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Existence and Convergence of Pareto Minima[1]

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Abstract. In the context of vector optimization for functions with values in an ordered topological vector space, we give a result for the existence of global minima. Moreover, we find a set of conditions ensuring the convergence of minimal points and minimal values. More general assumptions are excluded by several counterexamples.

Key Words. Vector optimization, cones, Pareto minima, stability of minima.

1. Introduction

The concept of an efficient point for a set in an ordered topological vector space has been investigated widely by many authors (Refs. 1–4). Usually, the literature considers the problem from only the point of view of an efficient point for a general nonempty set A of a locally convex vector space ordered by a cone, largely ignoring the case where A is the image of some function.

Only few articles (Refs. 5–6) have studied the stability of convergence for sets of efficient points. However, as far as we know, no paper has addressed the problem of convergence of minima for sequences of vector-valued functions. Section 3 aims at giving a contribution to the existence of global minima for functions satisfying a suitable coerciveness condition; to this end, we adopt a general semicontinuity notion introduced by Corley (Ref. 2). In Section 4, in order to obtain convergence for minimum points and minimum values, we assume a more restrictive notion of semicontinuity and adopt the concept of weak Pareto minimum point or weakly efficient point; see Ref. 7.

The weak minimum is introduced in connection with new definitions of weak minorant and weak infimum. This definition is interesting because, as we prove in Theorem 4.2, a nonempty set with weak minorants has a nonempty weak infimum.

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More general assumptions of regularity or convergence are excluded by a collection of simple counterexamples.

2. Preliminaries

Throughout this paper, we denote by X a topological space and by Y a topological vector space endowed with a filter of neighborhoods of 0.

Definition 2.1. A set $C \subset Y$ is a pointed cone iff:

(i) $\lambda C \subset C, \quad \forall \lambda \geq 0;$
(ii) $(\mathrm{cl}\, C) \cap (-\mathrm{cl}\, C) = \{0\};$
it is a convex pointed cone iff additionally
(iii) $C + C \subset C;$

We set $C_o = C \backslash \{0\}$ and, for $y, y' \in Y$, we write

$$y < y', \quad \text{iff } y' - y \in C_o,$$
$$y \leq y', \quad \text{iff } y' - y \in C.$$

Moreover, if Y^* is the topological dual of Y,

$$C^* := \{p \in Y^* : \langle p, y \rangle \geq 0, \ \forall y \in C\}$$

is the (positive) polar or dual of C and

$$C_o^* := \{p \in Y^* : \langle p, y \rangle > 0, \ \forall y \in C \backslash \{0\}\}$$

is the strict (positive) polar of C.

Definition 2.2. See Ref. 2. A function $f : X \to Y$ is said to be C-lower semicontinuous iff

$$f^{-1}(y - \mathrm{cl}\, C) \text{ is closed in } X, \quad \forall y \in Y.$$

When $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, we have the usual notion of lower semicontinuity.

Definition 2.3. If $A \subset Y, b \in Y$ is said to be a C-minorant for A iff $(b - C_o) \cap A = \emptyset$. We define

$$\mu_C(A) = \{b \in Y : b \text{ is a } C\text{-minorant for } A\}.$$

If $\mu_C(A) \neq \emptyset$, let

$$\inf_C A := \{b \in \mu_C(A) : (b + C_o) \cap \mu_C(A) = \emptyset\}.$$

If $A \neq \emptyset$, an element of the set

$$\min_C A := A \cap \inf_C A$$

is said to be a minimum for A.

Remark 2.1.

(i) When $A \subset Y$ is nonempty, it results clearly from Definition 2.3 that $b \in \min_C A$ if and only if $b \in A$ and $A \cap (b - C) = \{b\}$.
(ii) Even if $A \subset Y$ and $\mu_C(A)$ are nonempty, the set $\inf_C A$ might be empty.

For instance, consider the case where

$$Y = \mathbb{R}^2, \quad C = [0, \infty[\times[0, \infty[,$$
$$A = \{(x, y) \in \mathbb{R}^2 : 0 < x, y < -1/x\}.$$

Clearly, $\inf_C A = \emptyset$. However, $\mu_C(A) =]-\infty, 0] \times \mathbb{R}$.

