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A Discrete-Time Dynamic Game of Seasonal Water Allocation[1,2]

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Abstract. We present a method for the derivation of feedback Nash equilibria in discrete-time finite-horizon nonstationary dynamic games. A particular motivation for such games stems from environmental economics, where problems of seasonal competition for water levels occur frequently among heterogeneous economic agents. These agents are coupled through a state variable, which is the water level. Actions are strategically chosen to maximize the agents individual season-dependent utility functions. We observe that, although a feedback Nash equilibrium exists, it does not satisfy the (exogenous) environmental watchdog expectations. We devise an incentive scheme to help meeting those expectations and calculate a feedback Nash equilibrium for the new game that uses the scheme. This solution is more environmentally friendly than the previous one. The water allocation game solutions help us to draw some conclusions regarding the agents behavior and also about the existence of feedback Nash equilibria in dynamic games.

Key Words. Environmental management, feedback Nash equilibrium, diagonally strict concavity.

1. Introduction

In this paper, we solve a stylized problem of intertemporal environmental management and use its solution to study the existence of solutions to some dynamic games.

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The problem is a dynamic game between two groups of strategic economic agents competing for a renewable resource. In fact, our game is inspired by the intertemporal conflict between fishermen and watercress producers in the French region of Camargue (Ref. 3). We adopt a simple model, in which the utility of the agents in the first group (fishermen, say) is maximized by a high level of water, while the other agents (cress producers) prefer a low level of water, but only at harvest times. All agents can change the water levels in a costly manner to suit their utilities. Because of the allowance for seasonal competition and other nonstationary attributes of the year-by-year water apportionment problem, our model (explained in Section 2) differs from the usual common-property efficient-allocation models (for the latter, see Ref. 4).

Our intention is to establish equilibrium water levels, i.e., such that no agent would unilaterally deviate from. However, there is a third player in this game, which can be a regional government, which will object to some water levels. If the equilibrium water levels are damaging to the environment, the government will act to curtail the damage. We propose an incentive scheme that the government can use to induce the agents to agree upon environmentally friendly water levels.

The dynamic game at hand is finite-horizon and nonstationary with agents coupled through a state equation. We establish the conditions under which the game can be solved for a feedback Nash (Markovian) equilibrium. This is a difficult problem which does not possess a general solution. Coupled-dynamics game models were analyzed in Ref. 5 for the infinite-horizon case. Finite-horizon games with government subsidies and taxes with open-loop equilibria were studied in Ref. 6. In Ref. 7, feedback Nash equilibria were analyzed for a noncoupled-dynamics case. We contribute to that discussion by presenting a method to establish a feedback Nash equilibrium in finite-horizon coupled-state games. The method is recursive and consists of solving stage games backward in time; it is based also on the Rosen existence and uniqueness theorems (Ref. 8). The feedback Nash equilibrium solution thus obtained is numerical. As our stylized environmental game model is linear-quadratic, we can verify the recursive solution by comparing it to the one obtained through the Riccati equations formalism. The solutions coincide.

The paper is organized as follows. In Section 2, a stylized two-representative-agent game is presented to model a competition process for water levels. In Section 3, we discuss a plausible solution concept for the problem at hand and also the availability of a solution method for that concept. A stage-game feedback Nash equilibrium is suggested and computed in Section 4. See below Sections 5 and 6. This solution happens to be environmentally unfriendly as judged by some environmental standards. A different solution, more acceptable to the regional government, is motivated and discussed in Section 7. Some concluding remarks are provided in Section 8.

2. Competition for Different Water Levels

Consider a region where two groups of agents (e.g., fishermen and watercress producers) compete for water levels. Consider an annual cycle comprising K seasons. The level of water in season k is $x_k \geq 0$, $k = 1, \ldots, K+1$. The natural seasonal water level movements [e.g., caused by evaporation and retention (i.e., net recharge)] are modeled through the time-dependent (seasonal) parameter a_k, $0 < a_k$. In the pristine and deterministic environment, the water levels would change as follows:

$$x_{k+1} = a_k x_k, \quad x_1 \text{ given}, \quad k = 1, \ldots, K. \tag{1}$$

Assume there are $i = 1, \ldots, N$ productive agents. In the computational part of the paper, we will consider two representative players ($N = 2$); they will be fishermen F and watercress producers P. However, in the more theoretical sections, we will keep the notation i and $-i$ to refer to the agent at hand and to the other agent(s), respectively. This will enable us to define some notions needed for the game solution for any number of agents.

In each season, an agent may (costly) release or let in an amount of water[5] $u_k^{(i)}$, $u_k^{(i)} \geq 0$ or < 0, to control the level x_{k+1} in the next season.

