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Weak Convergence Theorem by an Extragradient Method for Nonexpansive Mappings and Monotone Mappings

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Abstract. In this paper, we introduce an iterative process for finding the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone, Lipschitz-continuous mapping. The iterative process is based on the so-called extragradient method. We obtain a weak convergence theorem for two sequences generated by this process.

Key Words. Extragradient method, fixed points, monotone mappings, nonexpansive mappings, variational inequalities.

1. Introduction

Let C be a closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C. A mapping A of C into H is called monotone if

$$\langle Au - Av, u - v \rangle \geq 0,$$

for all $u, v \in C$. The variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0,$$

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for all $v \in C$. The set of solutions of the variational inequality problem is denoted by VI (C, A). A mapping A of C into H is called α-inverse-strongly-monotone if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2,$$

for all $u, v \in C$; see Refs. 1–2. It is obvious that any α-inverse-strongly-monotone mapping A is monotone and Lipschitz continuous. A mapping S of C into itself is called nonexpansive if

$$\|Su - Sv\| \leq \|u - v\|,$$

for all $u, v \in C$; see Ref. 3. We denote by $\mathrm{F}(S)$ the set of fixed points of S. For finding an element of $\mathrm{F}(S) \cap \mathrm{VI}(C, A)$ under the assumption that a set $C \subset H$ is closed and convex, a mapping S of C into itself is nonexpansive and a mapping A of C into H is α-inverse-strongly-monotone, Takahashi and Toyoda (Ref. 4) introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C (x_n - \lambda_n A x_n), \tag{1}$$

for every $n = 0, 1, 2, \ldots$, where $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in (0,1), and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that, if $\mathrm{F}(S) \cap \mathrm{VI}(C, A)$ is nonempty, then the sequence $\{x_n\}$ generated by (1) converges weakly to some $z \in \mathrm{F}(S) \cap \mathrm{VI}(C, A)$. On the other hand, for solving the variational inequality problem in the finite-dimensional Euclidean space $\mathbb{R}^n$ under the assumption that a set $C \subset \mathbb{R}^n$ is closed and convex, a mapping A of C into $\mathbb{R}^n$ is monotone and k-Lipschitz-continuous and $\mathrm{VI}(C, A)$ is nonempty, Korpelevich (Ref. 5) introduced the following so-called extragradient method:

$$\begin{aligned} x_0 &= x \in C, \\ \bar{x}_n &= P_C (x_n - \lambda A x_n), \\ x_{n+1} &= P_C (x_n - \lambda A \bar{x}_n), \end{aligned}$$

for every $n = 0, 1, 2, \ldots$, where $\lambda \in (0, 1/k)$. He showed that the sequences $\{x_n\}$ and $\{\bar{x}_n\}$ generated by this iterative process converge to the same point $z \in \mathrm{VI}(C, A)$.

In this paper, motivated by the idea of extragradient method, we introduce an iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone, Lipschitz continuous mapping in a real Hilbert space. We obtain a weak convergence theorem for two sequences generated by this process.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that $\{x_n\}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y\rangle \geq 0, \tag{2}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{3}$$

for all $x \in H$, $y \in C$; see Ref. 3 for more details. Let A be a monotone mapping of C into H. In the context of the variational inequality problem, this implies

$$u \in \mathrm{VI}(C, A) \Leftrightarrow u = P_C(u - \lambda A u), \ \forall \lambda > 0. \tag{4}$$

It is also known that H satisfies the Opial condition (Ref. 6); i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

A set-valued mapping $T : H \to 2^H$ is called monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in T_y$ imply

$$\langle x - y, f - g\rangle \geq 0.$$

A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone, k-Lipschitz-continuous mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$; i.e.,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}.$$

Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$; see Ref. 7.

3. Weak Convergence Theorem

In this section, we prove a weak convergence theorem for nonexpansive mappings and monotone mappings. To prove it, we need two lemmas. The first lemma was proved Schu (Ref. 8) in a uniformly convex Banach space.

Lemma 3.1. Let H be a real Hilbert space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for all $n = 0, 1, 2, \ldots$, and let $\{v_n\}$ and $\{w_n\}$ be sequences in H such that

$$\limsup_{n\to\infty} \|v_n\| \leq c,$$
$$\limsup_{n\to\infty} \|w_n\| \leq c,$$
$$\lim_{n\to\infty} \|\alpha_n v_n + (1-\alpha_n) w_n\| = c,$$

for some $c \geq 0$. Then,

$$\lim_{n\to\infty} \|v_n - w_n\| = 0.$$

The second lemma was proved by Takahashi and Toyoda (Ref. 4).

