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TECHNICAL NOTE

Strong Duality for Proper Efficiency in Vector Optimization[1]

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Abstract. In this paper, we give counterexamples showing that the strong duality results obtained in Refs. 1–5 for several dual problems of multiobjective mathematical programs are false. We provide also the conditions under which correct results can be established.

Key Words. Multiobjective programs, properly efficient points, duality.

1. Introduction

Given a nonempty subset S of an Euclidean space R^n and the functions $f_i : R^n \to R, i \in I = \{1, 2, \dots, p\}$, we consider the following vector optimization problem:

$$\text{(VP)} \quad \min \quad f(x) := (f_1(x), f_2(x), \dots, f_p(x)), \tag{1}$$

$$\text{s.t.} \quad x \in S. \tag{2}$$

Let us give some definitions. A point x^0 is called a feasible point if $x^0 \in S$. A feasible point x^0 is called an efficient point of (VP) if there exists no other feasible point x such that $f_i(x) \le f_i(x^0), i \in I$, and if at least one of these inequalities is strict. An efficient point x^0 is called (Ref. 6) a properly efficientpoint of (VP) if

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there exists a positive number M such that, for each $i \in I$, we have

$$[f_i(x^0) - f_i(x)]/[f_j(x) - f_j(x^0)] \leq M,$$

for some $j \in I$ such that $f_j(x) > f_j(x^0)$ whenever $x \in S$ and $f_i(x) < f_i(x^0)$. We say that x^0 is an efficient point [resp. properly efficient point] of the problem of maximizing $f(x)$ subject to $x \in S$ if it is an efficient point [resp. properly efficient point] of the problem of minimizing $-f(x)$ subject to the same constraint $x \in S$.

In the duality theory of multiobjective mathematical programs, several problems, called dual problems of (VP), were proposed in several papers. In Refs. 1–5, it was pointed out that, under suitable assumptions, a solution of the dual problems, which is constructed from a properly efficient solution of the primal problem (VP), is a properly efficient solution. In this paper, we give counterexamples showing that this claim is false for all mentioned dual problems. We provide also conditions under which correct results are obtained.

We observe that our results are different from the corresponding ones of Ref. 7. This is because, in the present paper, we use the same notion of properly efficient solution in both the primal and dual problems, while in Ref. 7 the solutions of these problems are understood in a different sense: in the primal problem they are properly efficient; in the dual problem, they are efficient only.

Strong duality results where in both the primal and dual problems the proper efficiency property is replaced by the same weaker notion such as efficiency or weak efficiency were also established in the literature. Recent developments in this direction can be found e.g. in Refs. 8–11 (see also the references therein). In Ref. 12, efficiency is used in the primal problem, but duality is understood in the sense of generalized Lagrange duality. It is shown in Ref. 12 that, in this case, strong duality results can be obtained via the image space analysis if a suitable dual problem with set-valued objectives (Ref. 13) is introduced. It is known that the theory of vector variational inequalities, which was first considered in Ref. 14, is closely related to vector optimization theory. The reader who is interested in duality for vector variational inequalities is referred to Refs. 15–16.

For the reader convenience, let us recall a result of Geoffrion (Ref. 6).

Theorem 1.1. Let $x^0 \in S$. If there exists a vector $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_p)$ with positive components γ_i, $i \in I$, such that x^0 is a minimizer of the function $\sum_{i=1}^{p} \gamma_i f_i$ on S, then x^0 is a properly efficient point of (VP). The converse statement is true if S is a convex set and each f_i is a convex function on S.

2. Counterexamples

In this paper, we assume that

$$S = \{x : x \in C, g_k(x) \leq 0, k = 1, 2, \ldots, m\},$$

where C is a nonempty convex subset of R^n and $g_k : R^n \to R$ are given functions. In Ref. 2, it is assumed that f_i and g_k are composite functions,

$$f_i(x) = h_i(F_i(x)),$$
$$g_k(x) = l_k(G_k(x)),$$

where $h_i : R^q \to R$ and $l_k : R^q \to R$ are convex functions with subdifferentials $\partial h_i(\cdot)$ and $\partial l_k(\cdot)$, and where $F_i : R^n \to R^q$ and $G_k : R^n \to R^q$ are locally Lipschitz and differentiable maps with derivatives $F_i^q(\cdot)$ and $G_k'(\cdot)$. In Ref. 2, the following vector optimization problem (VD) was defined as the dual problem of (VP):