3. Existence of Minima

Let us introduce the polar set with respect to Y^*, $\lambda \in \mathbb{R}$,

$$E(K, \lambda) := \{x \in X : \langle p, f(x)\rangle \leq \lambda, \ \forall p \in K\}.$$

Theorem 3.1. Let C be a closed, convex, and pointed cone in Y and let $f : X \to Y$ be such that

$$x \in X \mapsto \langle p, f(x)\rangle \in \mathbb{R} \tag{1}$$

is lower semicontinuous for every $p \in K$. If there exist $\lambda \in \mathbb{R}$ and $K \subset C_o^*$, with $K \neq \emptyset$, such that $E(K, \lambda)$ is nonempty and compact, then $\min_C f(X) \neq \emptyset$.

Proof. By the Weierstrass theorem, for every $p \in K$ there exists an $x_p \in E(K, \lambda)$ such that $\langle p, f(x_p)\rangle$ is a global minimum for the map (1) on $E(K, \lambda)$. We see that

$$f(x_p) \in \min_C f(X).$$

In fact, we have

$$f(x_p) \in \min_C f(E(K, \lambda)),$$

because, if

$$y' < f(x_p), \quad y' = f(x'), \quad x' \in E(K, \lambda),$$

then

$$\langle p, f(x')\rangle < \langle p, f(x_p)\rangle,$$

in contradiction with the definition of x_p. Now, if $\bar{x} \in X\backslash E(K,\lambda)$, there exists $\bar{p} \in K$ such that $\langle \bar{p}, f(\bar{x})\rangle > \lambda$. Then, the relation $f(\bar{x}) < f(x_p)$ is false, because it would imply

$$\langle \bar{p}, f(\bar{x})\rangle < \langle \bar{p}, f(x_p)\rangle \leq \lambda.$$ □

Theorem 3.2. Let C be a closed, convex, and pointed cone in Y and let $f : X \to Y$ be C-lower semicontinuous. If there exist $\lambda \in \mathbb{R}$ and $K \subset C^*\backslash\{0\}$, with $K \neq \emptyset$, such that $E(K,\lambda)$ is nonempty and compact, then $\min_C f(X) \neq \emptyset$.

Proof. From Ref. 2, Corollary 3.1, it follows that there exists $b \in \min_C f(E(K,\lambda))$ satisfying $b = f(x)$ for some $x \in E(K,\lambda)$. The proof that $b \in \min_C(f(X))$ follows as in the previous theorem. □

Remark 3.1.

(i) In the following example, it is easy to verify the compacteness hypothesis on the set $E(K,\lambda)$, made in Theorems 3.1 and 3.2. Let

$$f : \mathbb{R} \to \mathbb{R}^2, \quad f(t) = (t, t^2), \quad C = [0,+\infty) \times [0,+\infty), \quad K = \{\varphi\},$$

with

$$\varphi(x, y) = x + y.$$

Then, for any $\lambda \in \mathbb{R}$, we have $E(K,\lambda) = \{t \in \mathbb{R} : t + t^2 \leq \lambda\}$.

(ii) In the following example, f is not C-lower semicontinuous, but satisfies the hypotheses of Theorem 3.1. Let

$$X = \{0\} \cup \{1/n, n \in \mathbb{N}, n \geq 1\}$$

with the topology induced by $\mathbb{R}$. Let

$$Y = l_2 = \left\{a = (a_n)_{n\geq 1} : \sum_{n=1}^{\infty} a_n^2 < +\infty\right\},$$

endowed with the usual Hilbert structure. Let

$$C = \{a \in l_2 : a_n \geq 0, n \in \mathbb{N}, n \geq 1\}$$

and let

$$\alpha = (\alpha_n)_{n\geq 1}, \quad \alpha_n = 1/n, \quad \forall n \geq 1.$$

We define $f : X \to Y$ by

$$f(x) = \begin{cases} e_n/n, & \text{if } x = 1/n, \\ \beta, & \text{if } x = 0, \end{cases}$$

where $\{e_n, \in \mathbb{N}, n \geq 1\}$ is the usual basis of l_2 and $\beta \in l_2$, whose components are

$$\beta_n = (-1)^{n+1}/n, \quad \text{if } n \geq 2, \quad \beta_1 = -\sum_{n=2}^{\infty}(-1)^{n+1}/n^2.$$

We verify that f is not C-lower semicontinuous.