Lemma 3.2. Let H be a real Hilbert space and let D be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $u \in D$,

$$\|x_{n+1} - u\| \leq \|x_n - u\|,$$

for every $n = 0, 1, 2, \ldots$. Then, the sequence $\{P_D x_n\}$ converges strongly to some $z \in D$.

Now, we can state a weak convergence theorem.

Theorem 3.1. Let C be a closed convex subset of a real Hilbert space H. Let A be a monotone k-Lipschitz continuous mapping of C into H and let S be a nonexpansive mapping of C into itself such that $\mathrm{F}(S) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ be sequences generated by

$$\begin{aligned}
&x_0 = x \in C,\\
&y_n = P_C(x_n - \lambda_n A x_n),\\
&x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n),
\end{aligned}$$

for every $n = 0, 1, 2, \ldots,$ where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, the sequences $\{x_n\}$, $\{y_n\}$ converge weakly to the same point $z \in \mathrm{F}(S) \cap \mathrm{VI}(C, A)$, where $z = \lim_{n\to\infty} P_{\mathrm{F}(S)\cap\mathrm{VI}(C,A)} x_n$.

Proof. Put $t_n = P_C(x_n - \lambda_n A y_n)$ for every $n = 0, 1, 2, \ldots.$ Let $u \in \mathrm{F}(S) \cap \mathrm{VI}(C, A)$. From (3), we have

$$\begin{aligned}
\|t_n - u\|^2 &\le \|x_n - \lambda_n A y_n - u\|^2 - \|x_n - \lambda_n A y_n - t_n\|^2\\
&= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, u - t_n\rangle\\
&= \|x_n - u\|^2 - \|x_n - t_n\|^2\\
&\quad +2\lambda_n (\langle A y_n - Au, u - y_n\rangle + \langle Au, u - y_n\rangle + \langle A y_n, y_n - t_n\rangle)\\
&\le \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, y_n - t_n\rangle\\
&= \|x_n - u\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2\\
&\quad +2\lambda_n \langle A y_n, y_n - t_n\rangle\\
&= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2\\
&\quad +2\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle.
\end{aligned}$$

Further, from (2), we have

$$\begin{aligned}
&\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle\\
&= \langle x_n - \lambda_n A x_n - y_n, t_n - y_n\rangle + \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle\\
&\le \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle\\
&\le \lambda_n k \|x_n - y_n\| \, \|t_n - y_n\|.
\end{aligned}$$

So, we obtain

$$\begin{aligned}\|t_n-u\|^2 &\le \|x_n-u\|^2-\|x_n-y_n\|^2-\|y_n-t_n\|^2\\ &\quad +2\lambda_n k\|x_n-y_n\|\,\|t_n-y_n\|\\ &\le \|x_n-u\|^2-\|x_n-y_n\|^2-\|y_n-t_n\|^2\\ &\quad +\lambda_n^2k^2\|x_n-y_n\|^2+\|y_n-t_n\|^2\\ &\le \|x_n-u\|^2+(\lambda_n^2k^2-1)\|x_n-y_n\|^2\\ &\le \|x_n-u\|^2.\end{aligned}$$

We have also

$$\begin{aligned}\|x_{n+1}-u\|^2 &= \|\alpha_n x_n+(1-\alpha_n)St_n-u\|^2\\ &= \|\alpha_n(x_n-u)+(1-\alpha_n)(St_n-u)\|^2\\ &\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)\|St_n-u\|^2\\ &\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)\|t_n-u\|^2\\ &\le \alpha_n\|x_n-u\|^2+(1-\alpha_n)(\|x_n-u\|^2+(\lambda_n^2k^2-1)\|x_n-y_n\|^2)\\ &= \|x_n-u\|^2+(1-\alpha_n)(\lambda_n^2k^2-1)\|x_n-y_n\|^2\\ &\le \|x_n-u\|^2.\end{aligned}$$

