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On Central-Path Proximity Measures in Interior-Point Methods[1]

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Abstract. One of the main ingredients of interior-point methods is the generation of iterates in a neighborhood of the central path. Measuring how close the iterates are to the central path is an important aspect of such methods and it is accomplished by using proximity measure functions. In this paper, we propose a unified presentation of the proximity measures and a study of their relationships and computational role when using a generic primal-dual interior-point method for computing the analytic center for a standard linear optimization problem. We demonstrate that the choice of the proximity measure can affect greatly the performance of the method. It is shown that we may be able to choose the algorithmic parameters and the central-path neighborhood radius (size) in such a way to obtain comparable results for several measures. We discuss briefly how to relate some of these results to nonlinear programming problems.

Key Words. Primal-dual interior-point methods, central path, proximity measures.

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1. Introduction

The concept of the central path has played a crucial role in the development and analysis of interior-point methods. All primal-dual interior-point methods for linear programming require the iterates to remain in an appropriate neighborhood of the central path. This adherence to the central path promotes global convergence of the duality gap sequence and it ensures a polynomial bound on the number of iterations required to produce an appropriate solution of the problem. Different polynomial bound are obtained for linear programming depending on which proximity measure a function measuring the distance to the central path is used in the analysis; see Refs. 1–3.

Proximity measures can be categorized in several ways (see Ref. 4). Some measures are used to measure distance to a specific point on the central path. Such measures depend often on the perturbation or barrier parameter. Other measures, deal with the distance to the central path as a set and thus they are parameter-free measures. Some measures have the barrier property; i.e., the value of the measure tends to infinity at a sequence of points approaching the boundary of the positive orthant. Yet other measures are finite for all points in the positive orthant. With all these different attributes of proximity measures, it is not clear which measures are practically useful.

In interior-point implementations for linear programming [e.g. LIPSOL (Ref. 5) and PCx (Ref. 6)], the iterates are confined to a reasonably large neighborhood. On the other hand, in certain applications, as the one considered in Refs. 7–8, the iterates are required to remain in a relatively small neighborhood of the central path.

The concept of central path or closely related definitions has been useful for designing interior-point algorithms for other optimization problems such as linear complementarity, semidefinite programming, convex nonlinear programming, and even general nonlinear programming problems (e.g. Refs. 9–10). The proximity to the central path can be used to improve the global convergence behavior of primal-dual interior-point methods for nonlinear programming as shown by Argaez, Tapia, and Velasquez (Ref. 11).

Although proximity measures have been used extensively in interior-point methods, there does not seem to be a comprehensive and unified presentation of them nor a study of their computational role. In this paper, we propose such a unified presentation using a standard linear optimization problem. Therefore, we define the concept of proximity measure in this context. We select several proximity measures from the literature and introduce also new measures in an attempt to sample the several categories of proximity measures discussed above. We study some of their properties and the relationship between these measures.

We present a generic primal-dual interior-point algorithmic framework based on the long-step shrinking-neighborhood algorithm (LSSN) proposed by Gonzalez-Lima, Tapia, and Potra (Ref. 12) for effectively computing the analytic center of the solution set in linear programming. We show that the search direction of this algorithm (Newton direction for the perturbed optimality conditions) is of descent for all the measures considered and we investigate the numerical effect of the choice of the proximity measures on the performance of the algorithm. We consider this algorithm as a natural setting for our study, since its behavior depends strongly on the proximity to the central path. We demonstrate that the choice of the proximity measure can affect greatly its performance.

The structure of the paper is as follows. In Section 2, we introduce the optimization problem and some relevant concepts as well as the notation that we use. In Section 3, we define the proximity measures and present the ones to be considered in this paper. In Section 4, we present the algorithmic framework and show that the search direction is of descent for all the proximity measures. Properties and relationships among the measures are presented in Section 5. In Section 6, we present our numerical experience with the proposed algorithm. We analyze its performance when using the proximity measures defined in Section 3. We modify these measures such that comparable performances of the algorithm are obtained for some of the tested problems. Section 7 is devoted to the presentation of an interesting class of measures and to study its properties. Finally, some conclusions and remarks are given in Section 8.

2. Preliminaries and Notation

We consider the following linear programming problem in standard form:

$$\min c^t x, \quad \text{s.t. } Ax = b, x \geq 0, \tag{1}$$

where $c, x \in R^n$, $b \in R^m$, $A \in R^{m\times n}$, $m < n$, and A has full rank m.

The Karush-Kuhn-Tucker (KKT) optimality conditions for (1) are

$$F(x, y, z) = ((Ax - b), (A^t y + z - c), (XZe))^t = 0, \quad (x, z) \geq 0, \tag{2}$$

where $X = \operatorname{diag}(x)$, $Z = \operatorname{diag}(z)$, and e is the n-vector of all ones.

The feasibility set of problem (2) is

$$\mathcal{F} = \left\{(x, y, z) : Ax = b, A^t y + z = c, (x, z) \geq 0\right\}.$$

A feasible point $(x, y, z) \in \mathcal{F}$ is said to be strictly feasible if x and z are strictly positive. The set of all the strictly feasible points is denoted by $\mathcal{F}^+$. In this paper, we assume that $\mathcal{F}^+ \neq \emptyset$.

We denote the solution set of problem (2) by $\mathcal{S}$, so we have

$$\mathcal{S} = \{(x, y, z) : F(x, y, z) = 0, (x, z) \geq 0\}.$$

The central path $\mathcal{P}$ associated with problem (2), parameterized by the parameter $\mu > 0$, is defined as the collection of points $(x(\mu), y(\mu), z(\mu)) \in \mathcal{F}^+$ that solve the perturbed KKT system

$$F_\mu(x, y, z) = F(x, y, z) + \mu(0, \ldots, 0, e)^t, \ \ (x, z) \geq 0. \tag{3}$$

Then,

$$\mathcal{P} = \{(x(\mu), y(\mu), z(\mu)) \text{ is a strictly feasible point satisfying } X(\mu)Z(\mu)e = \mu e\}.$$

Notice that (x, y, z) belongs to the central path if and only if

$$XZe = \mu^* e, \quad \text{with } \mu^* = x^t z/n.$$

Due to the assumption that $\mathcal{F}^+ \neq \emptyset$, the central path is well defined and as μ goes to zero (see Ref. 1), the central-path point $(x(\mu), y(\mu), z(\mu))$ converges to a unique solution where the product of the positive components in the relative interior of $\mathcal{S}$ is maximized, the so-called analytic-center solution.