$$\text{(VD)} \quad \max \ f(u) := (f_1(u), f_2(u), \dots, f_p(u)), \tag{3}$$

$$\text{s.t.} \quad 0 \in \sum_{i=1}^{p} \tau_i \partial h_i(F_i(u))F_i'(u) + \sum_{k=1}^{m} \lambda_k \partial l_k(G_k(u))G_k'(u) - (C - u)^+, \tag{4}$$

$$\lambda_k l_k(G_k(u)) \geq 0, \quad k \in K, \tag{5}$$

$$u \in C, \quad \tau_i > 0, i \in I, \quad \lambda_k \geq 0, k \in K, \tag{6}$$

where $K = \{1, 2, \dots, m\}$ and A^+ denotes the set of all points $x \in R^n$ such that, for all $\xi \in A \subset R^n$, the inner product $\langle x, \xi \rangle$ is nonnegative. We denote by τ [resp. λ] the vector with components τ_i [resp. λ_k]. Thus, a feasible point of (VD) is a point $v = (u, \tau, \lambda) \in R^n \times R^p \times R^m$ satisfying (4)–(6).

Let $K(x^0)$ be the index set corresponding to the active constraints at $x^0 \in S$,

$$K(x^0) = \{k \in K : l_k(G_k(x^0)) = 0\}.$$

In Ref. 2, it is said that the generalized Slater condition holds at x^0 if

$$\exists y \in \text{cone}(C - x^0) \text{ s.t.} < v, G_k'(x^0)y >< 0, \forall v \in \partial l_k(G_k(x^0)), \forall k \in K(x^0),$$

where

$$\text{cone}\, A := \{\alpha a : a \in A, \ \alpha > 0\}$$

denotes the cone generated by A. In Ref. 2, it is said that condition (GRC) holds if, for all $x, y \in C$, there exist positive numbers $\alpha_i(x, y)$ and $\beta_k(x, y)$ and a point $\mu(x, y) \in C - y$ such that

$$F_i(x) - F_i(y) = \alpha_i(x, y)\, F_i'(y)\, \mu(x, y), \qquad i \in I,$$
$$G_k(x) - G_k(y) = \beta_k(x, y)\, G_k'(y)\, \mu(x, y), \qquad k \in K.$$

The following result was proved in Ref. 2, Theorem 8.6.2.

Theorem 2.1. Assume that the generalized Slater condition holds at x^0 and that condition (GRC) is verified at each feasible point of (VP) and (VD).

If x^0 is a properly efficient point of (VP), then there exists (τ^0, λ^0) such that $v^0 := (x^0, \tau^0, \lambda^0)$ is a properly efficient point of (VD) and the objective values at these points are equal.

The following counterexample shows that this result is false.

Counterexample 2.1. Let

$$n = m = 1,\ p = 2,\ f_1(x) = x,\ f_2(x) = x^2,\ g_1(x) = x,\ C = (-\infty, -1|.$$

Under these assumptions, this problem can be interpreted as problem (VP) with $q = 1$, $l_1(x) = G_1(x) = x$, and

(i) $h_1(x) = h_2(x) = x, \quad F_1(x) = x, \quad F_2(x) = x^2.$

Another approach to consider this problem as problem (VP) is that, instead of (i), we set

(ii) $h_1(x) = x, \quad h_2(x) = x^2, \quad F_1(x) = F_2(x) = x.$

In both cases (i) and (ii), the corresponding dual problem (VD) is described as follows:

$$\begin{array}{lll}
\text{(VD)} & \max & (f_1(u), f_2(u)) := (u, u^2), \\
 & \text{s.t.} & 0 \in \tau_1 + 2\tau_2 u + \lambda_1 - (C - u)^+, \\
 & & \lambda_1 u \geq 0, \\
 & & u \in C, \tau_1 > 0, \tau_2 > 0, \lambda_1 \geq 0.
\end{array}$$

We can check that $x^0 = -1$ is a global minimizer of $(2/3)f_1 + (1/3)f_2$ on the whole real line. Hence, by Theorem 1.1, $x^0 = -1$ is a properly efficient point of (VP). Observe that the generalized Slater condition holds at x^0 since $K(x^0)$ is an empty set. Condition (GRC) holds in both cases (i) and (ii). Indeed, this is obvious for case (ii) by the linearity of F_1 and F_2. Condition (GRC) holds also in case (i) by taking

$$\alpha_1(x, y)=1, \quad \alpha_2(x, y) = (x + y)/2y, \quad \beta_1(x, y) = 1, \quad \mu(x, y) = x - y,$$
for all $x, y \in C = (-\infty, -1]$.