So, the coupled-dynamics state equation for the water level in season $k+1$ is

$$x_{k+1} = a_k x_k + \sum_{i=1}^{N} u_k^{(i)}, \quad x_1 \text{ given}, \quad k = 1, \ldots, K.$$

Suppose that the agents are interested in maximizing their intertemporal utility functions defined as

$$J_i\big(x_1, K; u^{(i)}, u^{(-i)}\big) = \sum_{k=2}^{K+1} h^{(i)}\big(x_k, \bar{x}_k^{(i)}, u_{k-1}^{(i)}\big), \tag{2}$$

where $h^{(i)}\big(x_k, \bar{x}_k^{(i)}, u_{k-1}^{(i)}\big)$ is a one-period utility function concave in x_k and $u_{k-1}^{(i)}$ and continuous in all three arguments. In (2), symbols indexed $K+1$ refer to the next year's first season. The variables $\bar{x}_k^{(i)}$ (with bars) represent the preferred water levels by agent i and may depend on a season, $k = 2, \ldots, K+1$. Presumably, a one-season utility $h^{(i)}\big(x_k, \bar{x}_k^{(i)}, u_{k-1}^{(i)}\big)$ would be maximized if $x_k = \bar{x}_k^{(i)}$ and $u_{k-1}^{(i)} = 0$. Hence, the expression (2) would be maximized for each player if the desired water levels were achieved and there would be no need for changing them. However, the desired levels are different for each player,

$$\bar{x}_k^{(i)} \neq \bar{x}_k^{(-i)}. \tag{3}$$

[5] In this model, the letters x and u will denote physical/output quantities; utilities/payoffs will be represented by f, F, V, Φ, J.

This suggests that the agents will be in conflict. We are interested in examining whether there are water levels, which the agents could accept as nonimprovable or equilibrium and what strategy would guarantee them.

Notice that we have omitted a discount factor in the utility function. This means that the agents value equally each season's utility. This helps producing solutions that do not depreciate the future effects of the present actions. On the other hand, including a discount factor in (2) would not change the solution procedure.

3. Solution Method

We present a few definitions and theorems that help us to establish a solution to the above game.

3.1. Solution Concept. We are looking for a feedback Nash equilibrium policy (Markovian) $\{u^{(i)}(x_k, k)\}$, $k = 1, \ldots, K$ and $i = 1, \ldots, N$, defined as

$$\{u^{(i)}(x_k, k)\}_{k=1\ldots K} = \arg\text{equil}\ \{J_i(x_1, K; (\cdot), (\cdot)), J_{-i}(x_1, K; (\cdot), (\cdot))\}. \tag{4}$$

A policy of that kind would be based on the available state observation (x_k, k), hence realistic in that it would allow the players to react to the other players' decisions and also to the natural environment changes.[6] Moreover, the policy would maximize the agents utility function (2) and be unilaterally nonimprovable. There are no guarantees that such an equilibrium exists.

3.2. Sufficiency and Uniqueness Conditions. We know that a concave game, i.e., such that each player's utility function is continuous in all players' actions and concave[7] with respect to its own strategy while the other players' strategies remain fixed, must have at least one Nash equilibrium.[8]

However, for the environmental game described in Section 2, we want to establish not only an equilibrium existence but also its uniqueness. The need for the solution uniqueness is typical of environmental management problems. The regional government needs to know what the equilibrium is. For, if there were many, some of them less environmentally friendly than some others, the government would not know which measures, to take, if any.

[6] See Ref. 4 for a discussion on open-loop versus feedback equilibria in a context of ground water management. We believe that the advantages of a feedback Nash equilibrium above the other dynamic games solution concepts carry over from their model to ours.

[7] These assumptions can be weakened; see e.g. Ref. 9.

[8] Notice that, for our dynamic game, checking the payoff concavity assumption might be nontrivial as the utility function is a sum and depends on the coupling equation (2). We will address this problem by examining stage games in Section 4.2.

The seminal paper by Rosen (Ref. 8) formulates the conditions for the uniqueness of a concave game equilibrium

$$\hat{u}_i = \arg\ \text{equil}\ \left\{f^{(i)}(u), f^{(-i)}(u)\right\}, \tag{5}$$

where $f^{(i)}(u)$ is player i's utility function, u is the combined policy vector,

$$u = \begin{bmatrix} u_i \\ \\ u_{-i} \end{bmatrix} \in \mathbb{X} \subset \mathcal{R}^m,$$

and u_i, $i = 1, \ldots, N$, is the part of u, which contains actions controlled by player i. The set $\mathbb{X}$ is compact and $m \geq N$ denotes the total number of actions of all players. Crucial for the equilibrium conditions is the concept of diagonal strict concavity (DSC) of the joint payoff function. We will define and explain those notions.

The joint payoff function is defined as[9]

$$\phi(u) \equiv \sum_{i=1}^{n} f^{(i)}(u). \tag{6}$$

In broad terms, a game whose joint payoff is DSC (for short, a game which is DSC) is one in which each player has more control over his payoff than the other players have over it. This is a rather common and desired feature of many economic models. The formal definition of DSC is as follows.

Definition 3.1. The game is called diagonally strictly concave (DSC) for $u \in X$ if, for every $u^0, u^1 \in X$, we have

$$(u^1 - u^0)' g(u^0) + (u^0 - u^1)' g(u^1) > 0, \tag{7}$$

where the prime denotes transposition and $g(u)$ is the pseudogradient of $\phi(u)$,

$$g(u) \begin{bmatrix} \partial f^{(1)}(u)/\partial u_1 \\ \vdots \\ \partial f^{(N)}(u)/\partial u_N \end{bmatrix}. \tag{8}$$

If the utility functions $f^{(i)}$ are twice differentiable, a criterion for DSC is simple and consists of checking whether the pseudo-Hessian of $\phi(u)$,

$$\mathcal{H} = H + H', \tag{9}$$

[9] Rosen's definition is more general. In particular, it would allow for a regional government appraisal of the individual agent utilities; however, we treat all agents' utilities equally.

where H, the Jacobian of $g(u)$,

$$H = \begin{bmatrix} \partial^2 f^{(1)}(u)/\partial u_1^2 & \partial^2 f^{(1)}(u)/\partial u_2 \partial u_1 & \cdots & \partial^2 f^{(1)}(u)/\partial u_N \partial u_1 \\ \partial^2 f^{(2)}(u)/\partial u_1 \partial u_2 & \partial^2 f^{(2)}(u)/\partial u_2^2 & \cdots & \partial^2 f^{(2)}(u)/\partial u_N \partial u_2 \\ \cdots & \cdots & \cdots & \cdots \\ \partial^2 f^{(N)}(u)/\partial u_1 \partial u_N & \partial^2 f^{(N)}(u)/\partial u_2 \partial u_N & \cdots & \partial^2 f^{(N)}(u)/\partial u_N^2 \end{bmatrix}. \quad (10)$$

is negative definite. This will be true for games in which the utility function concavity in the agents's own policy cannot be destroyed by the opponents' actions.