This fact plays a critical role result in the development of most primal-dual interior-point algorithms. These methods attempt to follow the central path by solving approximately the optimality conditions (3) using a damped Newton method. Hence, to measure the proximity to the central path and to define the neighborhoods of the central path becomes a very important issue. In Section 3 we will study formally these concepts.

In this paper, we use the following notations. If $w \in R^n$, we denote by $1/w$ or w^{-1} the vector $(1/w_1, \ldots, 1/w_n)^t$ and by w^2 the vector $(w_1^2, \ldots, w_n^2)^t$; $\max w$ denotes $\max_{1 \leq i \leq n} w_i$; (similarly, for $\min w$). If $u \in R^n$, we denote by uw the vector $(u_1 w_1, \ldots, u_n w_n)^t$. The notations $\|\cdot\|_1$ and $\|\cdot\|$ are used for the L_1-norm and the Euclidean L_2-norm respectively.

The natural logarithmic function is denoted by $\log(\cdot)$. Let f, g be two functions with the same domain. Then, we say that $f = O(g)$ if there exists a constant $C > 0$ such that $|f| \leq C|g|$ for g sufficiently small.

3. Proximity Measures

In the context of primal-dual interior-point methods, proximity or centrality measures are functions that measure in some way the distance between any strictly feasible point and the central path. Several functions have been proposed in the interior-point literature to measure the proximity to the central path; see for example Refs. 4 and 13–14. Most of these functions attain their optimum value at either a specific point of the central path or on all points of the central path.

Because of the central path definition, the most standard measure used in the literature is the one based in the Euclidean norm between the vectors xz and μe for any $\mu > 0$; this is to say

$$f_\mu^2(x, y, z) = \|(xz - \mu e)/\mu\|.$$

Most of the existent theoretical and practical results for short-steps algorithms rely on the use of this L_2-norm proximity measure (Refs. 1–14).

Another measure considered by several authors in the context of linear and nonlinear programming is, for any $\mu > 0$,

$$f_\mu^S(x, y, z) = \left\| \sqrt{xz/\mu} - \sqrt{\mu/xz} \right\|.$$

This measure can be written as a scaled version of the Euclidean L_2-norm measure, since

$$f_\mu^S(x, y, z) = \left\| (XZ)^{-0.5}(xz - \mu e) \right\| \Big/ \sqrt{\mu} = \left\| \sqrt{\mu/xz}(xz/\mu - e) \right\|.$$

Therefore, in this paper, we call it the nonlinearly scaled L_2-norm.

Jansen (Ref. 13) proposed this measure in the analysis of a primal-dual method for linear programming problems. Argaez and Tapia (Ref. 9) introduced this measure in the context of primal-dual methods in nonlinear programming. Nesterov and Todd (Ref. 15) proposed several proximity measures based on self-concordance functions of self-scaled cones and their derivatives. The above measure can be deduced from their definition.

Finally, we consider the logarithmic barrier function as presented in Ref. 13. This is, for any $\mu > 0$,

$$f_\mu^{Log}(x, y, z) = x^t z/\mu - \sum_{i=1}^{i=n} \log(x_i z_i) - n + n \log(\mu).$$

Notice that, up to the constant $-n + n\log(\mu)$, this is the well-known logarithmic barrier function used by many authors for analyzing primal-dual interior-point methods in linear and nonlinear programming (Ref. 11–16).

An interesting characteristic of the last two measures with respect to the first one is that the values of $f_{\mu}^{S}(x,y,z)$ and $f_{\mu}^{Log}(x,y,z)$ tends to infinity when $x_i z_i$ tends to zero for any $i=1,\ldots,n$. Therefore, when any point is too close to a boundary of the feasible set, the measures are quite large.

All the previous measures depend explicitly on the barrier parameter μ and attain their minimal value at the central point $(x(\mu), y(\mu), z(\mu))$. We call the functions that posses this property as point-proximity measures. A formal definition follows.

Definition 3.1. For any $\mu > 0$, a point-proximity measure is a function $f_{\mu} : \mathcal{F}^{+} \to \{0\} \cup R^{+}$ such that $f(x,y,z)=0$ if and only if $(x,y,z) = (x(\mu), y(\mu), z(\mu))$.

There are other measures that do not depend explicitly on the parameter μ. Such functions measure the distance between a given strictly feasible point and the central path set. We call such functions as path-proximity measures. Any point-proximity measure f_{μ} induces a path-proximity measure f defining $f = f_{\mu^*}$, with $\mu^* = \arg\min_{\mu>0} f_{\mu}$.

As an illustration, let us consider the logarithmic barrier function $f_{\mu^*}^{Log}$ for $\mu^* = x^T z/n$. This measure is found in the interior-point literature (e.g. Kojima et al, Ref. 14) and is related to the function

$$f^R = (x^T z/n)/\Pi_{1\le i\le n}(x_i z_i)^{1/n},$$

since

$$f_{\mu^*}^{Log} = n \log f^R.$$

The function f^R has been used by Tanabe (Ref. 16) in the context of general nonlinear systems of equations for measuring the distance to the central variety.

Path-proximity measures can be derived also from projective metrics. A formal definition is in order.

Definition 3.2. A path-proximity measure is a function $f : \mathcal{F}^{+} \to \{0\} \cup R^{+}$ such that $f(x,y,z)=0$ if and only if $(x,y,z) \in \mathcal{P}$.

In this paper, we focus our attention on the following path-proximity measures.

(i) L_2-norm measure evaluated at $\mu_1 = x^t z/n$,

$$f^{L_2}(x, y, z) = \|xz - x^t z/n\ e\|/(x^t z/n).$$

It is clear that

$$f^{L_2}(x, y, z) \leq f_{\mu}^{L_2}(x, y, z), \quad \text{for all } \mu > 0 \text{ and for all } (x, y, z).$$

(ii) Nonlinearly scaled L_2-norm measure squared evaluated at $\mu_2 = \sqrt{\|xz\|_1/\|(1/xz)\|_1}$,

$$f^{S^2}(x, y, z) = \sqrt{x^T z \|(1/xz)\|_1} - n.$$

It can be seen that μ_2 minimizes the function $(f_{\mu}^{S}(x, y, z))^2$ for all $\mu > 0$.

(iii) ℓ_1/ℓ_2 ratio measure defined by

$$f^{L}(x, y, z) = 1 - \left(1/\sqrt{n}\right) x^T z/\|xz\|.$$

This measure can be deduced from the projective metric used by Lagarias (Ref. 18) in analyzing the Karmarkar algorithm. The same measure can be deduced also from the definition of proximity measure given in Nesterov and Todd (Ref. 15).

