &\geq \theta(1 - 2\theta L_k/\tau') \langle g(y_k), y_k - z_k \rangle \\ &\geq \alpha\theta \langle g(y_k), y_k - z_k \rangle, \end{aligned}$$

when

$$1 - 2\theta L_k/\tau' \geq \alpha,$$

or equivalently,

$$\theta \leq (1-\alpha)\tau'/2L_k.$$

On account of (9), we conclude that

$$\theta_k \geq \min\{\beta(1-\alpha)\tau'/2L_k, \tilde{\theta}\} > 0.$$

Moreover, if $\{y_k\}$ is bounded, so is $\{z_k\}$ due to (16). Hence, we must have $L_k \leq L' < \infty$ and $\theta_k \geq \theta' > 0$, for $k = 0, 1, \cdots$; i.e., assertion (ii) is true.

(iii) Take any $y^* \in \Omega^*$. Then, Proposition 2.2 implies that

$$f(x, y^*) \leq 0, \quad \forall x \in \Omega.$$

By the convexity of $f(x_k, \cdot)$, we have

$$\langle g_k, y^* - y_k \rangle \leq f(x_k, y^*) - f(x_k, y_k) \leq -f(x_k, y_k);$$

i.e., (17) holds.

It is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|Y\|^2 - \lambda(1-\lambda)\|x - y\|^2,$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. Hence, using (17), we have

$$\begin{aligned}\|y_{k+1} - y^*\|^2 = {} & \|(1-\alpha_k-\beta_k)(y_k - y^*) \\ & + \alpha_k(\pi_\Omega[y_k - \gamma(f(x_k, y_k)/\|g_k\|^2)g_k] - y^*) + \beta_k(e_k - y^*)\|^2 \\ \leq {} & (1-\alpha_k-\beta_k)\|y_k - y^*\|^2 + \alpha_k\|y_k \\ & - \gamma(f(x_k, y_k)/\|g_k\|^2)g_k - y^*\|^2 + \beta_k\|e_k - y^*\|^2 \\ \leq {} & (1-\alpha_k-\beta_k)\|y_k - y^*\|^2 + \alpha_k\|y_k - y^*\|^2 \\ & - 2\alpha_k\gamma(f(x_k, y_k)/\|g_k\|^2)\langle g_k, y_k - y^*\rangle \\ & + \alpha_k(\gamma f(x_k, y_k)/\|g_k\|)^2 + \beta_k\|e_k - y^*\|^2 \\ \leq {} & \|y_k - y^*\|^2 - \alpha_k\gamma(2-\gamma)(f(x_k, y_k)/\|g_k\|)^2 + \beta_k\|e_k - y^*\|^2.\end{aligned}$$

Thus, (18) is also fulfilled. □

We are ready now to obtain the convergence for the modified CRM.

Theorem 3.1. Suppose that Assumptions (A1)–(A6) hold and that an infinite sequence $\{y_k\}$ is generated by Algorithm 3.1. Assume additionally that the sequences $\{\alpha_k\}$, $\{\beta_k\}$, $\{e_k\}$ satisfy the following conditions:

(i) $\delta \leq \alpha_k \leq 1$, $\forall k \geq 0$, for some $\delta \in (0, 1]$;
(ii) $\sum_{k=0}^{\infty} \beta_k < \infty$;
(iii) $\{e_k\}$ is bounded.

Then, $\{y_k\}$ converges weakly to a solution of the EP (1).

Proof. From (9) and (16), it follows that

$$f(x_k, y_k) \geq \alpha\theta_k\tau'\|z_k - y_k\|^2/2.$$

By using (18), we obtain

$$\|y_{k+1} - y^*\|^2 \leq \|y_k - y^*\|^2 + \beta_k\|e_k - y^*\|^2.$$

Since $\sum_{k=0}^{\infty} \beta_k < \infty$ and $\{e_k\}$ is bounded, we get

$$\sum_{k=0}^{\infty} \beta_k \|e_k - y^*\|^2 < \infty.$$

From Lemma 2.3, we know that $\lim_{k\to\infty} \|y_k - y^*\|$ exists. Hence, $\{y_k\}$ is bounded. On account of Lemma 3.2 (ii), it now follows that

$$\theta_k \geq \theta' > 0, \quad \text{for } k = 0, 1, \cdots.$$

Note that $\delta \leq \alpha_k \leq 1, \forall k \geq 0$. Combining the above inequality with (18) now gives

