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Local Convexity on Smooth Manifolds[1,2]

T. RAPCSÁK[3]

Abstract. Some properties of the spaces of paths are studied in order to define and characterize the local convexity of sets belonging to smooth manifolds and the local convexity of functions defined on local convex sets of smooth manifolds.

Key Words. Smooth manifolds, space of paths, linear connections, local convexity.

1. Introduction

A convex function has convex less-equal level sets. That the converse is not true was realized by De Finetti (Ref. 1.). The problem of level sets, formulated and discussed first by Fenchel in 1951, is as follows (see Ref. 2, p. 117): Under what conditions a nested family of closed convex sets is the family of level sets of a convex function?

Fenchel (Refs. 2–3) gave necessary and sufficient conditions for a convex function with prescribed level sets, furthermore, for a smooth convex function under the additional assumption that the given subsets are the level sets of a twice continuously differentiable function. In the first case, seven conditions were deduced; while the first six are simple and intuitive, the seventh is rather complicated. This fact and the additional assumption in the smooth case, according to which the given subsets are the level sets of a twice continuously differentiable function, seem to be the motivation of Roberts and Varberg (Ref. 4, p. 271) is raising the following question of level sets among some unsolved problems:"What nice conditions on a nested family of convex sets will ensure that it is the family of level sets of a convex function?"

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[3]Professor, Computer and Automation Research Institute, Hungarian Academy of Sciences, Budapest, Hungary.

The Fenchel problem of level sets consists of different subproblems. If the union of the level sets $A \subseteq R^n$ is a convex set, then a quasiconvex function can be constructed with the prescribed level sets (Fenchel, Ref. 2), so the original question now reads as finding the conditions under which the level sets of a quasiconvex function are those of a convex function. In the case of a convex set $A \subseteq R^n$ and a continuous quasiconvex function, the question is to characterize the convex image transformable functions.

In the theory of economics, Debreu (Ref. 5) proved his famous theorem on the representation of a continuous and complete preference ordering by a utility function. In economics, it is important to express a continuous, complete, and convex preference ordering by a concave utility function, or in other words to transform a continuous quasiconcave function into a concave function preserving the same upper-level set mapping. Crouzeix (Ref. 6) and Kannai (Refs. 7–8) studied the problem of concavifiability of convex preference orderings (i.e., the problem of the existence of a concave function having the same level sets as a given continuous quasiconcave function) and they developed the Fenchel results further.

In the smooth case, the original problem is divided into two parts. The first part is to give conditions for the existence of a smooth pseudoconvex function with the prescribed level sets; the second part is to characterize the smooth convex image transformable functions.

Rapcsák (Ref. 9) gave an explicit formulation of the gradient of the class of smooth pseudolinear functions, which results in the solution of the first part of the Fenchel problem in the case of a nested family of convex sets whose boundaries are hyperplanes defining an open convex set. This result was generalized by Rapcsák (Refs. 10–11) for the case where the boundaries of the nested family of convex sets in R^{n+1} are given by n-dimensional differentiable manifolds of class C^3 and the convex sets determine an open or closed convex set in R^{n+1}.

A first complete set of necessary and sufficient conditions for the second part of the level set problem was derived by Fenchel (Refs. 2–3). Later, several contributions were published by different authors and these results are presented within a unified framework in the book of Avriel et al. (Ref. 12).

In this paper and in a companion paper (Ref. 20), a new and nice geometric necessary and sufficient condition is given for the existence of a smooth convex function with the level sets of a given smooth pseudoconvex function, this is a new solution for the second part of the Fenchel problem of level sets in the smooth case. The main theorem is proved by using a general differential geometric tool, the geometry of paths defined on smooth manifolds which is the subject of this paper. This approach provides a complete geometric characterization of a new subclass of pseudoconvex functions originating from analytical mechanics, an extension of the local-global property of nonlinear optimization to nonconvex open sets, a powerful tool (a linear connection which does not depend on either the original data or the Riemannian metric) to improve the structure of a function or a problem from the optimization point of view, and a new view on convexlike and generalized convexlike mappings in the image analysis; see e.g. Giannessi (Refs. 13–14), and Mastroeni et al. (Ref. 15).