(iv) Hilbert measure defined by

$$f^{H}(x, y, z) = \log\left(\max(xz)/\min(xz)\right).$$

This measure is closely related to $\max(xz)/\min(xz)$, used by researchers in interior-point methods; see for example Roos, Terlaky, and Vial (Ref. 15). It is deduced from the definition of the Hilbert projective metric and extends naturally to more general optimization problems such as semidefinite programming and certain infinite-dimensional optimization problem. El-Bakry (Ref. 19) presents a detailed study of this function.

(v) $\ell_{-\infty}$-norm measure, defined by

$$f^{-\infty}(x, y, z) = 1 - \mathrm{n}\min(xz)/\|xz\|_1.$$

This measure has been used extensively in the analysis, development, and implementations of primal-dual interior-point methods; see Ref. 1. It defines a larger neighborhood and we include it here as a reference. However, we focus on the study of the other measures, since they represent more restrictive central-path following strategies.

Then, we call a proximity measure either a point-proximity measure or a path-proximity measure.

Using the concept of proximity measure, we can define a neighborhood of the central path as follows.

Definition 3.3. A β-neighborhood of the central path is defined as

$$\mathcal{N}_f(\beta) = \left\{(x, y, z) : (x, y, z) \in \mathcal{F}^+,\ f(x, y, z) \leq \beta\right\},$$

where $\beta > 0$ and f is a proximity measure.

In the next section, we show that all the proximity measures presented here are such that the Newton direction for the perturbed conditions (3) is of descent for each of them. We describe also the algorithm that we use as a framework to investigate the effect of different choices of the proximity measures.

4. Algorithmic Framework

Let us notice that the proximity measures are really defined depending on the vector xz; the vector y does not appear in the definition. It is well known that, under the standard assumptions, the vector y is uniquely defined depending on the vectors x, z if the point (x, y, z) is a strictly feasible point. Therefore, from now on, we suppress the vector y when working with the proximity measures. Moreover, for convenience, for all the measures $I = L_2, Log, S, S^2, L, H$, we write f^I instead of $f^I(x, z)$ and ∇f^I or δf^I to denote the gradient $\nabla_{(x,z)} f^I$ or the subgradient $\delta_{(x,z)} f^I$ for non-differentiable functions. We refer to Rockafellar (Ref. 20) for the definition of subgradient.

Lemma 4.1. Let $\mu > 0$, let $(x, y, z) \in \mathbb{R}^{2n+m}$, with $(x, z) > 0$ and $xz - \mu e \neq 0$. Let $\Delta w = (\Delta x, \Delta z)^t$ be a direction satisfying the equation

$$Z\Delta x + X\Delta z = -xz + \mu e. \tag{4}$$

Then, the following properties hold:

(i) $(\nabla (f_\mu^{L_2})^2)^t \Delta w = -(f_\mu^{L_2})^2$,

(ii) $(\nabla (f_\mu^{S})^2)^t \Delta w = -(f_\mu^{S})^2 - \|(XZ)^{-1}(xz - \mu e)\|$,

(iii) $(\nabla f_\mu^{Log})^t \Delta w = -(f_\mu^{S})^2$,

(iv) $(\nabla (f^{L_2})^2)^t \Delta w \leq -4(f^{L_2})^2$, if $\mu \leq x^t z/n$,
(v) $\nabla (f^{S^2})^t \Delta w = 1/2(x^t z\|1/xz\|_1)^{-1/2}\mu(n\|1/xz\|_1 - x^t z\|1/xz\|^2)$,
(vi) $(\nabla f^L)^t \Delta w = -\mu\sqrt{n}/\|xz\|(n - ((x^t z)/\|xz\|)^2)$,
(vii) $(\delta f^H)^t \Delta w = -\mu(1/\min(xz) - 1/\max(xz))$.

Proof. The proof comes from the definition of the proximity measures and the computation of their gradient or subgradient functions. We omit the proof here for brevity. The interested reader is referred to Ref. 21 for details. □

Using this lemma, we obtain the following result.

Theorem 4.1. Let $\mu > 0, (x, z) > 0$, such that $xz - \mu e \neq 0$ and let $(\Delta x, \Delta z)$ be a direction satisfying the complementarity equation (4). Then, $(\Delta x, \Delta z)$ is descent direction for (i) the point-proximity measures $f_\mu^{L_2}$. $f_\mu^S f_\mu^{Log}$, (ii) the path-proximity measures f^{S^2}, f^H, f^L, and (iii) if $\mu \leq x^t z/n$ also for f^{L_2}.

Proof. Except for items (v) and (vi), the proof is straightforward from the previous lemma. To obtain the result for f^L, observe that

$$\|xz\|_1^2 \leq \|xz\|^2 n.$$

Similar observation is used for f^{S^2}; therefore,

$$\nabla (f^{S^2})^t \Delta w \leq (1/2)(x^t z)^{-1/2}\|1/xz\|_1^{1/2}\,(\mu/n)\,(n^2 - x^t z\|1/xz\|_1).$$

Using the Cauchy-Schwarz inequality, we obtain that

$$n^2 - x^t z\|1/xz\|_1 < 0, \quad \text{if } xz \neq \mu e.$$ □

The next algorithm is based on the long-step shrinking neighborhood LSSN algorithm by González-Lima et al (Ref. 12). It has two main ingredients; approaching the central path and decreasing the gap. Proximity to the central path is obtained considering a fixed value of μ and approximately solving the perturbed nonlinear system (3). At each iteration, the gap is decreased because of the way the algorithmic parameters are chosen. In order to solve approximately (3), a damped Newton method is applied. Then, a merit function is used to measure the progress towards the central path. Also, a proximity measure is used to compute how close are the iterates from the central path. The central-path strategy idea used in the LSSN algorithm has been extended to the design of algorithms for (general) nonlinear programming; see Refs. 9–10.

Algorithm 4.1. LSSN Generic Algorithm. Consider $w^0 = (x^0, y^0, z^0) \in \mathcal{F}^+$ and $\beta > 0$. Let f_μ be a point-proximity measure and let g be any proximity measure. Choose $\eta \in (0, 1/2)$ and $0 < l < u < 1$. Do until convergence the steps below.

Step 1. Choose $\mu > 0$.

Step 2. Proximity to the Central Path. If

$$g(w^k) \leq \beta, \tag{5}$$

go to Step 6.

Step 3. Compute the New Iterate.