$$\begin{aligned}
&\sum_{k=0}^{n} [\delta\gamma(2-\gamma)\alpha^2\theta'^2\tau'^2/4].(\|z_k - y_k\|^2/\|g_k\|)^2 \\
&\leq \sum_{k=0}^{n} \alpha_k\gamma(2-\gamma)\cdot[f(x_k, y_k)/\|g_k\|]^2 \\
&\leq \sum_{k=0}^{n} (\|y_k - y^*\|^2 - \|y_{k+1} - y^*\|^2) + \sum_{k=0}^{n} \beta_k\|e_k - y^*\|^2 \\
&\leq \|y_0 - y^*\|^2 + \sum_{k=0}^{n} \beta_k\|e_k - y^*\|^2,
\end{aligned}$$

which implies that

$$\lim_{k\to\infty} \|z_k - y_k\| = 0.$$

Since $\{y_k\}$ is bounded, so is $\{z_k\}$. Therefore, both $\{y_k\}$ and $\{z_k\}$ have weak limit points; i.e., $\omega_w(y_k) \neq \emptyset$ and $\omega_w(z_k) \neq \emptyset$.

On the one hand, we claim that $\omega_w(y_k) \subseteq \Omega^*$; i.e., each weak limit point of $\{y_k\}$ is a solution of the EP (1). Indeed, let $z_0 \in \omega_w(y_k)$ a weak limit point of $\{y_k\}$ and let $\{y_{ki}\}$ be a subsequence of $\{y_k\}$ such that $w - \lim_{i\to\infty} y_{k_i} = z_0$. Since $\lim_{i\to\infty} \|z_{k_i} - y_{k_i}\| = 0$, we deduce that

$$w - \lim_{i\to\infty} z_{k_i} = z_0.$$

Now, take any $x \in \Omega$. Then, from (8), it follows that

$$\langle g(y_{k_i}) + A_{k_i} z_{k_i} - A_{k_i} y_{k_i}, x - z_{k_i}\rangle \geq 0.$$

Utilizing (7), we obtain

$$\begin{aligned}
\langle g(y_{k_i}), x - z_{k_i}\rangle &\geq \langle A_{k_i} y_{k_i} - A_{k_i} z_{k_i}, x - z_{k_i}\rangle \\
&\geq -\tau''\|yk_{k_i} - z_{k_i}\| \cdot \|x - z_{k_i}\|.
\end{aligned} \tag{19}$$

Observe that

$$\begin{aligned}
&|\langle g(y_{k_i}), x - z_{k_i}\rangle - \langle g(z_0), x - z_0\rangle| \\
&|\langle g(y_{k_i}), y_{k_i} - z_{k_i}\rangle + \langle g(y_{k_i}), x - y_{k_i}\rangle - \langle g(z_0), x - z_0\rangle| \\
&\leq |\langle g(y_{k_i}), y_{k_i} - z_{k_i}\rangle| + |\langle g(y_{k_i}), x - y_{k_i}\rangle - \langle g(z_0), x - z_0\rangle| \\
&\leq \|g(y_{k_i})\| \cdot \|y_{k_i} - z_{k_i}\| + \|\langle g(y_{k_i}), x - y_{k_i}\rangle - \langle g(z_0), x - z_0\rangle|. \qquad (20)
\end{aligned}$$

According to Assumption (A6) and Lemma 2.1, we know that $\{g(y_{k_i})\}$ is bounded and that

$$\lim_{i\to\infty} |\langle g(y_{k_i}), x - y_{k_i}\rangle - \langle g(z_0), x - z_0\rangle| = 0.$$

Hence, it follows from (20) that

$$\lim_{i\to\infty} \langle g(y_{k_i}), x - z_{k_i}\rangle = \langle g(z_0), x - z_0\rangle.$$

Therefore, taking the limit on two sides of (19) as $i \to \infty$, we can see that

$$\langle g(z_0), x - z_0\rangle \geq 0, \quad \forall x \in \Omega.$$

In view of Proposition 2.3, we get $z_0 \in \Omega^*$.

On the other hand, we claim that the sequence $\{y_k\}$ converges weakly to a solution of the EP (1); that is, $\omega_w(y_k)$ is a singleton. Indeed, let $z_0, z_1 \in \omega_w(y_k)$ and let $\{y_{k_i}\}$, $\{y_{l_j}\}$ be two subsequences of $\{y_k\}$ such that

$$w - \lim_{i\to\infty} y_{k_i} = z_0, \quad w - \lim_{j\to\infty} y_{l_j} = z_1.$$