As the following theorem specifies it, confirming the pseudo-Hessian negative definiteness is sufficient for the uniqueness of a Nash equilibrium (see Ref. 8 or Ref. 10).

Theorem 3.1. In a game

equil $\left\{ f^{(i)}(u), f^{(-i)}(u) \right\}$,

if the joint payoff function $\phi(u)$ is DSC, then there exists a unique Nash equilibrium.

4. Feedback Nash Equilibrium

4.1. Method. We know from Section 3.2 how to determine whether a game has a unique equilibrium. However, our game (4) is in J_i and J_{-i} [and subject to a dynamic coupling equation (2)] and not in $f^{(i)}$ and $f^{(-i)}$ [see (5)], which are concave functions. To solve our dynamic game (4), we will combine the existence and uniqueness theorem (Theorem 3.1) with the Bellman optimality principle. In broad terms, this means that we will examine the uniqueness of stage games for each stage $k = K, K-1, \ldots 1$ (backward in time). At each stage, the role of the utilities $f^{(i)}$ will be played by the utility-to-go functions, defined below as $F^{(i)}(x_k, k; \cdot, \cdot)$.

4.2. Stage Games. Define $V_k^{(i)}(x_k, k)$, the stage optimal value function for player i, as

$$V_k^{(i)}(x_k, k) = \max_{u_k^{(i)}} F^{(i)}\left(x_k, k; u_k^{(i)}, u_k^{(-i)}(x_k, k)\right), \quad k = K-1, \ldots, 1, \quad (11)$$

where

$$F^{(i)}\left(x_k, k; u_k^{(i)}, u_k^{(-i)}(x_k, k)\right) \equiv h^{(i)}\left(x_{k+1}, \bar{x}_{k+1}^{(i)}, u_k^{(i)}\right) + V_{k+1}^{(i)}(x_{k+1}, k+1), \quad (12)$$

$$V_K^{(i)}(x_K, K) = \max_{u_k^{(i)}} h^i\left(x_{K+1}, \bar{x}_{K+1}^{(i)}, u_K^{(i)}\right). \quad (13)$$

The following theorem[10] establishes a basis for using dynamic programming as a computational technique for feedback Nash equilibria (subgame perfect) in dynamic games.

Theorem 4.1. If there exist value functions $V_k^{(i)}(x^k, k)$ and strategies $\hat{u}_k^{(i)}(x_k, k)$ which satisfy equations (11)–(13) for $k = K, K-1, \ldots, 1$, where x is a vector of state variables, then the strategy pair

$$\hat{u} = \left(\hat{u}^{(i)}, \hat{u}^{(-i)}\right), \quad \hat{u}^{(i)} = \left\{\hat{u}_k^{(i)} : k = K, K-1, \ldots, 1\right\}$$

constitutes a feedback Nash equilibrium of the dynamic game with the feedback information pattern[11] . Moreover, the value functions $V_k^{(i)}(x_k, k)$ represent the optimal utility of player i for the game starting at (x_k, k). In particular,

$$V_1^{(i)}(x_1, 1) = J_i\left(x_1, K; \hat{u}^{(i)}, \hat{u}^{(-i)}\right). \tag{14}$$

If each stage game $F^{(i)(\cdot)}, F^{(-i)(\cdot)}$ is concave, then it makes sense to ask whether the stage games have unique equilibria. We can see that the last stage utility function $h^{(i)}\left(x_{K+1}, \bar{x}_{K+1}^{(i)}, u_K^{(i)}\right)$ is concave. We will verify the diagonal strict concavity of all stage games using backward induction.

Technically speaking, we establish the uniqueness of stage equilibria by using first Theorem 3.1 at each stage $k = K, \ldots 1$ to see if the equilibrium is unique and then Theorem 4.1 to compute the equilibrium strategy. Indeed, if a unique equilibrium

$$\begin{aligned}&\left\{\hat{u}^{(i)}(x_k, k)\right\} \\ &= \arg\ \mathrm{equil}\left\{F^{(i)}\left(x_k, k; u_k^{(i)}, u_k^{(-i)}(x_k, k)\right), F^{(-i)}\left(x_k, k; u_k^{(-i)}, u_k^{(i)}(x_k, k)\right)\right\}\end{aligned} \tag{15}$$

exists for each $k = K, \ldots, 1$, then by construction, the unique feedback-Nash equilibrium (4) also exists.