Let

$$y = (y_n)_{n\geq 1}, \quad \text{with } y_1 = \beta_1/2 \quad \text{and} \quad y_n = 1/n, \quad \text{if } n \geq 2.$$

Hence,

$$1/n \in f^{-1}(y - C) = \{x \in X : f(x) \leq y\}, \forall n \geq 2,$$
$$\text{but } 0 \notin f^{-1}(y - C).$$

Now, if we define $\varphi : l_2 \to \mathbb{R}$ by $\varphi(a) = \langle a, \alpha\rangle$, then $\varphi_\circ f : X \to \mathbb{R}$ is continuous, because

$$\varphi(f(x)) = \begin{cases} 1/n^2, & \text{if } x = 1/n, \\ \langle \beta, \alpha\rangle = 0, & \text{if } x = 0. \end{cases}.$$

Moreover, if $K = \{\varphi\}$, the set

$$E(K, \lambda) = \{x \in X : \varphi_\circ f(x) \leq \lambda\}$$

is compact for any $\lambda \in \mathbb{R}$.

Theorem 3.3. Let C be a closed, convex, and pointed cone in Y and let $f : X \to Y$ be C-lower semicontinous. If there exists $K \subset C^*\backslash\{0\}$, with $K \neq \emptyset$ and bounded in Y, such that $E(K, \lambda)$ is nonempty and compact for every $\lambda \in \mathbb{R}$, then

$$\inf{}_C(X) = \min{}_C f(X).$$

Proof. Let $b \in \inf_C f(X)$ and $k_o \in C_o$. We prove that there exists $\alpha \in \mathbb{R}$, such that

$$f^{-1}(b + k_o/n - C) \subset E(K, \alpha), \quad \forall n \in \mathbb{Z}_+.$$

In fact, if

$$f(x) = b + k_o/n - \varepsilon \quad \text{and} \quad p \in K,$$

then

$$\langle p, f(x)\rangle = \langle p, b + k_o/n\rangle - \langle p, \varepsilon\rangle \leq \langle p, b\rangle + (1/n)\langle p, k_o\rangle \leq \langle p, b + k_o\rangle;$$

hence,

$$f^{-1}(b + k_o/n - C) \subset \bigcap_{p\in K}\{x \in X : \langle p, f(x)\rangle \leq \langle p, b + k_o\rangle\}. \tag{2}$$

From the boundedness of K, if

$$V = \{p \in Y^* : \langle p, b + k_o\rangle < 1\},$$

then there exists $\alpha \in \mathbb{R}_+$ such that $K \subset \alpha V$; so, by (2), it follows that

$$f^{-1}(b + k_o/n - C) \subset E(K, \alpha).$$

Since

$$b + k_o/n \notin \mu_C(f(X)),$$

it follows that

$$(b + k_o/n - C_o) \cap f(X) \neq \emptyset;$$

hence,

$$f^{-1}(b + k_o/n - C) \neq \emptyset.$$

Moreover, it is clear that

$$f^{-1}(b + k_o/(n+1) - C) \subset f^{-1}(b + k_o/n - C), \quad \forall n \in \mathbb{Z}_+.$$

Then, by compactness, there exists

$$\bar{x} \in \bigcap_{n\in\mathbf{Z}_+} f^{-1}(b + k_o/n - C).$$

Hence, it follows that

$$f(\bar{x}) = b + k_o/n - c_n, \quad \text{for some } c_n \in C.$$

But C is closed, so

$$b - f(\bar{x}) = \lim_{n\to\infty} b + k_o/n - f(\bar{x}) = \lim_{n\to\infty} c_n \in C.$$

Therefore, $b \geq f(\bar{x})$ but $b \in \inf_C f(X)$, and we conclude that $b = f(\bar{x})$. □

4. Weak Minima and Convergence

Lemma 4.1. Let $C \subset Y$ be a pointed cone having $\text{int}\, C \neq \emptyset$. If $\varepsilon \in \text{int}\, C$ and $z \in Y$, then the sets

$$\{y \in Y : y > z - \varepsilon\}, \quad \{y \in Y : y < z + \varepsilon\}$$

are neighborhoods of z.

Proof. Clearly, we can suppose $z = 0$. Let V_ε be an open set such that $\varepsilon \in V_\varepsilon \subset C_o$. Then, $V_\varepsilon - \varepsilon$ is a neighborhood of 0 and

$$V_\varepsilon - \varepsilon \subset C_o - \varepsilon = \{y \in Y : y + \varepsilon \in C_o\} = \{y \in Y : y > -\varepsilon\}. \qquad \square$$

Definition 4.1. Let Y be a topological vector space ordered by a closed, convex, and pointed cone C with $\operatorname{int} C \neq \emptyset$.

