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On the E-Epigraph of an E-Convex Function

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Abstract. In Ref 1, Yang shows that some of the results obtained in Ref. 2 on E-convex programming are incorrect, but does not prove that the results which make the connection between an E-convex function and its E-epigraph are incorrect. In this note, we show that the results obtained in Ref. 2 concerning the characterization of an E-convex function f in terms of its E-epigraph are incorrect. Afterward, some characterizations of E-convex functions using a different notion of epigraph are given.

Key Words. E-convex sets, E-convex functions, epigraphs, slack 2-convex sets, counterexamples.

1. Introduction

The concepts of E- convex set and E-convex function were introduced in Ref. 2. For convenience, we recall these definitions.

Definition 1.1. Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a function. A subset M of $\mathbb{R}^n$ is said to be E-convex if

$$(1 - t)E(x) + tE(y) \in M, \tag{1}$$

for all $x, y \in M$ and all $t \in [0, 1]$.

Definition 1.2. Let M be a nonempty subset of $\mathbb{R}^n$ and let $E : \mathbb{R}^n \to \mathbb{R}^n$ be a function. A function $f : M \to \mathbb{R}$ is said to be E-convex on M if M is E-convex and

$$f((1 - t)E(x) + tE(y)) \leq (1 - t)f(E(x)) + tf(E(y)), \tag{2}$$

for all $x, y \in M$ and all $t \in [0, 1]$.

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In Ref. 2, some of the properties of E-convex sets and E-convex functions are given. These concepts are used in the study of E-convex programming.

Unfortunately, some of the results obtained in Ref. 2 are incorrect. Indeed, in Ref. 1, Yang shows that some of the results obtained in Ref. 2 on E-convex programming are incorrect, but does not prove that the result which makes the connection between an E-convex function and its E-epigraph is incorrect. Some of the incorrect results of Ref. 2 on E-convex programming have been proved using the connection between the E-convex functions and their E-epigraphs, which is an incorrect one.

In this note, we show that the result obtained in Ref. 2 on the characterization of an E-convex function f in terms of its E-epigraph, $E - e(f)$, is incorrect. We give also some characterizations of E-convex function using a different notion of epigraph.

2. Counterexample

If M is a nonempty subset of $\mathbb{R}^n$ and if $f : M \to \mathbb{R}$ and $E : \mathbb{R}^n \to \mathbb{R}^n$ are two functions, then in Ref. 2 the set

$$E - e(f) = \{(x, \alpha) \mid x \in M, \alpha \in \mathbb{R}, f(E(x)) \leq \alpha\}$$

is called the E-epigraph of f.

If $S \subseteq \mathbb{R}^n \times \mathbb{R}$ and $E : \mathbb{R}^n \to \mathbb{R}^n$, then in Ref. 2 the set S is said to be E-convex if $(x, \alpha), (y, \beta) \in S$ imply

$$((1 - t)E(x) + tE(y), \quad (1 - t)\alpha + t\beta) \in S, \quad 0 \leq t \leq 1.$$

Then, the following theorem of characterization of an E-convex function f in terms of its E-epigraph $E - e(f)$ is given in Ref. 2.

Theorem 2.1. See Theorem 3.1 of Ref. 2. A numerical function f defined on an E-convex set $M \subseteq \mathbb{R}^n$ is E-convex on M iff $E - e(f)$ is E-convex in $\mathbb{R}^n \times \mathbb{R}$.

Counterexample 2.1. Let $M = [0, 1] \subseteq \mathbb{R}$, let $f : M \to \mathbb{R}$ be defined by

$$f(x) = x^2, \quad \text{for all } x \in \mathbb{R},$$

and let $E : \mathbb{R} \to \mathbb{R}$ be defined by

$$E(x) = \sqrt{|x|}, \quad \text{for all } x \in \mathbb{R}.$$

The set $M = [0, 1]$ is E-convex, because for every $x, y \in [0, 1]$ and every $t \in [0, 1]$, we have

$$(1 - t)E(x) + tE(y) \in M = [0, 1].$$

The function f is E-convex on M, because for every $x, y \in M$ and every $t \in [0, 1]$, we have

$$
\begin{aligned}
&(1-t)f(E(x)) + tf(E(y)) - f((1-t)E(x) + tE(y)) \\
&= (1-t)(\sqrt{|x|})^2 + t(\sqrt{|y|})^2 - ((1-t)\sqrt{|x|} + t\sqrt{|y|})^2 \\
&= t(1-t)(\sqrt{|x|} - \sqrt{|y|})^2 \geq 0.
\end{aligned}
$$