The uniqueness theorem (Theorem 3.1) says that, if the game's joint payoff function is diagonally strictly concave (DSC), then the equilibrium is unique. So, we will construct the function[12]

$$\begin{aligned}\Phi\left(x_k, k; u_k^{(i)}, u_k^{(-i)}\right) \equiv\ & F^{(i)}\left(x_k, k; u_k^{(i)}, u_k^{(-i)}(x_k, k)\right) \\ &+ F^{(-i)}\left(x_k, k; u_k^{(i)}(x_k, k), u_k^{(-i)}\right),\end{aligned} \tag{16}$$

for each $k = K, \ldots, 1$; next, through checking the pseudo-Hessian of Φ, we will establish the uniqueness of (15) and then of (4).

[10] Standard in dynamic games; see Ref. 11, Theorem 6.6, pp. 284–285.

[11] Such an equilibrium is subgame perfect and is often called Markovian.

[12] If there are more than two players, the symbol $^{(-i)}$ represents the sum of all other players' value functions.

5. Model

5.1. Utility. We will determine now a particular form of the function $h^{(i)}(x_k, \bar{x}_k^{(i)}, u_{k-1}^{(i)})$, which was introduced in (2) as concave in x_k and $u_{k-1}^{(i)}$ and continuous in all three arguments.

Assume that the firms draw utility from their income that is composed of a fixed term and a variable term, where the latter are the penalties for the water level x_k not being optimal $\big($i.e., $x_k \neq \bar{x}_k^{(i)}\big)$ and for using the controls $u_{k-1}^{(i)}$. We will consider the variable part of the income only. So, for agent i, the utility function (2) will now be as follows[13]:

$$J_i\big(x_1, K; u^{(i)}, u^{(-i)}\big) = -\sum_{k=2}^{K+1} \Big[\big(x_k - \bar{x}_k^{(i)}\big)^2 + q_k^{(i)}\big(u_{k-1}^{(i)}\big)\Big]. \tag{17}$$

The above utility function (17) describes agents that are averse to large efforts $u_k^{(i)}$ and interested in keeping x_k close to $\bar{x}_k^{(i)}$. The control cost might correspond to punishable (illegal) opening of the region sluices.

We notice again [see (3)] that, because the desired levels are generically different for each player $\bar{x}_k^{(i)} \neq \bar{x}_k^{(-i)}$, it is impossible to achieve

$$J_i\big(x_1, K; u^{(i)}, u^{(-i)}\big) = 0,$$

which is maximum.

5.2. Parameter Values. It turns out that the game at hand cannot be solved analytically. Instead, numerical solutions will be obtained. We propose some numerical values for the model parameters.

5.2.1. Preference Level Parameters $\bar{x}_k^{(i)}$. As said, watercress producers prefer lower water levels during certain times, because of the growth and harvest requirements. We index the cress producers P (and use F for the fishermen); we model the level preference parameters $\bar{x}_k^{(i)}$ for all players as follows:

$$\bar{x}_k^F = [\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_1], \tag{18}$$

$$\bar{x}_k^P = [\bar{x}_1, \beta\bar{x}_2, \bar{x}_3, \beta\bar{x}_4, \bar{x}_1]. \tag{19}$$

From now onward, $k = 1$ corresponds to winter. Consequently, $k = 2, 3, 4$ denote spring, summer, and autumn, respectively.

In this study, we have normalized the environmentally ideal level to 1 and will assume that fishermen like to have 1.2 all the time ($\bar{x}_k^F = 1.2, \forall k$). Regarding

[13] The thus defined utility is obviously negative. However, it would be positive if we allowed for a sufficiently large constant income. Adding it to (17) would not change the equilibrium conditions.

the cress producers, we suppose that the critical seasons for them are spring and autumn and that they prefer 75% of the water level the fishermen want. So, $\beta = 0.75$ and $\beta \bar{x}_2 = \beta \bar{x}_4 = 0.9$.

5.2.2. Cost Coefficients. The next assumption concerns the cost coefficients $q_k^{(i)}$. We assume that the cost of changing the water level is constant and identical for each player, i.e., $q_k^{(i)} = q_k^{(-i)} = q > 0$, which might correspond to equal likelihood of each agent being caught at tinkering with the sluices.

5.2.3. Evaporation and Retention Parameters a_k. We consider that, in relative terms, the spring and autumn are wet, the summer is dry, and the winter is neutral. We assume the following values for the vector a:

$$a = \left[\sqrt{5}/2, \quad 4/5, \quad \sqrt{5}/2, \quad 1.0000\right]. \tag{20}$$

This means that, if there were no human interventions, there would on average be 12% more water in spring that in winter, 20% less in summer than in spring and 12% more in autumn than in summer; finally, the amount of water carried from autumn to winter would be the same.

6. Numerical Solution

6.1. Reference Solution. The dynamic game with the utility function (17) is linear-quadratic and can be solved through the value function substitution in (12)–(13):

$$V_k^{(i)} = A_k^{(i)} x^2 + B_k^{(i)} x + C_k^{(i)}, \quad k = K, \ldots, 1. \tag{21}$$

Then, the first-order conditions can be calculated and substituted back in (12)–(13). Next, the difference equations for A_k (Riccati), B_k, C_k, could be established and solved. We have solved[14] the game in that way. These results were identical with those obtained through the more general method based on dynamic programming and described in Section 4. In the remainder of this paper, we will solve the game numerically through the latter method.

[14] For example, the last period control for player 1 and the final state are respectively

$$u_x^{(1)} = \frac{\left(-qa_K x_K + \bar{x}_1^{(1)} - \bar{x}_1^{(2)}\right)(1+U) + q\left(Ux_1 - \bar{x}_1^{(1)}\right)}{q(2U+q+2)},$$

$$x_{K+1} = \frac{qa_K x_K + 2Ux_1\bar{x}_1^{(1)} + \bar{x}_1^{(2)}}{(2U+q+2)},$$

where the parameters U and x_1 are the government incentive parameters and will be explained in Section 7. However, even for a horizon as short as $K = 4$ and $U = 0$, $x_1 = 0$ the formulas complicate substantially and as said already only numerical solutions are available for $k < K$.