(i) We say that $f : X \to Y$ is strongly lower [upper] C-semicontinuous at the point $x_o \in X$ iff, for any $\varepsilon \in \operatorname{int} C$, there exists $U_{x_o\varepsilon}$, a neighborhood of x_o such that

$$f(x) > f(x_o) - \varepsilon \quad [f(x) < f(x_o) + \varepsilon], \quad \forall x \in U_{x_o\varepsilon}.$$

(ii) Let $f_n : X \to Y$, $n \in \mathbb{N}$, and $x_o \in X$. We say that the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly strongly lower [upper] C-semicontinuous at the point $x_o \in X$ iff, for any $\varepsilon \in \operatorname{int} C$, there exists $U_{x_o\varepsilon}$, a neighborhood of x_o, such that

$$f_n(x) > f_n(x_o) - \varepsilon \quad [f_n(x) < f_n(x_o) + \varepsilon], \quad \forall x \in U_{x_o\varepsilon} \quad \forall n \in \mathbb{N}.$$

Theorem 4.1. If C is a closed, convex, and pointed cone and if $f : X \to Y$ is strongly lower C-semicontinuous at $x \in X$ for every $x \in X$, then f is C-lower semicontinuous.

Proof. We prove that $f^{-1}(y - C)$ is closed for every $y \in Y$. Let $x \notin f^{-1}(y - C)$, namely, $y - f(x) \notin C$. If $\varepsilon \in \operatorname{int} C$ and U_{xn} is a neighborhood of x such that

$$f(x') > f(x) - \varepsilon/n, \quad \forall x' \in U_{xn},$$

then there exists $\bar{n} \in \mathbb{Z}_+$ for which

$$U_{x\bar{n}} \cap f^{-1}(y - C) = \emptyset.$$

Otherwise, if for every $n \in \mathbb{Z}_+$, there exists $x_n \in U_{xn}$ such that $y - f(x_n) \in C$, it would imply

$$y \geq f(x_n) > f(x) - \varepsilon/n,$$

that is,

$$y - f(x) + \varepsilon/n \in C$$

and by the closedness of C,

$$y - f(x) \in C. \qquad \square$$

Remark 4.1. In the following example, we show that strong lower C-semicontinuity is more restrictive than C-lower semicontinuity.

Let

$$X = [0, \infty[, \quad Y = \mathbb{R}^2, \quad C = \{(x, y) \in \mathbb{R}^2 : x \geq 0, 0 \leq y \leq x\},$$

and let $f : X \to Y$ be defined by

$$f(t) = \begin{cases} (0, 0), & \text{if } t = 0, \\ (t, 1/t), & \text{if } t > 0. \end{cases}$$

It follows that

$$f^{-1}((\bar{x}, \bar{y}) - C) = \begin{cases} \emptyset, & \text{if } (\bar{x}, \bar{y}) \in \mathbb{R}^2 \backslash [0, \infty[^2, \\ \{0\}, & \text{if } (\bar{x}, \bar{y}) \in [0, \infty[^2 \quad \text{and} \quad \bar{x}\bar{y} < 1, \\ \{0\} \cup [a, b], & \text{if } \bar{x}\bar{y} \geq 1, \quad (\bar{x}, \bar{y}) \in [0, \infty[^2, \end{cases}$$

with

$$a = 1/\bar{y}, \quad b = (1/2)[\bar{x} - \bar{y} + \sqrt{(\bar{x} - \bar{y})^2 + 4}].$$

Obviously, f is not strongly lower C-semicontinuous at $t = 0$.

Definition 4.2. Let $A \subset Y$, where Y is a topological vector space ordered by a closed, convex, and pointed cone C with $\text{int}\, C \neq \emptyset$.

(i) We call $b \in Y$ a weak C-minorant for A if $(b - \text{int}\, C) \cap A = \emptyset$ and we denote by $w - \mu_C(A)$ the set of all weak C-minorants of A.
Moreover, if $w - \mu_C(A)$ is nonempty, we define the set of weak C-infima of A as

$$w - \inf_C A := \{b \in w - \mu_C(A) : (b + \text{int}\, C) \cap w - \mu_C(A) = \emptyset\}.$$

(ii) We say that $b \in A$ is a weak C-minimum for A if

$$(b - A) \cap \text{int}\, C = \emptyset.$$

Remark 4.2.