Then, in view of Theorem 2.1, the set

$$E - e(f) = \{(x, \alpha) | x \in M, \alpha \in \mathbb{R}, f(E(x)) \leq \alpha\}$$

is E-convex in $\mathbb{R} \times \mathbb{R}$. This means that, for every $(x, \alpha), (y, \beta) \in E - e(f)$ and every $t \in [0, 1]$, we have

$$((1-t)E(x) + tE(y),\ (1-t)\alpha + t\beta) \in E - e(f),$$

or equivalently,

$$(1-t)E(x) + tE(y) \in M, \qquad (1-t)\alpha + t\beta \in \mathbb{R}, \tag{3}$$

and

$$f(E((1-t)E(x) + tE(y))) \leq (1-t)\alpha + t\beta. \tag{4}$$

Let now

$$x = 2^{-4}, \quad y = 2^{-2}, \quad t = 2^{-1}, \quad \alpha = 2^{-4}, \quad \beta = 2^{-2}.$$

On one hand, we have that (x, α) and (y, β) belong to $E - e(f)$. On the other hand,

$$(1-t)E(x) + tE(y) = 3/8 \in M = [0, 1], \qquad (1-t)\alpha + t\beta = 5/32 \in \mathbb{R};$$

hence, (3) holds, while

$$f(E((1-t)E(x) + tE(y))) = 3/8;$$

hence, (4) does not hold. This implies that Theorem 2.1 (see Theorem 3.1 of Ref. 2) is incorrect.

3. Some Characterizations of *E*-Convex Functions

In this section, some characterizations of E-convex functions using a different notion of epigraph are given.

If M is a nonempty subset of $\mathbb{R}^n$ and if $E : M \to M$ and $f : M \to \mathbb{R}$ are two functions, then we consider the following two sets:

$$
\begin{aligned}
&\text{epi}(f) = \{(x, a) \in M \times \mathbb{R} | f(x) \leq a\}, \\
&\text{epi}_E(f) = \{(z, a) \in E(M) \times \mathbb{R} | f(z) \leq a\}.
\end{aligned}
$$

Obviously, the three sets epi(f), epi_E, and $E - e(f)$ are not equal. Indeed, let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^2, \quad \text{for all } x \in \mathbb{R},$$

and let $E : \mathbb{R} \to \mathbb{R}$ be defined by

$$E(x) = \sqrt{|x|}, \quad \text{for all } x \in \mathbb{R}.$$

We have that

$$\text{epi}(f) = \{(x, a) \in \mathbb{R} \times \mathbb{R} : x^2 \leq a\},$$
$$\text{epi}_E(f) = \{(x, a) \in [0, +\infty) \times \mathbb{R} : x^2 \leq a\},$$
$$E - e(f) = \{(x, a) \in \mathbb{R} \times \mathbb{R} : |x| \leq a\};$$

hence,

$$\text{epi}(f) \neq \text{epi}_E(f) \neq E - e(f) \neq \text{epi}(f).$$

We remark also that

$$E - e(f) \subseteq M \times \mathbb{R} \quad \text{and} \quad \text{epi}_E(f) \subseteq E(M) \times \mathbb{R}.$$

We recall that the function $f : M \to \mathbb{R}$ is convex on the nonempty convex subset M of $\mathbb{R}^n$ if and only if its epigraph epi(f) is a convex subset of $\mathbb{R}^n \times \mathbb{R}$.

The following theorem gives a sufficient condition for f to be an E-convex function using the set $\text{epi}_E(f)$.

Theorem 3.1. Let M be a nonempty subset of $\mathbb{R}^n$ and let $f : M \to \mathbb{R}$ and $E : \mathbb{R}^n \to \mathbb{R}^n$ be two functions. If M is an E-convex set and $\text{epi}_E(f)$ is a convex set, then f is an E-convex function on M.