For a horizon of length $K = 4$ (four seasons, say), the closed form formulas for the stage games joint payoffs, pseudo-Hessians, strategies, etc., are obtainable through MATLAB SYMBOLIC MATHS OR MATHEMATICA. However, they are long and complicated as well as their Riccati equations counterparts. Hence, we have solved the game numerically for the parameter values which are specified in Section 5.2. The results are presented in the following figures, where they are shown to coincide with the solutions obtained by the more general stage-games backward-induction method, explained in Section 4 and applied below.

6.2. Existence and Uniqueness. We check first the existence and uniqueness of the stage games strategies (15). The following Figure 1 shows the definiteness of the pseudo-Hessian[15] of $-\Phi\big(x_k, k; u_k^{(i)}, u_k^{(-i)}\big)$. This is a symmetric 2×2 matrix for each stage $k = 4, \ldots, 1$. Sufficient for its positive definiteness are: positive entry 1,1 and the determinant. The left panels show the 1,1 entries for each stage game; the right panels show the corresponding determinants, all as functions of the cost coefficient q. The most upper panels are drawn for the last stage game. The next ones are for the last two-stage game, etc. We observe that the requested strict positive definiteness is guaranteed; however, it depends on the cost coefficient q and worsens for smaller q. This is not surprising because, for the no-cost case ($q = 0$), the players could use any control and an equilibrium would unlikely exist. We also notice that definiteness improves slightly for shorter horizons.

We know that, if the equilibrium (15) is unique, then we can compute it by solving (11) simultaneously for $i = 1, 2$.

6.3. Strategies. In the above, we have confirmed the positive definiteness of the pseudo-Hessian of $-\Phi\big(x_k, k; u_k^{(i)}, u_k^{(-i)}\big)$ and thus established the existence and uniqueness of feedback Nash equilibrium strategies in a dynamic game of competition for a renewable resource. We believe that this is an important result for the regulator. Provided the cost of (or penalty for) moving the sluices is positive, the regulator should not expect dramatic changes in the agents' behavior.

In Figure 2, we present the computed equilibrium strategy realizations and the corresponding state evolution in the upper and bottom panels, respectively, for two values of the control cost parameter q and for $\bar{x}_k^F = 1.2, \forall k, \beta = 0.75, x_1 = 1$, and the natural level fluctuations governed by (20). The positive bars on the strategy graph (upper panel) are for fishermen and the negative bars are for cress-producers. The dashed lines correspond to $q = 0.5$ and the dash-dotted ones to a cheaper control characterized by $q = 0.1$. The natural water level fluctuations are shown as the thin solid line in the bottom panel. The fishermen's preference level is 1.2,

[15] The negative pseudo-Hessian needs to be positive definite.

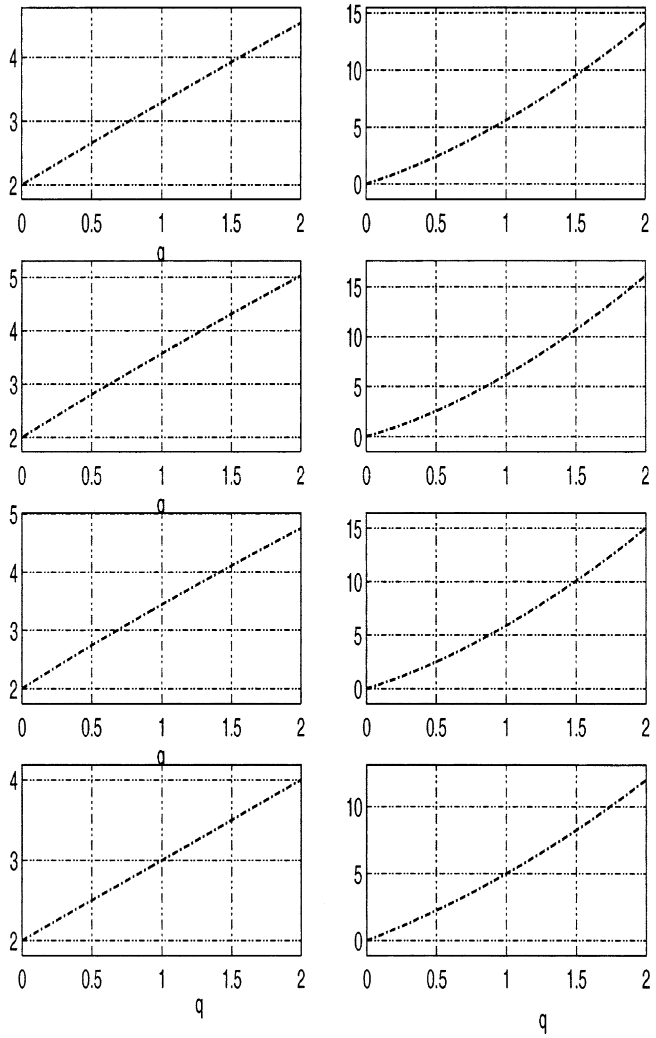

Fig. 1. Pseudo-Hessian definiteness.