Step 3.1. Solve for $\Delta w^k = (\Delta x^k, \Delta y^k, \Delta z^k)$, $F'(w^k)(\Delta w^k)^t = -F_\mu(w^k)$.

Step 3.2. Choose $\tau^k \in (0, 1)$ and compute the steplength $\alpha^k = \min(1, \tau^k \hat{a}^k)$, where $\hat{a}^k = -1/\min(t^k)$, with $t^k = (\Delta x^k / w^k (1, \ldots, n), \Delta z^k / w^k (n+m+1, \ldots, n))$.

Step 3.3. Form $w^{k+1} = w^k + \alpha^k \Delta w^k$.

Step 4. Line Search.

Step 4.1. If

$$f_\mu(w^{k+1}) \leq f_\mu(w^k) + \eta \alpha^k \nabla f_\mu(w^k)^t (\Delta w^k)^t, \tag{6}$$

go to Step 5.

Step 4.2. If not, reduce $\alpha^k := \rho \alpha^k$, with $\rho \in [l, u]$, and form $w^{k+1} = w^k + \alpha^k \Delta w^k$.

Step 4.3. Go to Step 4.1.

Step 5. Set $k := k+1$ and go to Step 2.

Step 6. Reduce μ.

Step 7. Do Steps 3.1, 3.2, 3.3.

Step 8. Set $k := k+1$ and go to Step 1.

In the following section, we study some properties and relationships between the different measures and their corresponding neighborhoods.

5. Properties

The following result establishes a useful relationship between the Euclidean proximity measures considered in this paper. A proof for the first part in the case when $\mu = x^t z / n$ can be found in Ref. 4.

Proposition 5.1. Let $\mu > 0$. Then,

$$f_\mu^{L_2} (\mu/\max(xz))^{1/2} \le f_\mu^S \le (\mu/\min(xz))^{1/2} f_\mu^{L_2}.$$

From this, the following results are obtained:

(i) For any $\beta > 0$, the set of points $\{(x, y, z) \in \mathcal{N}^{L_2}(\beta) : \min(xz)/\mu > 1\} \subset \mathcal{N}^S(\beta)$. The set of points $\{(x, y, z) \in \mathcal{N}^S(\beta) : \max(xz)/\mu < 1\} \subset \mathcal{N}^{L_2}(\beta)$.

(ii) There holds that

$$\left| f_\mu^{L_2} - f_\mu^S \right| \le \left(f_\mu^S \right)^2.$$

(iii) If $\mu = \sigma x^t z/n$, with $\sigma \in (0, 1)$, then

$$f_\mu^S \le \sqrt{\sigma}\sqrt{1/(1 - f^{-\infty})}\, f^{L_2}.$$

Besides, if $\min(xz) \ge \gamma(x^T z/n)$, with $\gamma \in (0, 1)$, then

$$f_\mu^S \le \sqrt{\sigma/\gamma}\, f_\mu^{L_2}.$$

Proof. Let us recall that

$$\left(f_\mu^S \right)^2 = (1/\mu) \left\| (XZ)^{-0.5} (xz - \mu e) \right\|^2. \tag{7}$$

Then,

$$\left(f_\mu^S \right)^2 \le (1/\mu) \left\| (XZ)^{-0.5} \right\|^2 \| xz - \mu e \|^2,$$

which implies that

$$\left(f_\mu^S \right)^2 \le \mu \left\| (XZ)^{-0.5} \right\|^2 \left(f_\mu^{L_2} \right)^2.$$

Since X and Z are diagonal matrices,

$$\left\| (XZ)^{-0.5} \right\|^2 = \max_{1 \le j \le n} \left(1/x_j z_j \right) = 1/\min(xz).$$

Besides, equation (7) can be stated as

$$\left(f_\mu^S \right)^2 = (1/\mu) \sum (x_i z_i - \mu)^2 / x_i z_i.$$

Therefore,

$$\left(f_\mu^S \right)^2 (x, y, z) \ge (\mu/\max(xz)) \, \| xz/\mu - e \|^2$$

and the first part of the proposition is obtained. Items (i) and (ii) follow in a straight forward way from this result.

In order to prove (ii), we just need to obtain bounds for the values of $\mu/\min(xz)$ and $\mu/\max(xz)$. Let us call

$$v=\sqrt{xz/\mu},\, v_{\min}=\min(v),\, v_{\max}=\max(v).$$

It can be seen easily that

$$f_\mu^S=\left\| v-v^{-1}\right\|.$$

Then,

$$f_\mu^S\geq\left|v_{\max}-v_{\max}^{-1}\right| \text{ and } f_\mu^S\geq\left|v_{\min}-v_{\min}^{-1}\right|.$$

Assume $v_{\max}\geq 1$. Then,

$$\left|v_{\max}-v_{\max}^{-1}\right|=-1/v_{\max}+v_{\max}.$$

Therefore, we obtain

$$v_{\max}\leq 1+f_\mu^S.$$

This inequality is obviously true if $v_{\max}<1$.

Assume $v_{\min}\leq 1$. Then,

$$\left|v_{\min}-v_{\min}^{-1}\right|=1/v_{\max}-v_{\min}.$$

Therefore, we obtain

$$v_{\min}\geq 1-f_\mu^S.$$

This inequality is obviously true if $v_{\min}>1$.

From the first part of the proposition, we have that

$$f_\mu^S v_{\min}\leq f_\mu^{L_2}\leq f_\mu^S v_{\max}.$$

Therefore,

$$-\left(f_\mu^S\right)^2+f_\mu^S\leq f_\mu^{L_2}\leq f_\mu^S+\left(f_\mu^S\right)^2$$

and the proof of (ii) is complete. □

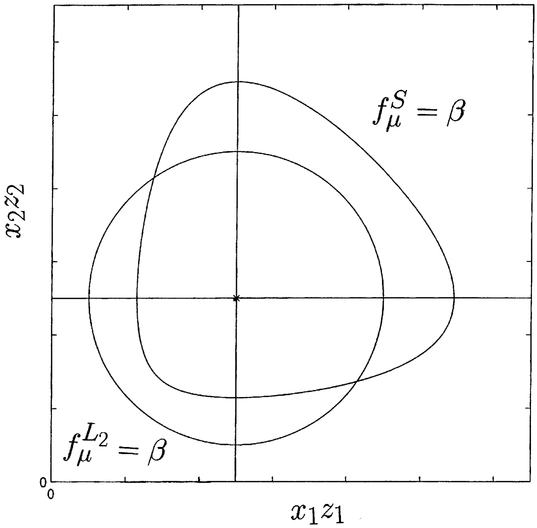

Fig. 1. Comparison of $\mathcal{N}_{f_\mu^{L_2}(\beta)}$ and $\mathcal{N}_{f_\mu^S(\beta)}$.