In Section 2, some properties of the spaces of paths are studied in order to define and characterize the local convexity of sets belonging to smooth manifolds and the local convexity of functions defined on the local convex sets of smooth manifolds.

2. Space of Paths

Let M be a smooth C^2 n-dimensional connected manifold and let m be a point in M. The tangent space TM_m at m is an n-dimensional vector space. A 2-covariant tensor at m is a real-valued 2-linear function on $TM_m \times TM_m$. A 2-covariant tensor is positive semidefinite [definite] at a point $m \in M$ if the corresponding matrix is positive semidefinite [definite] on $TM_m \times TM_m$ in any coordinate representation. A 2-covariant tensor field is positive semidefinite [definite] on a set $A \subseteq M$ if it is a positive semidefinite [definite] tensor at every point of A. A path γ on M is a smooth mapping $\gamma : [0, 1] \to M$. The space of paths is based on the differentiable structure of the manifold.

Definition 2.1. The mapping Γ is a linear connection on an open subset A of M under these conditions:

(i) a set of n^3 smooth (at least continuously differentiable) functions

$$\Gamma^{l_3}_{l_1 l_2}, \quad l_1, l_2, l_3 = 1, \dots, n,$$

is given in every system of local coordinates on A;

(ii) the sets of functions $\Gamma^{l_3}_{l_1 l_2}$ and $\overline{\Gamma}^{l_3}_{l_1 l_2}$, $l_1, l_2, l_3 = 1, \dots, n$, given in a coordinate representation x and u, respectively, are transformed by the rule

$$\Gamma^{l_3}_{l_1 l_2} = \overline{\Gamma}^{k_3}_{k_1 k_2} \frac{\partial x_{l_3}}{\partial u_{k_3}} \frac{\partial u_{k_1}}{\partial x_{l_1}} \frac{\partial u_{k_2}}{\partial x_{l_2}} + \frac{\partial^2 u_{l_3}}{\partial x_{l_1} \partial x_{l_2}} \frac{\partial x_{l_3}}{\partial u_{k3}},$$

$$\text{for all } l_1, l_2, l_3, k_1, k_2, k_3 = 1, \dots, n, \tag{1}$$

where two coinciding indices mean summation.

Definition 2.2. A Γ-geodesics, i.e. a geodesics of a linear connection Γ on an open set $A \subseteq M$, is a path such that each coordinate expression of it satisfies the differential equations

$$x''_{l_3}(t) + \Gamma^{l_3}_{l_1 l_2}(x(t))x'_{l_1}(t)x'_{l_2}(t) = 0, \quad t \in [0, 1], \ l_3 = 1, \ldots, n. \tag{2}$$

Let

$$\Gamma = \begin{pmatrix} \Gamma^1 \\ \Gamma^2 \\ \vdots \\ \Gamma^n \end{pmatrix},$$

where $\Gamma^i, i = 1, \ldots, n$, are $n \times n$ matrices. Then, the Γ-geodesics can be given in a coordinate neighborhood as follows:

$$x''(t) = -x'(t)^T \Gamma(x(t))x'(t), \quad t \in [0, 1]. \tag{3}$$

By the theory of differential equations, equation (3) has a solution at every point and in every direction. Let $R^n, R, R_+, R_\geq$ be the n-dimensional Euclidean space, the 1-dimensional Euclidean space consisting of real numbers, positive real numbers, and nonnegative real numbers, respectively.

Definition 2.3. Let Γ be a linear connection on an open subset $A \subseteq M$. Then, A is Γ-convex if, for all $m_1, m_2 \in A$, there exists a Γ-geodesics γ such that $\gamma(0) = m_1, \gamma(1) = m_2$, and $\gamma \subseteq A$.

A function $f : A \to R$ is strictly Γ-convex on a Γ-convex set A if it is (strictly) convex along all the Γ-geodesics belonging to A.

By the definition, the following inequalities hold for every Γ-geodesic belonging to A and joining two arbitrary points $m_1, m_2 \in A$:

$$f(\gamma(t)) \leq (1-t)f(\gamma(0)) + tf(\gamma(1)), \quad t \in [0, 1], \tag{4}$$

where $\gamma(0) = m_1, \ \gamma(1) = m_2$.