$\forall k$, while the cress producers preference level differs for $k = 2$ and $k = 4$ and equals 0.9. For clarity of the figure, these levels are omitted here; they will be sketched later in Figures 4 and 6.

The water levels shown in the lower panel of Figure 2 are a result of the joint agents' actions or actually of the intake-release balance. The upper panel of Figure 2 shows the equilibrium actions that lead to the levels. Notice that the

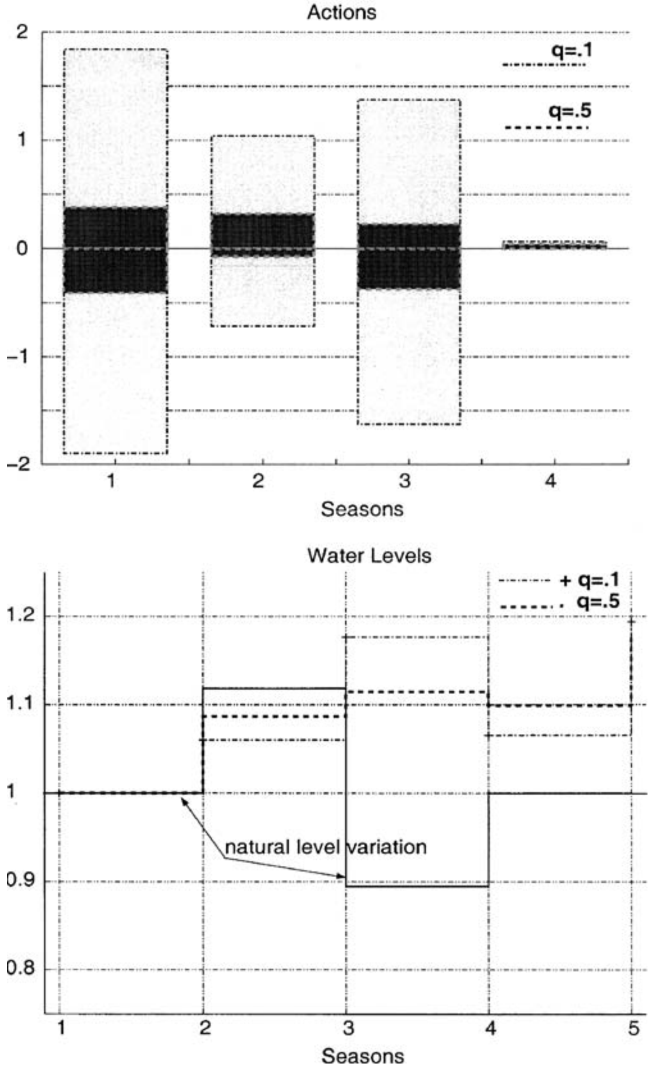

Fig. 2. Equilibrium strategy and state realizations.

actions are quite large, especially in the first three seasons. Also, observe that the water levels could remain the same if the players cooperated and reduced their actions proportionally, i.e., so that the balances were kept. In effect, the players would be penalized less. The situation, which consists of receiving a lower payoff at equilibrium than under cooperation, is similar to the prisoners' dilemma and common to many Nash equilibria. We do not attempt to resolve this dilemma in this paper. Instead, we introduce an *incentive scheme* in Section 7 that will improve environmental standards and also reduce the amounts of water released and let in; this will decrease the penalties paid by the agents.

Examination of the figure's lower panels makes it obvious to conclude that sustaining the natural water level $x_1 = x_5 = 1$ is impossible, especially for cheap controls or lax penalties for cheating or tinkering with the sluices. Therefore, a need for a government intervention becomes a necessity to lessen the environmental impact of the agents' economic activities. In Section 7, we examine whether an intervention of the regional government is likely to stabilize the water level around 1 and how much this may cost.

7. Modified Dynamic Game

7.1. One-Year Problem. Certainly, a regional government will be concerned if x_5 differs substantially from x_1. We will now examine how adding an incentive or penalty term to a player's utility function can modify the players' behavior. In particular, we want to show that the government can control the agents to an environmentally friendlier equilibrium through the use of an incentive scheme.

In Ref. 12, a leader controlled satisfactorily a water level through the use of certain correction parameters in the context of an optimal control problem. We will follow that approach and apply it here to improve our dynamic game's outcome.[16]

Consider the following scheme: Assuming that, after a year (i.e., K periods), the levels x_{K+1} and x_1 are close to each other (in the sense of an environmental watchdog standards), each player will be paid a bonus

$$(W - (x_{K+1} - x_1)^2)U, \quad U, W \geq 0, \tag{22}$$

where U is the government incentive parameter and W is an income scaling constant. If the difference between the levels is large, the bonus will become a penalty. So, now we will look for an equilibrium where the players maximize the following utility functions:

$$\begin{aligned} &I_i\left(x_1, K; u^{(i)}, u^{(-i)}\right) \\ &= [W - (x_{K+1} - x_1)^2]U - \sum_{k=2}^{K+1} \left[\left(x_k - \bar{x}_k^{(i)}\right)^2 + q_k^{(i)} \left(u_{k-1}^{(i)}\right)^2\right]. \end{aligned} \tag{23}$$

[16] We do not endeavor to compute the scheme implementation cost in this paper. However, once the costs became available, the government could try and choose a strategy such that its cost would be constrained; for more on an incentive scheme with constrained costs, see Ref. 13.