Therefore, there exists

$$c=\lim_{n\to\infty}\|x_n-u\|$$

and the sequence $\{x_n\}, \{t_n\}$ are bounded. From the last relations, we obtain also

$$(1-\alpha_n)(1-\lambda_n^2k^2)\|x_n-y_n\|^2\le\|x_n-u\|^2-\|x_{n+1}-u\|^2.$$

So we have

$$\|x_n-y_n\|^2\le\frac{1}{(1-\alpha_n)(1-\lambda_n^2k^2)}(\|x_n-u\|^2-\|x_{n+1}-u\|^2).$$

Hence,

$$x_n-y_n\to 0,\quad n\to\infty.$$

Further, we obtain

$$\begin{aligned}\|y_n-t_n\|^2 &= \|P_C(x_n-\lambda_n Ay_n)-P_C(x_n-\lambda_n Ax_n)\|^2\\ &\le \lambda_n^2k^2\|y_n-x_n\|^2\\ &\le \frac{\lambda_n^2k^2}{(1-\alpha_n)(1-\lambda_n^2k^2)}(\|x_n-u\|^2-\|x_{n+1}-u\|^2).\end{aligned}$$

Hence,

$$y_n - t_n \to 0, \quad n \to \infty.$$

From

$$\|x_n - t_n\| \leq \|x_n - y_n\| + \|y_n - t_n\|,$$

we have also

$$x_n - t_n \to 0, \quad n \to \infty.$$

Since A is Lipschitz continuous, we have

$$Ay_n - At_n \to 0, \quad n \to \infty.$$

As $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to some z. We obtain that $z \in \mathrm{F}(S) \cap \mathrm{VI}\,(C, A)$. First, we show that $z \in \mathrm{VI}\,(C, A)$. Since $x_n - t_n \to 0$ and $y_n - t_n \to 0$, we have $\{t_{n_i}\} \rightharpoonup z$ and $\{y_{n_i}\} \rightharpoonup z$. Let

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$; see Ref. 7. Let $(v, w) \in G(T)$. Then, we have

$$w \in Tv = Av + N_C v$$

and hence,

$$w - Av \in N_C v.$$

So, we have

$$\langle v - u, w - Av \rangle \geq 0, \quad \text{for all } u \in C.$$

On the other hand, from

$$t_n = P_C(x_n - \lambda_n Ay_n) \quad \text{and} \quad v \in C,$$

we have

$$\langle x_n - \lambda_n Ay_n - t_n, t_n - v \rangle \geq 0,$$

and hence,

$$\langle v - t_n, (t_n - x_n)/\lambda_n + Ay_n \rangle \geq 0.$$

Therefore from

$$w - Av \in N_C v \quad \text{and} \quad t_i \in C,$$

we have

$$\begin{aligned}\langle v - t_{n_i}, w \rangle &\geq \langle v - t_{n_i}, Av \rangle \\ &\geq \langle v - t_{n_i}, Av \rangle - \langle v - t_{n_i}, (t_{n_i} - x_{n_i})/\lambda_{n_i} + Ay_{n_i} \rangle \\ &= \langle v - t_{n_i}, Av - At_{n_i} \rangle + \langle v - t_{n_i}, At_{n_i} - Ay_{n_i} \rangle \\ &\quad - \langle v - t_{n_i}, (t_{n_i} - x_{n_i})/\lambda_{n_i} \rangle \\ &\geq \langle v - t_{n_i}, At_{n_i} - Ay_{n_i} \rangle - \langle v - t_{n_i}, (t_{n_i} - x_{n_i})/\lambda_{n_i} \rangle.\end{aligned}$$

Hence, we obtain

$$\langle v - z, w \rangle \geq 0, \quad \text{as } i \to \infty.$$

Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in$ VI(C, A).

We show that $z \in F(S)$. Let $u \in$ F(S) ∩ VI (C, A). Since

$$\|St_n - u\| \leq \|t_n - u\| \leq \|x_n - u\|,$$

we have

$$\limsup_{n \to \infty} \|St_n - u\| \leq c.$$

Further, we have

$$\lim_{n \to \infty} \|\alpha_n(x_n - u) + (1 - \alpha_n)(St_n - u)\| = \lim_{n \to \infty} \|x_{n+1} - u\| = c.$$