If $\Gamma^{l_3}_{l_1 l_2} = 0, l_1, l_2, l_3 = 1, \ldots, n$, then the Γ-convex set $A \subseteq R^n$ is a convex set and the Γ-convex function $f : A \to R$ is a convex function on A, where

$$\gamma(t) = m_1 + t(m_2 - m_1), \quad t \in [0, 1]. \tag{5}$$

A function $f : A \to R$ defined on a Γ-convex set $A \subseteq M$ is strictly Γ-concave if $-f$ is (strictly) Γ-convex.

Geodesic convexity derived from Riemannian metrics on Riemannian manifolds was studied in details from the differential geometric point of view in Udriste (Ref. 16) and from optimization theoretical point of view in Rapcsák (Ref. 11). Here, these approaches are developed further. The following statements are direct generalization of the results related to geodesic convexity.

Lemma 2.1.

(i) If $A \subseteq M$ is a Γ-convex set and if $g_i : A \to R$, $i = 1, \dots, l$, are Γ-convex functions, then the intersection of the level sets

$$\bigcap_{i=1}^{l} A_{gi}(m_0) = \{m \in A | g_i(m) \leq g_i(m_0), m_0 \in A\}, \quad i = 1, \dots, l, \tag{6}$$

is a Γ-convex set.

(ii) If $A \subseteq M$ is a Γ-convex set and if $g_i : A \to R$, $i = 1, \dots, l$, are Γ-convex functions, then the nonnegative linear combinations of Γ-convex functions are Γ-convex on A.

(iii) If $A \subseteq M$ is a Γ-convex set, if $f : A \to R$ a Γ-convex function, and if $\phi : R \to R$ a nondecreasing convex function, then ϕf is Γ-convex on A.

Theorem 2.1. If $A \subseteq M$ is a Γ-convex set and if $f : A \to R$ a Γ-convex function, then a local minimum of f is a global minimum.

Definition 2.4. Let Γ be a linear connection on an open subset $A \subseteq M$. Then A is locally Γ-convex if a neighborhood $U(x)$ of every point $x \in A$ exists such that all the pairs of points $m_1, m_2 \in U(x)$ can be joined by a unique Γ-geodesic belonging to $U(x)$; i.e.,

$$\gamma(0) = m_1, \quad \gamma(1) = m_2, \quad \gamma \subseteq U(x).$$

A function $f : A \to R$ is locally (strictly) Γ-convex on A if A is a locally Γ-convex set and f is (strictly) convex along all the Γ-geodesics belonging to a Γ-convex neighborhood of every point of A.

The next statement demonstrates the importance of Definition 2.4.

Theorem 2.2. Whitehead Theorem (Ref. 17). Let $W(M; \Gamma)$ be the set of all geodesics of some linear connection Γ on a smooth n-dimensional manifold M. Then, M is locally convex with respect to $W(M; \Gamma)$; i.e., M is locally Γ-convex.

Theorem 2.3. If $A \subseteq M$ is an open set, if $f : A \to R$ is a differentiable function, and if Γ is a linear connection on A, then f is locally (strictly) Γ-convex on A iff, for every pair of points $m_1 \in A$, $m_2 \in A$ in any Γ-convex neighborhood and for a connecting geodesics $\gamma(t), t \in [0, 1]$, $\gamma(0) = m_1, \gamma(1) = m_2$,

$$f(m_2) - f(m_1)(>) \geq df(m_1)/dt, \tag{7}$$

where $df(m_1)/dt$ denotes the derivative of $df(\gamma(t))/dt$ at the point 0.

Proof. By Definition 2.4, a function f is locally (strictly) Γ-convex on A if A is a locally Γ-convex set and if f is (strictly) convex along all the Γ-geodesics belonging to a Γ-convex neighborhood of every point of A. Since A is an open set, by the Whitehead theorem A is locally Γ-convex. Thus, it is sufficient to verify the statement only in an arbitrary Γ-convex neighborhood.