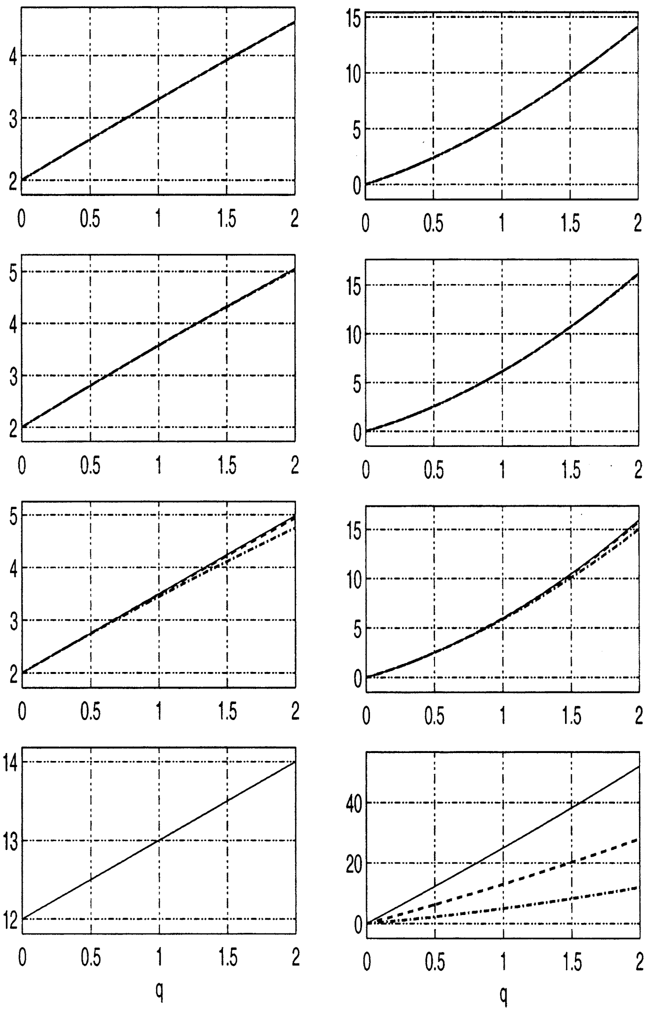

Fig. 3. Pseudo-Hessian definiteness for varying incentive levels.

First, we have to examine the existence of stage equilibria defined analogously to (23) where the functions J_i are replaced by I_i.

The following Figure 3 is analogous to Figure 1 and shows the definiteness of the stage games' pseudo-Hessians for varying values of the incentive parameter U.

As before, the left panels represent entry 1,1 values and the right panels show the pseudo-Hessian determinants. The dash-dotted lines correspond to $U = 0$ and are identical with those in Figure 1. The dashed lines correspond to $U = 2$. The solid lines are drawn for $U = 5$. It is obvious from the figure that unique equilibria exist for $U \geq 0$ and $q > 0$. Larger incentives clearly improve the (negative) pseudo-Hessian positive definiteness (in particular, see the right bottom panel, which corresponds to the full-horizon determinant of the pseudo-Hessian). This is an encouraging result, which suggests that the regulator will have a range of U values for which equilibria exist and that it will be able to select an environmentally friendly solution.

Indeed, Figure 4 shows (for $q = 0.5$) that the modified water levels are such that x_5 is much closer to x_1 than before. The dash-dotted line corresponds to $U = 0$ and is considered environmentally unfriendly (exactly as in Figure 2). The dashed line ($U = 2$) and the solid line ($U = 5$) show that the new equilibrium strategy can bring x_5 very close to x_1.

The equilibrium actions leading to the above water levels are shown in Figure 5 upper and bottom panels, for the fishermen (Player 1) and cress producers (Player 2), respectively. Notice that, under the incentive scheme, the actions are a fraction of the original actions in Section 6.3. This means that the prisoners' dilemma mentioned in Section 6.3 has been attenuated (albeit not eliminated completely).

We notice that the players behave rationally in that they react to the government control parameter U by modifying mostly their last period's actions. This seems sufficient to fulfill the government's aim to bring the final water level close to x_1.

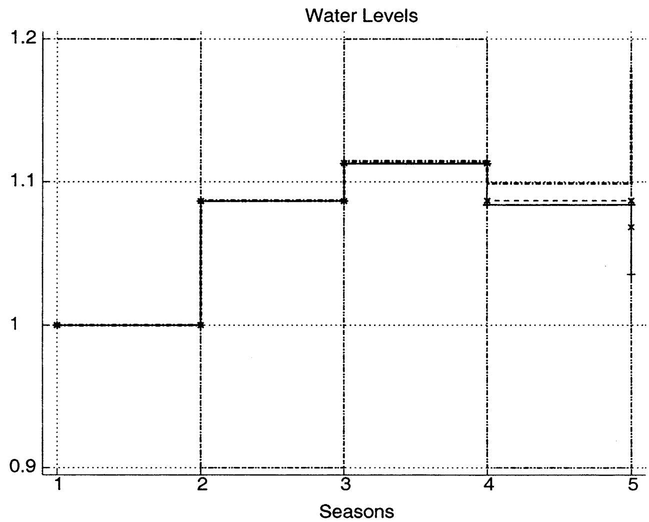

Fig. 4. Modified state realizations.