By Lemma 3.1, we obtain

$$\lim_{n \to \infty} \|St_n - x_n\| = 0.$$

Since

$$\|Sx_n - x_n\| \leq \|Sx_n - St_n\| + \|St_n - x_n\| \leq \|x_n - t_n\| + \|St_n - x_n\|,$$

we have

$$\lim_{n \to \infty} \|Sx_n - x_n\| = 0.$$

From the demiclosedness of $I-S$, we know that $\{x_{n_i}\} \rightharpoonup z$ and $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$ imply $z \in \mathrm{F}(S)$; see Ref. 3.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $\{x_{n_j}\} \rightharpoonup z'$. Then, $z' \in \mathrm{F}(S) \cap \mathrm{VI}\ (C, A)$. Let us show that $z = z'$. Assume that $z \neq z'$. From the Opial condition (Ref. 6), we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - z\| &= \liminf_{i\to\infty} \|x_{n_i} - z\| < \liminf_{i\to\infty} \|x_{n_i} - z'\| \\ &= \lim_{n\to\infty} \|x_n - z'\| = \liminf_{j\to\infty} \|x_{n_j} - z'\| \\ &< \liminf_{j\to\infty} \|x_{n_j} - z\| = \lim_{n\to\infty} \|x_n - z\|.\end{aligned}$$

This is a contradiction. Thus, we have $z = z'$. This implies

$$x_n \rightharpoonup z \in \mathrm{F}(S) \cap \mathrm{VI}(C, A).$$

Since $x_n - y_n \to 0$, we have also

$$y_n \rightharpoonup z \in \mathrm{F}(S) \cap \mathrm{VI}(C, A).$$

Now, put

$$u_n = P_{F(S)\cap \mathrm{VI}(C,A)} x_n.$$

We show that

$$z = \lim_{n\to\infty} u_n.$$

From

$$u_n = P_{\mathrm{F}(S)\cap \mathrm{VI}(C,A)} x_n \text{ and } z \in \mathrm{F}(S) \cap \mathrm{VI}(C, A),$$

we have

$$\langle z - u_n, u_n - x_n \rangle \geq 0.$$

By Lemma 3.2, $\{u_n\}$ converges strongly to some $z_0 \in \mathrm{F}(S) \cap \mathrm{VI}(C, A)$. Then, we have

$$\langle z - z_0, z_0 - z \rangle \geq 0$$

and hence $z = z_0$. □

4. Applications

Using Theorem 3.1, we prove two theorems in a real Hilbert space.

Theorem 4.1. Let H be a real Hilbert space. Let A be a monotone k-Lipschitz continuous mapping of H into itself and let S be a nonexpansive mapping of H into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in H$ and let

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S(x_n - \lambda_n A(x_n - \lambda_n A x_n)),$$

for every $n = 0, 1, 2, \dots$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, the sequence $\{x_n\}$ converges weakly to some point $z \in F(S) \cap A^{-1}0$, where

$$z = \lim_{n\to\infty} P_{F(S)\cap A^{-1}0} x_n.$$

Proof. We have $A^{-1}0 = VI(H, A)$ and $P_H = I$. By Theorem 3.1, we obtain the desired result. □

Remark 4.1. Notice that $F(S) \cap A^{-1}0 \subset VI(F(S), A)$. See also Yamada (Ref. 9) for the case when A is a strongly monotone and Lipschitz continuous mapping of a real Hilbert space H into itself and S is a nonexpansive mapping of H into itself.

Theorem 4.2. Let H be a real Hilbert space. Let A be a monotone k-Lipschitz-continuous mapping of H into itself and let $B : H \to 2^H$ be a maximal monotone mapping such that $A^{-1}0 \cap B^{-1}0 \neq \emptyset$. Let J_r^B be the resolvent of B for each $r > 0$. Let $\{x_n\}$ be a sequence generated by

$$x_0 = x \in H,$$
$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_r^B (x_n - \lambda_n A(x_n - \lambda_n A x_n)),$$

for every $n = 0, 1, 2, \dots$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, the sequence $\{x_n\}$ converges weakly to some point $z \in A^{-1}0 \cap B^{-1}0$, where

$$z = \lim_{n\to\infty} P_{A^{-1}0\cap B^{-1}0} x_n.$$

Proof. We have $A^{-1}0 = VI(H, A)$ and $F(J_r^B) = B^{-1}0$. Putting $P_H = I$, by Theorem 3.1 we obtain the desired result. □

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