In the x_iz_i space, Figure 1 illustrates the relationship, between the β-neighborhoods for the central-point Euclidean proximity measures $f_\mu^{L_2}$ and f_μ^S. The neighborhoods defined by these Euclidean measures are never totally included one into the other, as it is stated in the previous proposition.

The neighborhoods defined by the logarithmic barrier function have a similar shape as the ones defined by the nonlinearly scaled L2-norm measure. They differ essentially in the size of the neighborhood. The next proposition establishes a relationship between these two measures.

Proposition 5.2. Let $\mu > 0$. Then, $f_\mu^{Log} \leq \left(f_\mu^S\right)^2$. The equality is satisfied if and only if the (x, y, z) satisfies $xz = \mu e$. Therefore,

$$\mathcal{N}_\mu^S(\beta) \subset \mathcal{N}_\mu^{Log}(\beta), \quad \text{for } \beta \in (0, 1).$$

Proof. For notational convenience, let us denote $v_i = x_iz_i/\mu$. Because of the definitions of the measures, it is suffices to prove that

$$v - 1 - \log(v) \leq \left(v^{1/2} - v^{-1/2}\right)^2, \quad \text{for all } v \in \mathbb{R}, v > 0,$$

since

$$f_\mu^{Log} = \sum_{i=n}^{n} (v_i - 1 - \log(v_i))$$

and

$$\left(f_\mu^S\right)^2 = \sum_{i=1}^{n} \left(v_i^{1/2} - v^{-1/2}\right)^2.$$

Let us call

$$f(v) = v^{-1} - 1 + \log(v).$$

Then, we have just to prove that

$$f(v) \geq 0, \quad 0 = f(1), \text{for all } v > 0.$$

But

$$f^1(v) = -v^{-2} + v^{-1} \geq 0, \text{ if } v \geq 1 \text{ and } f^1(v) \leq 0 \text{ if } v \leq 1$$

and the proof is complete. □

In order to understand and being able to compare the different measures, for any $(x, y, z) \in \mathcal{F}^+$ we consider the angle between vectors xz and e. Let us call it θ. It is clear that, if θ is close to zero, then (x, y, z) is close to the central path $\mathcal{P}$. The angle does not depend directly on the scaling or the norms used, so we consider this quantity as the preferred measure to quantify the distance between any point and the central path set. Therefore, in the next propositions, we characterize the measures introduced in terms of θ.

Proposition 5.3. Let $\mu > 0$. Then,

$$\begin{aligned}\left(f_\mu^{L_2}\right)^2 &= n + \|xz\|^2/\mu^2 - 2\sqrt{n}\|xz\| \cos\theta/\mu \\ &= n + (x^t z)^2/n(\mu^2)\cos^2\theta - 2x^t z/\mu.\end{aligned}$$

If $\mu = \sigma(x^t z)/n$ with $\sigma > 0$, then

$$\left(f_\mu^{L_2}\right)^2 = n\left(-1 + 1/\sigma^2 \cos^2\theta\right).$$

Therefore,

$$f^{L_2} = \sqrt{n}\tan\theta.$$

Proof. The proof comes directly from applying the law of cosines to the triangle formed by the vector μe, xz, and $xz - \mu e$. □

The relationship between f^{L_2} and the angle θ has been presented in Sturm (Ref. 22).

The next propositions characterize the L and S^2 measures in terms of θ.

Proposition 5.4. The measure $f^L = 1 - \cos\theta = 2\sin^2\theta/2$.

Proof. The proof comes directly by using the fact that

$$(xz)^t e = \|xz\|\sqrt{n}\cos\theta.$$

□

Proposition 5.5. The measure $f^{S^2} = C(x, y, z)\cos^{1/2}\theta - n$, with $C(x, y, z) = \sqrt{\|xz\|\,\|1/xz\|_1\sqrt{n}}$.

Proof. Using the law of cosines, we have

$$(xz)^t e = \|xz\|\sqrt{n}\cos\theta \quad \text{and} \quad (1/xz)^t e = \|1/xz\|\sqrt{n}\cos\theta',$$

with θ' equal to the angle between the vectors $1/xz$ and e. Multiplying both equalities and observing that

$$(xz)^t e = x^t z \quad \text{and} \quad (1/xz)^t e = \|1/xz\|_1,$$

we obtain the following equation:

$$x^t z\|1/xz\|_1 = \|xz\|\,\|1/xz\| n\cos\theta\cos\theta'.$$

Then,

$$\left(f^{S^2} + n\right)^2 = \|xz\|\,\|1/xz\|\cos\theta\cos\theta'.$$

On the other hand,

$$\cos\theta' = \|1/xz\|_1/(\|1/xz\|\sqrt{n}).$$

Substituting this equality in the definition of f^{S^2}, we have the desired result. □

In the following result, we characterize the measures for small values of θ. Although the next result holds only for small values of θ, we consider it relevant, since it shows that, even then, the proximity measures have very different values which produces different numerical performances of the generic LSSN algorithm. We will use the next result to define some modifications of the proximity measures considered such that similar performances of the algorithm are obtained for the modified measures.

Theorem 5.1. Consider $(x, y, z) \in \mathcal{F}^+$ and let us call $w = xz$. Let θ be the angle between vectors w and e, i.e.,

$$\cos\theta = w^t e / \|w\| \sqrt{n}.$$

For any $\mu > 0$, let us denote

$$C1_\mu(w) = \left(\|w\|/\mu - \sqrt{n}\right)^2 \quad \text{and} \quad C2_\mu(w) = \sqrt{n}\|w\|/\mu.$$

Let us call

$$C3(w) = \sqrt{\|w\| \|1/w\|_1 \sqrt{n}} - n \quad \text{and} \quad C4(w) = (-C3(w) - n)/4.$$

Then,

(i) $\left(f_\mu^{L_2}\right)^2 = C1_\mu(w) + C2_\mu(w)\theta^2 + O(\theta^4)$,
(ii) $f^{L_2} = \theta\sqrt{n} + O(\theta^3)$,
(iii) $f^{S^2} = C3(w) + C4(w)\theta^2 + O(\theta^4)$,
(iv) $f^L = \frac{\theta^2}{2} + O(\theta^4)$,
(v) $f^H \leq \theta\sqrt{n} + O(\theta^3)$.