Observe that $\alpha_2(x, y) > 0$, since $x < 0$ and $y < 0$. Thus by Theorem 2.1, for suitable (τ^0, λ^0), the point $v^0 = (x^0, \tau^0, \lambda^0) = (-1, \tau^0, \lambda^0)$ is a properly efficient point of (VD).

Now, for any number $a \in (2/3, 1)$, consider the points

$$u(a) = -a/2(1 - a) \in R, \quad \tau(a) = (a, 1 - a) \in R^2.$$

Then,

$$v(a) := (u(a), \tau(a), 0)$$

is a feasible point of (VD). Let us set $i = 2$, $j = 1$ in the definition of properly efficient point v^0. Then, it is a simple matter to check that

$$[f_2(u(a)) - f_2(v^0)]/[f_1(v^0) - f_1(u(a))] = (2 - a)/2(1 - a) \to +\infty, \quad \text{as } a \to 1,$$

a contradiction to the proper efficiency of v^0. Since this is proved in both cases (i) and (ii), we see that Theorem 2.1 is false, even though either all h_i and l_k or all F_i and G_k are linear functions.

In Ref. 4, it is assumed that $C = R^n$ and that f_i and g_k are of class C^1 with Fréchet derivatives $f'_u(\cdot)$ and $g'_k(\cdot)$. Instead of (VD), the following dual problem (VD)′ is proposed in Ref. 4:

$$\text{(VD)}' \quad \max \quad \bar{f}(v) := (\bar{f}_1(v), \bar{f}_2(v), \ldots, \bar{f}_p(v)), \tag{7}$$

$$\text{s.t.} \quad 0 = \sum_{i=1}^{p} \tau_i f'_i(u) + \sum_{k=1}^{m} \lambda_k g'_k(u), \tag{8}$$

$$u \in C = R^n, \quad \tau_i > 0, i \in I, \quad \lambda_k \geq 0, k \in K, \tag{9}$$

$$\sum_{i=1}^{p} \tau_i = 1, \tag{10}$$

where $v = (u, \tau, \lambda) \in R^n \times R^p \times R^m$ and

$$\bar{f}_i(v) = f_i(u) + \sum_{k=1}^{m} \lambda_k g_k(u).$$

A feasible point of (VD)′ is a point v satisfying (8)–(10). The following result is proved in Theorem 2.5 of Ref. 4.

Theorem 2.2. Let $C = R^n$. Let f_i and g_k be convex functions and let x^0 be a properly efficient point of (VP) at which a constraint qualification is satisfied. Then, there exists (τ^0, λ^0) such that (x^0, τ^0, λ^0) is a properly efficient point of (VD)′ and the objective values of (VP) and (VD)′ are equal.

Weir (Ref. 4) does not define the constraint qualification notion that he uses in Theorem 2.2. From the proof of Theorem 2.5 of Ref. 4, it is understood that a constraint qualification condition is any condition under which there exists (τ^0, λ^0) such that $v^0 := (x^0, \tau^0, \lambda^0)$ is a feasible point of (VD)′, $\sum_{k=1}^{m} \lambda_k^0 g_k(x^0) = 0$, and

(x^0, λ^0) is a maximizer of the following scalar optimization problem:

$$\begin{aligned} \max \quad & \sum_{i=1}^{p} \tau_i^0 f_i(u) + \sum_{k=1}^{m} \lambda_k g_k(u), \\ \text{s.t.} \quad & 0 = \sum_{i=1}^{p} \tau_i^0 f_i'(u) + \sum_{k=1}^{m} \lambda_k g_k'(u), \\ & \lambda_k \geq 0, \quad k \in K, \end{aligned}$$

where τ_i^0 [resp. λ_k] are the components of τ^0 [resp. λ]. Observe that, in this problem, τ^0 is fixed and (u, λ) is a variable.

The following counterexample shows that Theorem 2.2 fails to hold even in the case when all functions involved are linear.