Proof. Let $x, y \in M$ and $t \in [0, 1]$. Since M is an E-convex set, we have that $E(x), E(y) \in E(M) \subseteq M$. Since $(E(x), f(E(x)))$ and $(E(y), f(E(y)))$ belong to $\text{epi}_E(f)$ and since $\text{epi}_E(f)$ is convex, we have

$$(1 - t)(E(x), f(E(x))) + t(E(y), f(E(y))) \in \text{epi}_E(f).$$

From this, it follows that

$$(1 - t)E(x) + tE(y) \in E(M) \subseteq M$$

and

$$f((1 - t)E(x) + tE(y)) \leq (1 - t)f(E(x)) + tf(E(y)).$$

Consequently, f is an E-convex function on M.

We can exclude the convexity hypothesis of the set $\text{epi}_E(f)$ and in exchange we ask for the set $E(M)$ to be convex and for the set $\text{epi}_E(f)$ to be slack 2-convex with respect to $E(M) \times \mathbb{R}$.

To begin with, we recall the meaning of the set $A \subseteq \mathbb{R}^n$ being slack 2-convex with respect to $B \subseteq \mathbb{R}^n$. □

Definition 3.1. See Ref. 3 or Ref. 4. Let A and B be two subsets of $\mathbb{R}^n$. We say that A is slack 2-convex with respect to B if, for every $x, y \in A \cap B$ and every $t \in [0, 1]$ with the property that

$$(1 - t)x + ty \in B,$$

we have

$$(1 - t)x + ty \in A.$$

The following theorem gives a sufficient condition for f to be an E-convex function.

Theorem 3.2. Let M be a nonempty subset of $\mathbb{R}^n$ and let $f : M \to \mathbb{R}$ and $E : \mathbb{R}^n \to \mathbb{R}^n$ be two functions. If M is an E-convex set, $E(M)$ is a convex set, and $\text{epi}_E(f)$ is a slack 2-convex set with respect to $E(M) \times \mathbb{R}$, then f is an E-convex function on M.

Proof. Let $x, y \in M$ and $t \in [0, 1]$. Then,

$$(E(x), f(E(x))),\ (E(y),\ f(E(y))) \in (E(M) \times \mathbb{R}) \cap \text{epi}_E(f).$$

Since $E(M)$ is a convex set, we have

$$(1 - t)E(x) + tE(y) \in E(M);$$

hence,

$$((1 - t)E(x) + tE(y),\ (1 - t)f(E(x)) + tf(E(y))) \in E(M) \times \mathbb{R}.$$

Then,

$$((1 - t)E(x) + tE(y), (1 - t)f(E(x)) + tf(E(y))) \in \text{epi}_E(f), \tag{5}$$

because $\text{epi}_E(f)$ is a slack 2-convex set with respect to $E(M) \times \mathbb{R}$.

From (5), it follows that

$$f((1 - t)E(x) + tE(y)) \leq (1 - t)f(E(x)) + tf(E(y));$$

hence, f is an E-convex function on M. □

Remark 3.1. The hypothesis that $E(M)$ is a convex set is essential; if we exclude this hypothesis, then the conclusion of Theorem 3.2 may not be true. Indeed, let

$$M = [0, 1] \cup \{2, 4\} \subseteq \mathbb{R},$$

let $E : \mathbb{R} \to \mathbb{R}$ be defined by

$$E(x) = x/4, \quad \text{for all } x \in \mathbb{R},$$

and let $f : M \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \in E(M) = [0, 1/4] \cup \{1/2, 1\} \\ 10, & \text{if } x \in M \setminus E(M). \end{cases}$$

Obviously, M is an E-convex set and

$$\mathrm{epi}_E(f) = \{(z, \alpha) \in \mathbb{R} \times \mathbb{R} | z \in [0, 1/4] \cup \{1/2, 1\}, z^2 \leq \alpha\}$$

is a slack 2-convex set with respect to

$$E(M) \times \mathbb{R} = ([0, 1/4] \cup \{1/2, 1\}) \times \mathbb{R}.$$

But let

$$x = 2, \quad y = 4, \quad t = 2^{-1}.$$

Then,

$$f((1 - t)E(x) + tE(y)) = f(3/4) = 10,$$

while

$$(1 - t)f(E(x)) + tf(E(y)) = (1/2)f(1/2) + (1/2)f(1) = 5/8;$$

hence, (2) does not hold. Thus, f is not an E-convex function on M.

Consequently, in Theorem 3.2, we cannot exclude the hypothesis that $E(M)$ is a convex set.