(i) When $A \subset Y$ is nonempty, it clearly follows from Definition 4.2 that $b \in A$ is a weak C-minimum for A if and only if $b \in w - \inf_C(A) \cap A$.
(ii) Clearly a minimum for a set $A \subset Y$ is also a weak C-minimum, but the converse is not true.

A trivial example is given by $C = [0, \infty[\times [0, \infty[\subset \mathbb{R}^2$, $A = C$, and the point $(1, 0)$.

Theorem 4.2. When $A \subset Y$ and $w - \mu_C(A)$ are nonempty, then $w - \inf_C A$ is nonempty.

Proof. Let $a \in A$ and $b \in w - \mu_C(A)$. We set

$$\alpha := \sup\{\lambda \in [0, 1] : \lambda a + (1 - \lambda)b \in w - \mu_C(A)\}.$$

We prove that

$$\bar{a} = \alpha a + (1 - \alpha)b \in w - \inf_C A.$$

Indeed,

$$\bar{a} \in w - \mu_C(A),$$

because otherwise, if $(\bar{a} - \operatorname{int} C) \cap A \neq \emptyset$, we can find $\varepsilon \in \operatorname{int} C$ such that $\bar{a} - \varepsilon \in A$. We may assume $\alpha > 0$, so we can find $\alpha' \in [0, \alpha[$ close enough to α so that

$$\varepsilon - \bar{a} + \alpha' a + (1 - \alpha')b \in \operatorname{int} C \quad \text{and} \quad \alpha' a + (1 - \alpha')b \in w - \mu_C(A).$$

That is a contradiction, because

$$A \ni \bar{a} - \varepsilon = \alpha' a + (1 - \alpha')b - (\varepsilon - \bar{a} + \alpha' a + (1 - \alpha')b).$$

Now, we verify that

$$\bar{a} + \varepsilon \notin w - \mu_C(A), \quad \text{when } \varepsilon \in \operatorname{int} C.$$

We may assume $\alpha < 1$. By contradiction, if $\bar{a} + \varepsilon \in w - \mu_C(A)$, then

$$(\bar{a} + \varepsilon - \operatorname{int} C) \cap A = \emptyset.$$

On the other hand,

$$\bar{a} \in \operatorname{int}(\bar{a} + \varepsilon - \operatorname{int} C).$$

Therefore, there exists β such that

$$\alpha < \beta < 1 \quad \text{and} \quad \beta a + (1 - \beta)b \in \operatorname{int}(\bar{a} + \varepsilon - \operatorname{int} C).$$

But

$$\beta a + (1 - \beta)b - \operatorname{int} C \subset \bar{a} + \varepsilon - \operatorname{int} C,$$

so that

$$(\beta a + (1 - \beta)b - \operatorname{int} C) \cap A = \emptyset$$

in contradiction with the definition of α. □

Theorem 4.3. Let C be a closed convex pointed cone with $\operatorname{int} C \neq \emptyset$ and let $(\delta_n)_{n \in \mathbb{N}}$ be a sequence in C_o such that $\lim_{n \to \infty} \delta_n = 0$. Let $f : X \to Y$ and let $(f_n)_{n \in \mathbb{N}}$ be a sequence of uniformly strongly lower C-semicontinuous functions, $f_n : X \to Y$ such that $\lim_{n \to \infty} f_n(x) = f(x)$ for every $x \in X$.

We assume that, for each $n \in \mathbb{N}$ $\inf_C f_n(X) \neq \emptyset$, $b_n \in \inf_C f_n(X)$, and $x_n \in X$ such that $f_n(x_n) < b_n + \delta_n$. If there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}} \to \bar{x} \in X$, then $f(\bar{x})$ is a weak C-minimum for $f(X)$.

Moreover, if C is a normal cone (see Ref. 8, Definition 1.2) and $(f_n)_{n\in\mathbb{N}}$ is also uniformly strongly upper C-semicontinuous, then $\lim_{k\to\infty} f_{n_k}(x_{n_k}) = f(\bar{x})$.