Counterexample 2.2. Let

$$n = m = 1, \quad p = 2, \quad f_1(x) = x, \quad f_2(x) = -x, \quad g_1(x) = x - 1.$$

Then, the corresponding dual problem $(\text{VD})'$ is described as follows:

$$\begin{aligned} (\text{VD})' \quad \max \quad & \bar{f}(v) = (\bar{f}_1(v), \bar{f}_2(v)), \\ \text{s.t.} \quad & \tau_1 - \tau_2 + \lambda_1 = 0, \\ & \tau_1 > 0, \quad \tau_2 > 0, \quad \lambda_1 \geq 0, \\ & \tau_1 + \tau_2 = 1, \end{aligned}$$

where $u \in R$, $\tau = (\tau_1, \tau_2) \in R^2$, $\lambda_1 \in R$, and for any $v = (u, \tau, \lambda_1)$,

$$\bar{f}_1(v) = u + \lambda_1(u - 1), \quad \bar{f}_2(v) = -u + \lambda_1(u - 1).$$

Observe that

$$(1/2)f_1 + (1/2)f_2 \equiv 0,$$

on R. Hence, by Theorem 1.1, $x^0 = 0$ is a properly efficient point of (VP). Obviously, a constraint qualification is satisfied at x^0. Thus, by Theorem 2.2, there exists (τ^0, λ_1^0) such that $v^0 := (0, \tau^0, \lambda_1^0)$ is a properly efficient point of $(\text{VD})'$. Here, λ_1^0 cannot be equal to 1. Observe that, in our case, no constraint is imposed on the variable u of problem $(\text{VD})'$.

Now, for any positive number $a < 1$, let us set

$$\lambda_1(a) = a, \tau(a) = (\tau_1(a), \tau_2(a)),$$

where

$$\tau_1(a) = (1 - a)/2 \quad \text{and} \quad \tau_2(a) = (1 + a)/2.$$

Then, for any $u \in R$ sufficiently large, $v(a) := (u, \tau(a), \lambda_1(a))$ is a feasible point of $(\mathrm{VD})'$ and

$$\bar{f}_1(v(a)) - \bar{f}_1(v^0) = u(a+1) - a + \lambda_1^0 > 0,$$
$$\bar{f}_2(v^0) - \bar{f}_2(v(a)) = u(1-a) + a - \lambda_1^0 > 0.$$

Let us take $i = 1$ and $j = 2$ in the definition of the properly efficient point v^0. Then, we have

$$\begin{aligned}&[\bar{f}_1(v(a)) - f_1(v^0)]/[\bar{f}_2(v^0) - \bar{f}_2(v(a))]\\ &= [a + 1 + (\lambda_1^0 - a)/u]/[1 - a + (a - \lambda_1^0)/u]\\ &\to +\infty, \quad \text{as } a \to 1, u \to \infty.\end{aligned}$$

This contradicts the proper efficiency of v^0.

In Ref. 1, the following Mond-Weir dual problem is considered:

$$(\mathrm{VD})'' \quad \max\ f(u) := (f_1(u), f_2(u), \ldots, f_p(u)), \tag{11}$$
$$\text{s.t.} \quad 0 = \sum_{i=1}^{p} \tau_i f_i'(u) + \sum_{k=1}^{m} \lambda_k g_k'(u), \tag{12}$$
$$\sum_{k=1}^{m} \lambda_k g_k(u) \geq 0, \tag{13}$$
$$u \in C, \quad \tau_i > 0, i \in I, \quad \lambda_k \geq 0, k \in K, \tag{14}$$
$$\sum_{i=1}^{p} \tau_i = 1, \tag{15}$$

A feasible point of $(\mathrm{VD})''$ is a point $v = (u, \tau, \lambda) \in R^n \times R^p \times R^m$ satisfying (12)–(15).

Recall (Ref. 1) that, for an arbitrary map $\eta : S \times S \to R^n$, a differentiable function $h : R^n \to R$ is called η-invex on S at $u \in S$ if, for all $x \in S$,

$$h(x) - h(u) \geq h'(u)\eta(x, u). \tag{16}$$

In the case when the last inequality is replaced by the following implication:

$$h(x) - h(u) \leq 0 \Rightarrow h'(u)\eta(x, u) \leq 0, \tag{17}$$

then h is called η-quasiinvex on S at $u \in S$. Observe that a differentiable convex function (in particular, a linear function) h is always η-invex (and hence, η-quasiconvex), with $\eta(x, u) = x - u$.

The following strong duality property was proved in Theorem 2.4 of Ref. 1.