Proof. The proof comes from the previous propositions and the Taylor series. Let us consider the Taylor series of the function $\cos\theta$ around $\theta = 0$. This is,

$$\cos\theta = 1 + \theta^2/2 + O(\theta^4).$$

Substituting in Proposition 5.3, we obtain

$$f_\mu^{L_2} = n + (\|w\|/\mu)^2 - 2\sqrt{n}\|w\|/\mu + 2\sqrt{n}\,(\|w\|/\mu)\,\theta^2/2 + O(\theta^4).$$

Rearranging the first term, we obtain the proof of part (i). Parts (ii), (iii), and (iv) can be shown in a similar way, considering the Taylor series of

$\tan\theta$ and $\cos\theta$ around 0 and the previous propositions. Finally, let us observe that

$$1 - f^{L_2} \leq \min(xz)/\left(x^T z/n\right) \leq \max(xz)/\left(x^T z/n\right) \leq 1 + f^{L_2}.$$

Then,

$$\log\left(\max(xz)/\min(xz)\right) \leq \log\left(\left(1 + f^{L_2}\right)/\left(1 - f^{L_2}\right)\right).$$

But locally, $\log\left[\left(1+f^{L_2}\right)/\left(1-f^{L_2}\right)\right]$ is the same order of f^{L_2}. Using (ii) of this theorem, we get the result for the proximity measure f^H. This concludes the proof. □

6. Numerical Experience

In this section, we discuss the numerical results obtained from applying the generic LSSN algorithm to a subset of the NETLIB test problems and, considering the different proximity measures. The experiments were performed in 64 bit arithmetic using codes implemented in MATLAB[4]. The starting points for the algorithms are obtained following Lustig, Marsden, and Shanno (Ref. 24) and are not necessarily feasible. The line-search strategy (backtracking) defined in Step 4 of the generic LSSN algorithm was implemented using the value $\alpha = 10^{-4}$ and a fixed value $\rho = 1/2$.

In all problems, the parameters τ^k were chosen as

$$\tau^k = 1 - \min(0.05, 0.05(x^k)^T z^k).$$

Therefore, $\tau^k = 0.95$ far away from the solution set and is closer to 1 when the duality gap is small.

The parameter μ was chosen as in the LSSN algorithm (see Ref. 12 for a detailed explanation). This is to say that a parameter $\sigma^0 \in (0, 1)$ is chosen at the beginning of the algorithm. Then,

$$\mu = \sigma^0 (x^k)^t z^k/n, \quad \text{for } k \in \{0\} \cup \left\{k \geq 1 : g\left(x^{k-1}, y^{k-1}, z^{k-1}\right) \leq \beta \text{ or } g\left(x^k, y^k, z^k\right) \leq \beta\right\},$$

with g a proximity measure. The proximity measure used for the line search strategy was $\left(f_\mu^{L2}\right)^2$.

[4]MATLAB is a registered trademark of the MathWorks, Inc.

We say that a problem is solved to an accuracy of 10^{-8} if the algorithm is terminated when

$$\max\left(\frac{\left|c^t x^k - b^T y^k\right|}{1+\left|b^T y^k\right|}, \frac{\|Ax^k - b\|_1}{1+\|x\|_1}, \frac{\left\|A^T y^k + z^k - c\right\|_1}{1+\left\|y^k\right\|_1+\left\|z^k\right\|_1}, g\left(x^k, y^k, z^k\right)\right) \leq 10^{-8}.$$

The algorithm stops when the problem is solved to the given accuracy or when the number of iterations reaches 120. In this latter case, we say that the algorithm did not converge.

The results obtained showed that the algorithm can be very sensitive to the choice of the proximity measure in terms of total number of iterations and backtrackings. As an illustration, consider Tables 1 and 2. In these tables, we are showing the number of iterations required for the algorithm to converge for some of the tested problems, considering the different measures and different values of β, the neighborhood size. Observe that, in some cases, as for problems Sctap1 or Lotfi, the difference in the number of iterations, from one measure to another, can be very high. Moreover, it can be the case that the algorithm converges only for some of the measures and not for all of them, as is the case for problem Blend.

We tested also the problems using as a merit function in the line search strategy the logarithmic barrier function, which we denoted as f_μ^{Log}. The results obtained are similar to the ones obtained with the L2-norm merit function in terms of total number of iterations of the algorithm for each measure. Even for the cases where the algorithm did not converge using the L2-norm merit function, the same result was obtained using the logarithmic barrier function. However, we believe that this may not be the case for all problems or for all path following algorithms.

Table 1. Number of iterations using different proximity measures.

Problem	f^{L_2}	f^{S^2}	f^L	f^H	$f_\mu^{L_2}$	f_μ^S	f_μ^{Log}	β	σ^0
ISRAEL	68	64	58	68	68	68	64	0.01	0.01
”	64	59	67	61	64	64	61	0.25	0.01
”	68	56	67	68	54	54	53	10	0.01
SCSD1	NC*	25	23	NC	NC	NC	29	0.01	0.01
”	26	24	27	27	NC	NC	27	0.25	0.01
”	25	24	27	27	39	39	26	10	0.01
BLEND	44	32	31	44	44	44	32	0.01	0.001
”	NC	31	34	NC	NC	NC	30	0.25	0.001
”	35	29	34	36	34	34	29	10	0.001
”	34	31	29	34	34	34	32	0.01	0.01
”	32	29	27	32	32	33	29	0.25	0.01
”	41	34	31	41	41	41	36	0.01	0.1

*Nonconvergence

Table 2. Number of iterations using different proximity measures.

Problem	f^{L_2}	f^{S^2}	f^{L}	f^{H}	$f_\mu^{L_2}$	f_μ^{S}	f_μ^{Log}	β	σ^0
BLEND	36	33	28	35	37	37	33	0.25	0.1
”	29	28	28	29	28	28	28	10	0.1
”	72	60	45	66	98	98	74	0.25	0.5
”	47	45	45	47	47	47	45	10	0.5
SCTAP1	32	29	27	32	32	33	30	0.25	0.001
”	54	52	56	54	59	59	59	0.25	0.01
”	67	64	42	55	95	95	91	0.25	0.1
LOTFI	NC*	NC	45	NC	NC	NC	NC	0.01	0.01
”	92	62	48	88	92	95	63	0.25	0.001
”	96	66	47	92	96	99	64	0.25	0.01
”	106	76	53	102	106	109	76	0.25	0.1
”	96	66	47	92	96	99	49	10	0.01

*Nonconvergence

From Tables 1 and 2, we observe that, using the ℓ_1/ℓ_2 ratio measure, the algorithm requires less number of iterations to converge for most of the tested problems than using the other measures. Moreover, convergence using this measure is always attained. The logarithmic barrier measure is the best choice from the point-proximity measures and the performance of the algorithm is sometimes very similar to the one obtained using the ℓ_1/ℓ_2 ratio measure. Therefore, these two measures are the best choices among the considered proximity measures.