Proof. We prove that

$$f(\bar{x}) - f(x) \notin \operatorname{int} C, \quad \forall x \in X.$$

If we suppose, contrary to the assertion, that

$$\varepsilon = f(\bar{x}) - f(x_o) \in \operatorname{int} C, \quad \text{for some } x_o \in X,$$

then there exists $\bar{n} \in \mathbb{N}$ such that

$$f_n(\bar{x}) - f_n(x_o) \in \operatorname{int} C, \quad \text{if } n \geq \bar{n}.$$

Due to the uniform strong lower C-semicontinuity, there exists a neighborhood U of $\bar{x}$ such that

$$f_n(x) > f_n(\bar{x}) - \varepsilon/4, \quad \forall x \in U.$$

So, for $k \geq k_\varepsilon \in \mathbb{N}$, by letting

$$\varepsilon_n = f_n(\bar{x}) - f_n(x_o),$$

we get

$$f_{n_k}\big(x_{n_k}\big) > f_{n_k}(\bar{x}) - \varepsilon/4 = f_{n_k}(x_o) + \varepsilon_{n_k} - \varepsilon/4.$$

Then,

$$b_{n_k} + \delta_{n_k} - \varepsilon_{n_k} + \varepsilon/4 > f_{n_k}(x_o), \quad \forall k \geq k_\varepsilon.$$

On the other hand, we may assume that

$$\delta_{n_k} - \varepsilon_{n_k} + \varepsilon/4 < -\varepsilon/2, \quad \forall k \geq k_\varepsilon.$$

So, for these values of k, it follows that

$$b_{n_k} - \varepsilon/2 > f_{n_k}(x_o),$$

in contradiction with the definition of b_{n_k}.

With the further assumptions on C and $(f_n)_{n\in\mathbb{N}}$ let now $\varepsilon \in \operatorname{int} C$. Then, for every $j \in \mathbb{Z}_+$, we take $n_{k_j} \in \mathbb{N}$ such that

$$f_{n_{k_j}}(\bar{x}) + \varepsilon/j > f_{n_{k_j}}\big(x_{n_{k_j}}\big) > f_{n_{k_j}}(\bar{x}) - \varepsilon/j.$$

Then, it follows that

$$2\varepsilon/j > f_{n_{k_j}}\big(x_{n_{k_j}}\big) - f_{n_{k_j}}(\bar{x}) + \varepsilon/j > 0.$$

By Proposition 1.3, Chapter 2, of Ref. 8, we get

$$f_{n_{k_j}}\left(x_{n_{k_j}}\right) - f_{n_{k_j}}(\bar{x}) + \varepsilon/j \to 0,$$

so that

$$f_{n_{k_j}}\left(x_{n_{k_j}}\right) \to f(\bar{x}).$$

The same conclusion may be obtained if we consider any subsequence of $f_{n_k}(x_{n_k})$, so we have

$$\lim f_{n_k}(x_{n_k}) = f(\bar{x}).$$ □

Remark 4.3.

(i) By assuming C-lower semicontinuity instead of strong lower C-semicontinuity, Theorem 4.3 fails as the following example shows.

Let $f : \mathbb{R} \to \mathbb{R}^2$ such that

$$f(t) = \begin{cases} (t, 1/t), & \text{if } t > 0, \\ (t, -t^2), & \text{if } t \leq 0, \end{cases}$$

$$C = \{(x, y) \in \mathbb{R}^2 : 0 < y \leq x\} \quad \text{and} \quad f_n = f, \quad \forall n \in \mathbb{N}.$$

It is easy to verify that the sequence $(f_n)_{n\in\mathbb{N}}$ is (uniformly) C-lower semicontinuous and that, if

$$0 < t < \bar{t}(\bar{t} = (1/4)[-1 + \sqrt{65}]),$$

then t is a global minimum point for f_n. However, 0 is not a global minimum for f.

(ii) Without the assumption of upper C-semicontinuity for the sequence $(f_n)_{n\in\mathbb{N}}$, the assertion $\lim f_{n_k}(x_{n_k}) = f(\bar{x})$ in Theorem 4.3 fails.

For instance, if

$$X =]-\infty, 0], \quad Y = \mathbb{R}^2, \quad C = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y\},$$

and if $(f_n)_{n\in\mathbb{N}}$ is the constant sequence

$$f(t) = \begin{cases} (0, 0), & \text{if } t = 0, \\ (t, -1/t), & \text{if } t \leq -1, \\ (t, 1), & \text{if } -1 < t < 0, \end{cases}$$

then, for any $t \in X$, it follows that $f(t) \in \min_C(f(X))$. Moreover, f is strongly lower C-semicontinuous at every point $t \in X$, but is not strongly upper C-semicontinuous at $t = 0$ and $f(-1/n) \not\to f(0)$.

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