Theorem 2.3. Let C be an open set of R^n. Let x^0 be a properly efficient point of (VP) at which a constraint qualification holds at x^0. Then, there exists

(τ^0, λ^0) such that (x^0, τ^0, λ^0) is a feasible point of (VD)$''$ and $\sum_{k=1}^{m} \lambda_{k,}^0 g_k(x^0) = 0$. Also, if there is a map $\eta : S \times S \to R^n$ such that, for each feasible point (u, τ^0, λ) of (VD)$''$, $\sum_{i=1}^{p} \tau_i^0 f_i$ is η-invex on S at $u \in S$ and $\sum_{k=1}^{m} \lambda_k g_k$ is η-quasiinvex on on S at $u \in S$, then (x^0, τ^0, λ^0) is a properly efficient point of (VD)$''$.

The following counterexample shows that the claim of Theorem 2.3 is incorrect.

Counterexample 2.3. Let

$$n = m = p = 2, \quad C = R^2, \quad f_1(x) = x_1 + x_2, \quad f_2(x) = x_1,$$
$$g_1(x) = -x_1, \quad g_2(x) = -x_2,$$

where $x = (x_1, x_2) \in R^2$. Then, the corresponding dual problem (VD)$''$ is described as follows:

$$\begin{array}{lll} \text{(VD)}'' & \max & f(u) := (f_1(u), f_2(u)), \\ & \text{s.t.} & \tau_1 + \tau_2 - \lambda_1 = 0, \\ & & \tau_1 - \lambda_2 = 0, \\ & & \lambda_1(-u_1) + \lambda_2(-u_2) \geq 0, \\ & & \tau_1 + \tau_2 = 1, \\ & & \tau_1 > 0, \quad \tau_2 > 0, \quad \lambda_1 \geq 0, \quad \lambda_2 \geq 0, \end{array}$$

where

$$u = (u_1, u_2) \in R^2, \quad f_1(u) = u_1 + u_2, \quad f_2(u) = u_1.$$

Obviously, $x^0 = (0, 0) \in R^2$ is a minimizer of the function $(1/2)f_1 + (1/2)f_2$ on the feasible set of (VP). By Theorem 1.1, $x^0 = (0, 0)$ is a properly efficient point of (VP). Since the constraint functions of (VP) are linear, the Kuhn-Tucker constraint qualification holds at x^0 (see Ref. 17, p. 102). Also, the invexity and quasiinvexity properties required in Theorem 2.3 are satisfied, since all functions f_i and g_k are linear. Thus, by Theorem 2.3, there exists $(\tau^0, \lambda^0) \in R^2 \times R^2$ such that $v^0 := (x^0, \tau^0, \lambda^0) = (0, \tau^0, \lambda^0)$ is a properly efficient point of (VD)$''$.

Now, for any positive number $a > 1$, let us set

$$u(a) = (-a, a^2) \in R^2, \tau(a) = (1/a, 1 - 1/a) \in R^2, \lambda(a) = (1, 1/a) \in R^2.$$

Then, $v(a) := (u(a), \tau(a), \lambda(a))$ is a feasible point of (VD)$''$. By taking $i = 1$, $j = 2$ in the definition of the proper efficiency of v^0, we have

$$[f_1(u(a)) - f_1(u^0)]/[f_2(u^0) - f_2(u(a))] = a - 1 \to +\infty, \quad \text{as } a \to \infty,$$

a contradiction to the definition of proper efficiency of v^0.

Remark 2.1. Counterexample 2.2 shows also that the strong duality results of Theorems 2.2 and 3.1 of Ref. 1 and Theorem 4.5 of Ref. 5 are incorrect. Counterexample 2.3 shows also that the strong duality result obtained in Theorem 3.2 of Ref. 1 is incorrect.

Remark 2.2. Theorems 2.2 and 2.3 were generalized in Theorems 3.3 and 3.5 of Ref. 3 where the convexity, η-invexity, and η-quasiinvexity are replaced by the F-convexity, F-pseudoconvexity, and F-quasiconvexity with $F : S \times S \times R^n \to R$ being any function such that, for fixed $(x, u) \in S \times S$, $F(x, u, .)$ is sublinear. Recall (Ref. 3) that the function $h : R^n \to R$ is called F-convex [resp. F-quasiconvex] on S at $u \in S$ if, for all $x \in S$. condition (16) (resp. (17)] holds, with $F(x, u, h'(u))$ in place of $h'(u)\eta(x, u)$. The function h is called (Ref. 3) F-psendoconvex on S at u if, for all $x \in S$,

$$F(x, u, h'(u)) \geq 0 \;\Rightarrow\; h(x) \geq h(u). \tag{18}$$

Clearly, a linear function h is F-convex (hence, F-pseudoconvex and F-quasiconvex), with $F(x, u, y) =< y, x - u >$. From this and Counterexamples 2.2 and 2.3, it follows that the strong duality results of Theorems 3.3 and 3.5 of Ref. 3 are false.