Theorem 3.3. Let M be a nonempty subset of $\mathbb{R}^n$ and let $f : M \to \mathbb{R}$ and $E : \mathbb{R}^n \to \mathbb{R}^n$ be two functions. If M is an E-convex set and f is an E-convex function on M, then $\mathrm{epi}(f)$ is a slack 2-convex set with respect to $E(M) \times \mathbb{R}$.

Proof. Let $(x, \alpha), (y, \beta) \in (E(M) \times \mathbb{R}) \cap \mathrm{epi}(f)$, and $t \in [0, 1]$ be such that

$$(1 - t)(x, \alpha) + t(y, \beta) \in E(M) \times \mathbb{R}.$$

Then,

$$x, y \in E(M) \subseteq M, \quad f(x) \leq \alpha, \quad f(y) \leq \beta,$$

and

$$(1-t)x + ty \in E(M) \subseteq M.$$

From $x, y \in E(M)$, it follows that there exist $x', y' \in M$ such that

$$x = E(x'), \quad y = E(y').$$

Since f is E-convex on M, we have

$$\begin{aligned} f((1-t)x + ty) &= f((1-t)E(x') + tE(y')) \\ &\leq (1-t)f(E(x')) + tf(E(y')) \\ &= (1-t)f(x) + tf(y) \\ &\leq (1-t)\alpha + t\beta. \end{aligned}$$

Therefore,

$$(1-t)(x, \alpha) + t(y, \beta) = ((1-t)x + ty, (1-t)\alpha + t\beta) \in \text{ epi}(f).$$

Consequently, epi(f) is a slack 2-convex set with respect to $E(M) \times \mathbb{R}$. □

The following theorem gives another sufficient condition for f to be an E-convex function.

Theorem 3.4. Let M be a nonempty subset of $\mathbb{R}^n$ and let $f : M \to \mathbb{R}$ and $E : \mathbb{R}^n \to \mathbb{R}^n$ be two functions. If M is an E-convex set, $E(M)$ is a convex set, and epi(f) is a slack 2-convex set with respect to $E(M) \times \mathbb{R}$, then f is an E-convex function on M.

Proof. Let $x, y \in M$ and $t \in [0, 1]$. Then,

$$(E(x), f(E(x))), (E(y), f(E(y))) \in (E(M) \times \mathbb{R}) \cap \text{ epi}(f).$$

Since $E(M)$ is a convex set, we have

$$(1-t)E(x) + tE(y) \in E(M);$$

hence,

$$\begin{aligned} &(1-t)(E(x), f(E(x))) + t(E(y), f(E(y))) \\ &= ((1-t)E(x) + tE(y), (1-t)f(E(x)) + tf(E(y))) \in E(M) \times \mathbb{R}. \end{aligned}$$

Then,

$$((1-t)E(x) + tE(y),\ (1-t)f(E(x)) + tf(E(y))) \in \text{ epi}(f), \tag{6}$$

because epi(f) is slack 2-convex set with respect to $E(M) \times \mathbb{R}$. From (6), it follows that

$$f((1-t)E(x) + tE(y)) \leq (1-t)f(E(x)) + tf(E(y)).$$

Consequently, f is an E-convex function on M. □

Remark 3.2. In general, in Theorem 3.4, the hypothesis that $E(M)$ is a convex set cannot be replaced. To this end, suppose that M, E, f are those of Remark 3.1. In Remark 3.1, we show that M is an E-convex set and f is not an E-convex function on M. But the set

$$\begin{aligned}\mathrm{epi}(f) = &\{(z,\alpha) \in \mathbb{R} \times \mathbb{R} | z \in [0, 1/4] \cup \{1/2, 1\}, z^2 \leq \alpha\} \\ &\cup \{(z,\alpha) \in M \times \mathbb{R} | z \in M \setminus ([0, 1/4] \cup \{1/2, 1\}), 10 \leq \alpha\}\end{aligned}$$

is slack 2-convex with respect to $E(M) \times \mathbb{R}$.

Now, from Theorems 3.3 and 3.4, the following theorem of characterization of an E-convex function can be deduced.

Theorem 3.5. Let M be a nonempty subset of $\mathbb{R}^n$ and let $f : M \to \mathbb{R}$ and $E : \mathbb{R}^n \to \mathbb{R}^n$ be two functions. Assume that M is an E-convex set and $E(M)$ is a convex set. Then, f is an E-convex function on M if and only if epi(f) is a slack 2-convex set with respect to $E(M) \times \mathbb{R}$.

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