The difference in performance is related to the different values of the measures. From our numerical experimentation, we say that the values of the point-proximity measures are the largest from all of the measure values unless the iterate is very close to the central path. The ℓ_1/ℓ_2 ratio measure has the smallest value and it is always much smaller than the other measure values unless the iterate is very close to the central path set.

We also performed some tests including feasibility considerations in the proximity condition [condition (5)], the merit functions, and using a slightly different algorithm more adequate for nonlinear programming. For more details on these tests, we refer to Ref. 21. The experimentation showed that, although the total number of iterations changes, the relative performance among measures is preserved.

The parameters β and μ, or equivalently σ^0, play an important role in the behavior of the generic LSSN algorithm. Because of the difference in values of the proximity measures, the way of choosing the parameter β when defining the neighborhoods of the central path is not clear. In order to compare the behavior of the algorithm with respect to the different measures we need to choose the parameters such that the comparisons are as fair as possible. In the following section, we address this issue.

6.1. Normalization of the Proximity Measures. In an attempt to make the comparison among the measures less dependent to the choice of the neighborhood size β, we use Theorem 5.1. There, we found a description of the measures in terms of the angle between the vectors xz and e for any strictly feasible point (x, y, z). We denoted this angle by θ. We modified the measures so that locally they are of similar order, depending on this angle. Since the algorithm usually requires less number of iterations for convergence of the ℓ_1/ℓ_2 ratio measure, we modified the measures so that locally they behave as the ℓ_1/ℓ_2 ratio measure. This is equivalent to comparing the measures using different values of the size neighborhood in a fairer fashion.

The definition of the normalized measures was based in the description given by Theorem 5.1 and the performance of the algorithm. This is to say, from Theorem 5.1, the path-proximity measures f^{L_2}, f^H, f^L were easy to describe in terms of the angle θ. It is not the same case for the other measures, since the characterization in Theorem 5.1 depends also on other values. However, because of the relationships between the different measures and the performance of the LSSN algorithm we considered also a modification of the other measures. Because of the way the algorithm works, small values of θ correspond to small values of the distance between the vector xz and μe and this is what we used in the definition of the normalized measures. This fact (that does not have to be true in general) is true because of the way the parameter μ is chosen in the algorithm and because of the way in which the algorithm performs. For more details, see Ref. 12. We include Figure 2 to illustrate this comment. In this figure, if at iteration k of the algorithm we consider the corresponding vector xz, the vector $\hat{x}\hat{z}$ denotes the corresponding vector at iteration $k+1$. Observe that the distance between the vector $\hat{x}\hat{z}$ and the vector μe is always less than the distance between the vectors xz and μe when the angle θ reduces its value. In Theorem 5.1, this observation implies that, for small θ, $C1_\mu(w)$ and $C_3(w)$ are close to zero.

Table 3 shows the new proximity measures considered. For convenience, we refer to these measures as the normalized measures. We applied the generic LSSN algorithm using the normalized proximity measures on the same subset of NETLIB problems previously tested.

Table 4 shows the results obtained with the normalized proximity measures for some of the tested problems. Comparing the results in Table 4 with the ones in Tables 1 and 2, we observe that the proximity measures can be modified so that similar results are obtained in terms of

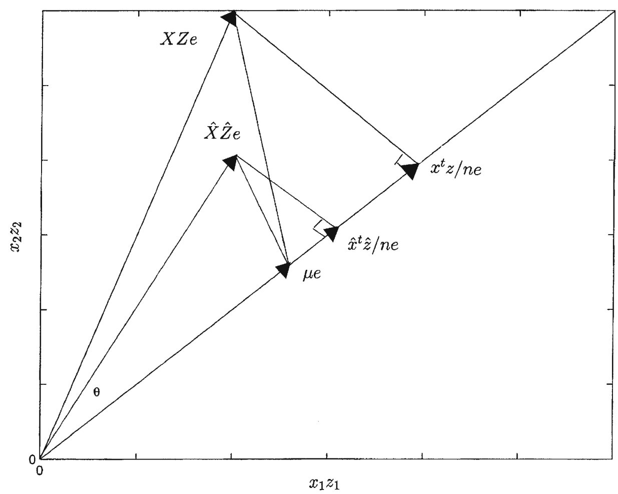

Fig. 2. Relationship of the consecutive vectors XZe and $\hat{X}\hat{Z}e$ generated by the algorithm.

Table 3. Normalized proximity measures.

$\tilde{f}^{L_2} = (f^{L_2})^2/2n$	$\tilde{f}^{S^2} = 2f^{S^2}/n$	$\tilde{f}^{L} = f^{L}$	$\tilde{f}^{H} = (f^{H})^2/2n$
$\tilde{f}_\mu^{L2} = f_\mu^{L2}/2n$	$\tilde{f}_\mu^{S} = f_\mu^{S}/2n$	$\tilde{f}_\mu^{Log} = \sqrt{f_\mu^{Log}}/2n$	

the number of iterations required for convergence. However, some of the measures can still be better than others in terms of the accuracy of the solution. This latter topic deserves further research.

Table 4. Number of iterations using normalized proximity measures.

Problem	f^{L_2}	f^{S^2}	f^L	f^H	$f_\mu^{L_2}$	f_μ^S	f_μ^{Log}	β	σ^0
ISRAEL	67	57	67	67	60	67	67	0.25	0.01
”	67	67	67	67	68	68	68	10	0.01
SCSD1	23	23	23	25	27	27	24	0.01	0.01
BLEND	33	29	34	34	28	31	31	0.25	0.001
”	45	45	45	45	46	46	46	0.25	0.5
SCTAP1	42	42	42	42	42	42	42	0.25	0.1
LOTFI	45	65	45	48	45	54	44	0.01	0.01
”	47	49	47	47	48	48	49	10	0.01

7. Comparison of Point-Proximity Measures

In this section, we consider the class of nonlinear scaled L_2-norm proximity measures defined as $\left\|(XZ)^{-q}(xz-\mu e)\right\|^2/\mu$, for $q \geq 0$. We observe that, for $q=1/2$, this is the nonlinearly scaled L_2-norm proximity measure (squared) considered in this paper. This is also the measure used in nonlinear programming problems; see Argaez and Tapia (Ref. 9). For $q=0$, this is the measure $\|xz-\mu e\|^2/\mu$ used in (general) nonlinear programming problems; see Ref. 11.