Remark 2.3. To illustrate Theorem 2.5 of Ref. 4 (i.e. Theorem 2.2 of this paper), Weir (Ref. 4) considers the problem (VP) of minimizing

$$f(x) = (f_1(x), f_2(x)) = (x, x^2),$$

subject to

$$g(x) = x \leq 0, \quad x \in C = R.$$

The corresponding dual problem (VD)$'$ is the problem of maximizing

$$\bar{f}(v) = (\bar{f}_1(v), \bar{f}_2(v)) = (u + \lambda_1 u, u^2 + \lambda_1 u),$$

subject to

$$\tau_1 + 2\tau_2 u + \lambda_1 = 0, \quad \tau_1 > 0, \quad \tau_2 > 0, \quad \lambda_1 \geq 0, \quad \tau_1 + \tau_2 = 1,$$

where $v = (u, \tau, \lambda_1)$ and $\tau = (T\tau_1, \tau_2)$. It is easily checked that any point $x^0 < 0$ is a properly efficient point of (VP) at which a constraint qualification holds. Weir (Ref. 4) claims that, for any $x^0 < 0$, the point $v^0 = (x^0, \tau^0, \lambda_1^0)$ is a properly efficient point of (VD)$'$, where $\tau^0 = (2x^0/(2x^0 - 1), -1/(2x^0 - 1))$, $\lambda_1^0 = 0$. This claim is incorrect. Indeed, for any positive number $a < 1$, the point $v(a)$ mentioned in counterexample 2.1 is a feasible point of (VD)′ and we have

$$[\bar{f}_\lambda(v(a)) - \bar{f}_2(v^0)]/[\bar{f}_1(v^0) - \bar{f}_1(v(a))] = [a/2(1-a)] - x^0$$
$$\to +\infty, \text{ as } a \to 1.$$

This contradicts the proper efficiency of v^0.

Remark 2.4. We now give a more geometrical explanation of Counterexample 2.2. To this end, we need the following proposition which originated in Ref. 18 and can be obtained from Theorems 3.1.2 and 3.1.4 of Ref. 19.

Proposition 2.1. A feasible point $v^0 = (x^0, \tau^0, \lambda^0)$ is a properly efficient point of problem (VD)′ if and only if there exists a convex cone $T \subset R^p$ such that $R_+^p \backslash \{0\} \subset \text{ int } T$ and

$$[\bar{f}(v^0) + T] \cap \overline{f}(Q') = \{\overline{f}(v^0)\},$$

where R_+^p denotes the nonengative orthant of R^P, int T stands for the interior of T, and Q' is the set of all feasible points of (VD)′.

Turning to Counterexample 2.2, we have seen that $x^0 = 0$ is a properly efficient point of (VP) at which a constraint qualification is satisfied. Observe that, if $v = (u, \tau, \lambda_1) \in Q'$, then $\lambda_1 \in [0, 1)$. Conversely, if $\lambda_1 \in [0, 1)$, then there exists (u, τ) such that $(u, \tau, \lambda_1) \in Q'$.

For any $\lambda_1 \in [0, 1)$, let us set

$$F(\lambda_1) = \{(u + \lambda_1(u-1), -u + \lambda_1(u-1)) : u \in R\}.$$

Then,

$$F(\lambda_1) = \{(\omega, [(\lambda_1 - 1)]/[(1+\lambda_1)]\,\omega - 2\lambda_1/(1+\lambda_1)) : \omega \in R\},$$
$$\overline{f}(Q') = \bigcup_{\lambda_1 \in [0,1)} F(\lambda_1).$$

Let $\tau^0 = (\tau_1^0, \tau_2^0)$ and λ_1^0 be such that $v^0 = (0, \tau, \lambda_1^0) \in Q'0$ Let $T \subset R^2$ be any convex cone such that $R_+^2 \backslash \{0\} \subset \text{ int } T$. Then, there exist $\lambda_1 \in [0, 1)$ and $(u, \lambda_1) \in F(\lambda_1)$ such that $(\bar{u}, \bar{\lambda}_1) \in \bar{f}(v^0) + T$ and $(\bar{u}, \bar{\lambda}_1) \neq \bar{f}(v^0)$. By Proposition 2.1 v^0 is not a properly eficient point of (VD)′. So, Theorem 2.2 does not work for Counterexample 2.2.