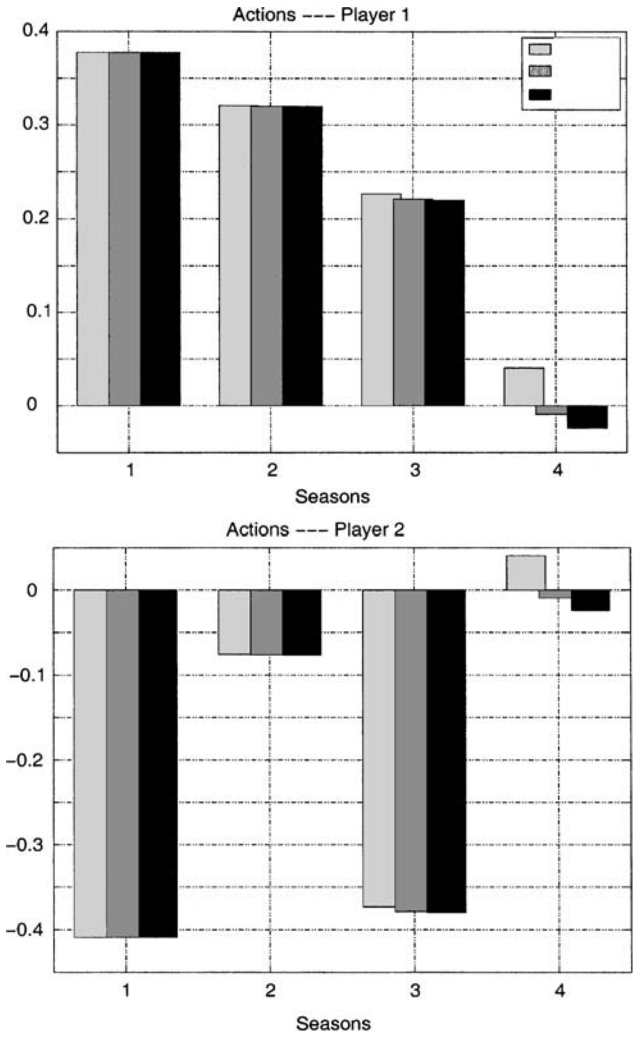

Fig. 5. Modified equilibrium strategy realizations.

To return the system to the same state (or close) as last year appears a satisfactory result under the assumption that the normalized water level 1 was environmentally acceptable. In the next section, we will see that the incentive scheme based on parameters U, W can lead the system to 1.

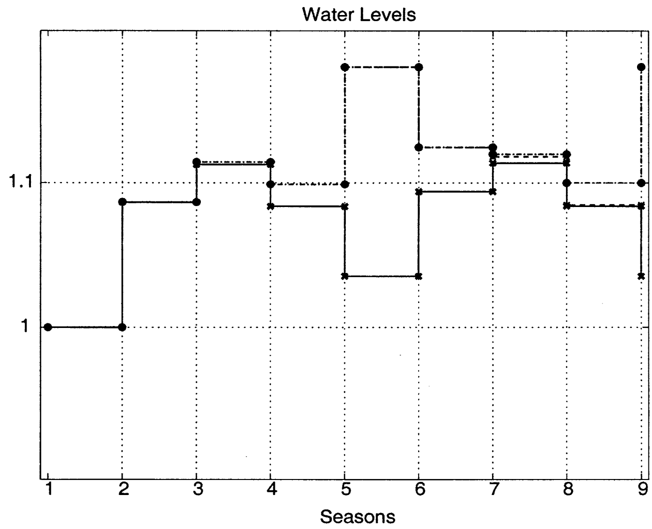

Fig. 6. Two-year water levels.

7.2. Multi-Year Extension. An obvious question to ask is whether the above dynamic four-season dynamic equilibrium will be maintained in the second and subsequent years. To answer this question, we have computed the equilibrium strategies for $U = 0$ and $U = 5$ ($q = 0.5$) for the second year, assuming the initial state equals to x_5 (which the government considers as sufficiently close to 1).

Figure 6 shows the corresponding water levels. The dash-dotted line corresponds to $U = 0$ and the solid line to $U = 5$. It is apparent that the equilibrium strategies (aided or not by the incentive scheme) are such that winter equilibrium levels remain largely unchanged.

The dashed line (mostly overlapping with the other lines but for watch periods 5, 6, 7) shows the water levels in the second year after $U = 0$ was applied in the first year and the incentive scheme with $U = 5$ was implemented from $x_5 = 1.1798$ onward. The government incentive appears efficient in that it motivates agents to such actions that bring the next winter's level close to the desired one, (largely) independently of the previous winter's level. This means that the incentive scheme can control successfully the next winter's level for a broad range of initial conditions[17]. This speaks well about the scheme robustness, i.e., it should work satisfactorily also in the presence of a stochastic noise or other uncertainties.

[17] This is obvious from (22). The winter state is largely independent of the autumn water level provided a large value of U is applied.

8. Concluding Remarks

The first aim of this paper was to solve a water level competition game. The obtained results tell us that it is possible for a regional government to induce agents to a feedback Nash equilibrium where environmental standards are obeyed. The sufficient features of the model that provides the desired equilibrium include increasing action costs and a bonus-penalty incentive function.

We choose the agent cost function and the government incentive function to be quadratic [see (22)]. The former corresponds to the natural way of modeling getting-away with small infraction (little water let in or out) and being caught and punished for large amounts of the water transfers. The latter reflects the government will to spend money on environmental improvements. Both choices appear politically realistic; hence, we should expect sensible results from our model, provided its parameters were properly calibrated.

Our second aim was to contribute to the methodology of solutions of state-coupled dynamic games. We have demonstrated that checking stage games for DSC and solving them recursively in backward time leads to the establishment of feedback Nash (Markovian) equilibria.

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