In Argaez et al. (Ref. 11), it is shown that the use of the latter proximity measure gives poorer results (in terms of total number of iterations) than the use of the former proximity measure for central-path following primal-dual methods for nonlinear programming problems. Our numerical results confirm what is presented in Ref. 11 (which is interesting, since the measure $\|xz-\mu e\|/\sqrt{\mu}$ is closely related to f_μ^{L2}), although we highlight that, in Ref. 11, different algorithms are used with the different proximity measures. In our case, the same algorithm (therefore, the same way of choosing the perturbation parameter μ) was used for comparing the measures.

Let us now denote

$$v=\sqrt{\mu/xz} \quad \text{and} \quad \phi(v_i)=(v_i^{2q-2}-v_i^{2q})^2.$$

Observe that $v_i=1$ for all $i=1,\dots,n$ if and only if if $xz=\mu e$. In this case, $\phi(v_i)=0$. A straightforward calculation shows that

$$\begin{aligned}\left\|(XZ)^{-q}(xz-\mu e)\right\|^2/\mu &= \mu^{(1-2q)}\left\|v^{2q-2}-v^{2q}\right\|^2\\ &= \mu^{(1-2q)}\sum_{i=1}^{i=n}(v_i^{2q-2}-v_i^{2q})^2\\ &= \mu^{(1-2q)}\sum_{i=1}^{i=n}\phi(v_i).\end{aligned}$$

In order to study this class of proximity measures and better understand the relationship among them, we use the next result which states some properties of the function ϕ.

Lemma 7.1. Let $q\in[0,1]$ and $\phi: R^+\to\{0\}\cup R^+$ defined as $\phi(v)=(v^{2q-2}-v^{2q})^2$. Then:

(i) We have that $\phi(1)=\phi'(1)=0$, $\phi''(v)>0$, and $\lim_{v\to 0}\phi''(v)=+\infty$ for $q\in[0,1)$. If $q=1$, then $\lim_{v\to 0}\phi''(v)=-4$.

(ii) If $q \in [0, 1/2)$, then $\lim_{v \to 0} \phi''(v) = 0$. If $q = 1/2$, then $\lim_{v \to \infty} \phi''(v) = 2$. If $q \in (1/2, 1]$, then $\lim_{v \to \infty} \phi''(v) = +\infty$.

Proof. In a straightforward computation, we obtain that

$$\begin{aligned}\phi''(v) &= 4((4q^2 - q)v^{4q-2} - (8q^2 - 10q + 3)v^{4q-4} + (4q^2 - 9q + 5)v^{4q-6}) \\ &= 4(a(q)v^{4q-2} + b(q)v^{4q-4} + c(q)v^{4q-6}),\end{aligned}$$

with

$$\begin{aligned}a(q) &= (4q - 1)q, \\ b(q) &= -8(q - 1/2)(q - 3/4), \\ c(q) &= 4(q - 1)(q - 5/4).\end{aligned}$$

If $q \in [0, 1)$, then,

$$\phi''(v) = 4v^{4q-6}((a(q)v^4 + b(q)v^2 + c(q)).$$

Since $4q - 6 < 0$ and $c(q) > 0$, then $\lim_{v \to 0} \phi''(v) = +\infty$. If $q = 1$, then $\phi''(v) = 4(3v^2 - 1)$ and we obtain part (i) of the lemma.

In order to prove part (ii), we use the same arguments. If $q \in [0, 1/2)$, then

$$4q - 6 < 4q - 4 < 4q - 2 < 0.$$

Then,

$$\lim_{v \to \infty} \phi''(v) = 0.$$

If $q = 1/2$, then a direct computation gives $\lim_{v_i \to \infty} \phi''(v) = 2$. If $q \in (1/2, 1)$, then

$$4q - 6 < 4q - 4 < 0 \quad \text{and} \quad 4q - 2 > 0.$$

Also, $a(q) > 0$. Then, $\lim_{v_i \to \infty} \phi''(v) = +\infty$. This is also true if $q = 1$, since we have

$$\phi''(v) = 4(3v^2 - 1).$$

This completes the proof. □

Therefore, the lemma is saying that the nonlinearly scaled L_2-norm measure (squared) is the one, from the q-class previously considered, with the smallest value of $q \in [0, 1]$ such that $\phi''(v_i)$ is bounded away from zero when $x_i z_i$ is approaching zero for some i. This property may explain the better performance for $q = 1/2$ when comparing with $q = 0$. However, if greater values of $q > 1/2$ are used, the best performance is also found for $q = 1/2$. This may be explained by the observation that $q = 1/2$ is the value that gives a balance in the exponents of the variables v_i.

We conclude this section by mentioning that the nonlinearly scaled L_2-norm naturally arises in the definition of primal-dual interior-point methods. See Peng, Roos, and Terlaky (Ref. 3) for more details. Based on this measure, a new class of proximity measures $f_\mu^{Sq} = \|w^{-q} - w\|$ for $q \geq 1$ with $W = \sqrt{XZ/\mu}$ has been proposed as well as a new class of polynomial primal-dual methods.

8. Conclusions

In this paper, we considered different proximity measures and studied their effect on the computation of central path points. It can be seen that the choice of the proximity measure impacts strongly the numerical behavior of primal-dual interior-point algorithms following the central path.

A generic algorithm for computing the analytic center, based on the LSSN algorithm proposed in Ref. 12, was introduced and used as a tool to study the differences between the measures.

We defined two different types of proximity measures, the point-proximity measure and path-proximity measure. Except the L_2-norm measure and the ℓ_1/ℓ_2 ratio proximity measure, all the other measures considered satisfy the so-called barrier property. Our numerical experimentation shows that this property is not strictly necessary in order to obtain a good numerical behavior of the algorithm.

In terms of the best choice of measure from our experimentation we can conclude that the logarithmic barrier function is the most adequate when using point-proximity measures, and the ℓ_1/ℓ_2 ratio measure is the best option when the closeness to the central path is more important than the closeness to a specific central point. However, it is clear that this conclusion relies on the choice considered of the neighborhood size. Then, we presented in this paper an attempt to relate the measures and normalize them, so that similar numerical behavior is obtained for the algorithm.

Finally, we pointed out some aspects and trends (as nonlinear programming or long-step algorithms) where the use and study of the proximity measures is relevant.

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