3. Sufficient Conditions for Strong Duality

The counterexamples given in the previous section prove that Theorems 2.1–2.3 are not valid without additional assumptions. The aim of this section is to provide conditions under which the conclusions of these theorems are correct.

The following result is taken from Theorem 3.1.4 of Ref. 19 applied to maximization problems.

Lemma 3.1. Consider the problem of maximizing a vector-valued function f subject to $x \in S$. Then, a point $x^0 \in S$ is a properly eflicient point of this problem if and only if

$$R_+^p \cap \text{cl cone}\,(f(S) - f(x^0) - R_+^p) = \{0\},$$

where cl denotes the closure.

Now, let us denote by Q, Q', Q'' the sets of feasible points of (VD), (VD)$'$, (VD)$''$ respectively. For each point $v = (u, \tau, \lambda) \in R^n \times R^p \times R^m$, let us write $\tilde{f}(v) = f(u)$. We say that (VD) satisfies the closedness assumption at x^0 if the set cone $(\tilde{f}(Q) - f(x^0) - R_+^p)$ is closed, where f is the vector-valued objective function of (VD).

We say that (VD)$'$ satisfies the closefiness assumption at x^0 if the set cone $(\bar{f}(Q)' - f(x^0) - R_+^p)$ is closed, where $\bar{f}$ is the vector-valued objective function of (VD)$'$.

We say that (VD)$''$ satisfies the closedness assumption at x^0 if the set cone $(\tilde{f}(Q'') - f(x^0) - R_+^p)$ is closed, where f is the vector-valued objective function of (VD)$''$.

Theorem 3.1. In addition to the conditions of Theorem 2.1 [resp. Theorem 2.2; Theorem 2.3] assume that (VD) [resp. (VD)$'$; (VD)$''$] satisfies the closdness assumption at x^0. Then, the conclusions of Theorem 2.1 (resp. Theorem 2.2; Theorem 2.3) are true.

Proof. Let us prove only the conclusions of Theorem 2.1 under the closedness assumption. The case of Theorems 2.2 and 2.3 can be considered similarly. It is known from Ref. 2, p. 264 that there exists (τ^0, λ^0) such that $v^0 = (x^0, \tau^0, \lambda^0)$ is a feasible point of (VD). It remains to prove that v^0 is a properly efficient point of (VD). In view of Lemma 3.1 applied to the maximization problem (VD), it suffices to show that

$$R_+^p \cap \text{cl cone}(\tilde{f}(Q) - \tilde{f}(v^0) - R_+^p) = \{0\},$$

or equivalently,

$$R_+^p \cap \text{cone}(\tilde{f}(Q) - f(x^0) - R_+^p) = \{0\}, \tag{19}$$

by the closedness assumption. Indeed, assume to the contrary that (19) does not hold. Then, we must find $\alpha > 0$, $v = (u, \tau, \lambda) \in Q$, and $y \in R_+^p$ such that

$$\alpha(\tilde{f}(v) - f(x^0) - y) \in R_+^p \backslash \{0\},$$

which implies that $f(x^0) - \tilde{f}(v) \in -R^p_+\backslash\{0\}$. This contradiets the weak duality property (see Theorem 8.6.1 of Ref. 2), which says that $f(x^0) - \tilde{f}(Q)) \cap (-R^p_+\backslash\{0\}) = \emptyset$. □

Remark 3.1. From Theorem 3.1 it follows that the closedness assumption is not satisfied in each of Counterexamples 2.1–2.3.

Remark 3.2. Since the dual problems are those with different objective and constraint functions, the closedness assumption may be satisfied for some dual problems, while it is not satisfied for other dual problems. To illustrate this remark, let us consider the problem (VP) where

$$n = m = 1, \quad p = 2, \quad C = R^n, \quad f_1(x) = x, \quad f_2(x) = -x, \quad g_1(x) = x.$$

It is easy to verify that the closedness assumption at $x^0 = 0$ is satisfied for (VD) and (VD)$''$, but is violated for (VD)$'$.

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