For each $y^* \in \Omega^*$, since $\lim_{k\to\infty} \|y_k - y^*\|$ exists, we have

$$\begin{aligned}
\lim_{k\to\infty} \|y_k - z_1\|^2 &= \lim_{i\to\infty} \|y_{k_i} - z_1\|^2 \\
&= \lim_{i\to\infty} \|y_{k_i} - z_0 + z_0 - z_1\|^2 \\
&= \lim_{i\to\infty} [\|y_{k_i} - z_0\|^2 + 2\langle y_{k_i} - z_0, z_0 - z_1\rangle + \|z_0 - z_1\|^2] \\
&= \lim_{i\to\infty} \|y_{k_i} - z_0\|^2 + \|z_0 - z_1\|^2 \\
&= \lim_{k\to\infty} \|y_k - z_0\|^2 + \|z_0 - z_1\|^2. \qquad (21)
\end{aligned}$$

Interchanging the role of z_0 and z_1 yields

$$\lim_{k\to\infty} \|y_k - z_0\|^2 = \lim_{k\to\infty} \|y_k - z_1\|^2 + \|z_1 - z_0\|^2. \qquad (22)$$

Adding (21) and (22), we obtain $z_0 = z_1$. This shows that $\omega_w(y_k)$ consists of one point. Therefore, $\{y_k\}$ converges weakly to a solution of the EP (1). □

References

1. BAIOCCHI, C., and CAPELO, A., *Variational and Quasivariational Inequalites: Applications to Free Boundary Problems*, John Wiley and Sons, New York, NY, 1984.
2. BLUM, E., and OETTLI, W., *From Optimization and Variational Inequalities to Equilibrium Problems*, Mathematics Student, Vol. 63, pp. 123–145, 1994.
3. HADJISAVVAS, N., and SCHAIBLE, S., *Quasimonotonicity and Pseudomonotonicity in Variational Inequalities and Equilibrium Problems*, Generalized Convexity, Generalized Monotonicity, Edited by J. P. Crouzeix, J. E. Martinez-Legaz, and M. Volle, Kluwer Academic Publishers, Dordrecht, Netherlands, pp. 257–275, 1998.
4. DEMYANOV, V. F., and PEVNYI, A. B., *Numerical Methods for Finding Saddle Points*, USSR Computational Mathematics and Mathematical Physics, Vol. 12, pp. 11–52, 1972.
5. GOLSHTEIN, E. G., and TRETYAKOV, N. V., *Augmented Lagrange Functions*, Nauka, Moscow, 1989 (in Russian).
6. POLYAK, B. T., *Introduction to Optimization*, Optimization Software, New York, NY, 1987.
7. KONNOV, I. V., *Combined Subgradient Methods for Finding Saddle Points*, Russian Mathematics, Vol. 36, pp. 30–33, 1992.
8. KONNOV, I. V., *A Two-Level Subgradient Method for Finding Equilibrium Points and Solving Related Problems*, USSR Computational Mathematics and Mathematical Physics, Vol. 33, pp. 453–459, 1993.
9. KONNOV, I. V., *Combined Relaxation Methods for Finding Equilibrium Points and Solving Related Problems*, Russian Mathematics, Vol. 37, pp. 44–51, 1993.
10. KONNOV, I. V., *Combined Relaxation Method for Monotone Equilibrium Problems*, Journal of Optimization Theory and Applications, Vol. 111, pp. 327–340, 2001.
11. ZOUTENDIJK, G., *Methods of Feasible Directions*, Elsevier, Amsterdam, Netherlands, 1960.
12. MORE, J., and TORALDO, G., *Algorithms for Bound-Constrained Quadratic Programming Problems*, Numerische Mahematik, Vol. 55, pp. 377–400, 1989.
13. MADSEN, K., NIELSEN, H. B., and PINAR, M. C., *Bound-Constrained Quadratic Programming via Piecewise-Quadratic Functions*, Mathematical Programming, Vol. 85, pp. 135–156, 1999.
14. MADSEN, K., NIELSEN, H. B., and PINAR, M. C., *A Finite Continuation Algorithm for Bound-Constrained Quadratic Programming*, SIAM Journal on Optimization, Vol. 9, pp. 62–83, 1999.
15. KONNOV, I. V., *Methods for Solving Finite-Dimensional Variational Inequalities*, DAS, Kazan, Russia, 1998 (in Russian).
16. FAN, K., *A Generalization of Tychonoff's Fixed-Point Problem*, Mathematische Annalen, Vol. 142, pp. 305–310, 1961.
17. TAN, K. K., and XU, H. K., *Approximating Fixed Points of Nonexpansive Mappings by the Ishikawa Iteration Process*, Journal of Mathematical Analysis and Applications, Vol. 178, pp. 301–308, 1993.