The local (strict) Γ-convexity of f in a Γ-convex neighborhood means the (strict) convexity of the single-variable function along the connecting Γ-geodesics for every pair of points in this Γ-convex neighborhood. By the first-order characterization, a differentiable function $f(\gamma(t)), t \in [0, 1]$, is (strictly) convex iff formula (7) holds, from which the statement follows. □

By formula (7), the local (strict) Γ-convexity of the function f is equivalent to the local Γ-invexity (Ref. 18); moreover, in the case of every pair of points $(m_1, m_2) \in A \times A$, the invexity map satisfies $\eta(m_1, m_2) \in TM_{m_1}$ and is equal to the tangent vector at m_1 of the Γ-geodesics joining the points m_1 and m_2.

Definition 2.5. A point m of the n-dimensional manifold M is said to be a critical (stationary) point of the smooth map $f : M \to R$ if the derivative of the function f at that point is equal to zero.

Corollary 2.1. Let $A \subseteq M$ be an open Γ-convex set and let $f : A \to R$ be a differentiable (strictly) Γ-convex function. Then, every stationary point of f is a (strict) global minimum point. Moreover, the set of global minimum points is Γ-convex.

Monotonicity notions studied for geodesic convex functions by Udriste (Ref. 16) can be applied directly to Γ-convex functions.

Definition 2.6. Let $A \subseteq M$ be an open set, and let $f : A \to R$ be a differentiable function. Then df/dt is locally (strictly) Γ-monotone on A

if, for every pair of points $m_1 \in A, m_2 \in A$, in any Γ-convex neighborhood, and for a connecting geodesics $\gamma(t), 0 \le t \le 1, \gamma(0) = m_1, \gamma(1) = m_2$,

$$df(m_1)/dt - df(m_2)/dt (<) \le 0. \tag{8}$$

If $M = R^n$ or if we consider a coordinate representation, then the (strict) Γ-monotonicity of df/dt means that

$$\nabla f(m_1)\gamma'(0) - \nabla f(m_2)\gamma'(1) (<) \le 0, \tag{9}$$

where the row vector ∇f denotes the gradient of the function f.

Theorem 2.4. Let $A \subseteq M$ be an open set and let $f : A \to R$ be a differentiable function. Then, f is locally (strictly) Γ-convex on A iff df/dt is locally (strictly) Γ-monotone on A.

Let us introduce the notation

$$V^T \Gamma = \sum_{i=1}^{n} v_i \Gamma^i, \tag{10}$$

where the vector $V = (v_1, \dots, v_n)^T$ belongs to an n-dimensional vector space; let $C^2(A, R)$ denote the set of all twice continuously differentiable functions of A into R.

Definition 2.7. The covariant derivative of a smooth function on the manifold M is equal to the derivative in any coordinate representation. A vector field V is defined on the manifold M as a smooth map $V : M \to R^n$ such that $V(m) \in TM_m$ for all $m \in M$. The covariant derivative with respect to a linear connection Γ on a covariant vector field V is equal to

$$D_\Gamma V = JV - V^T \Gamma, \tag{11}$$

in any coordinate representation, where JV denotes the Jacobian matrix of the corresponding vector field at each point of an arbitrary coordinate neighborhood.

Theorem 2.5. If $A \subseteq M$ is an open set, if $f \in C^2(A, R)$, and if Γ is a linear connection on A, then f is locally (strictly) Γ-convex on A iff the second covariant derivative tensor field $D^2_\Gamma f$ of the function f with respect to Γ is (strictly) positive semidefinite on A.

Proof. By Definition 2.4, a function f is locally (strictly) Γ-convex on A if A is a locally Γ-convex set and f is (strictly) convex along all the Γ-geodesics belonging to a Γ-convex neighborhood of every point of A. Since A is an open set, by the Whitehead theorem A is locally Γ-convex. Thus, it, is sufficient to verify the statement only in an arbitrary Γ-convex neighborhood.

Consider an arbitrary coordinate representation $x(u), (u) \in U \subseteq R^n$, of the manifold M in any Γ-convex neighborhood U of A. Then, a Γ-geodesics joining two arbitrary points in this neighborhood can be given in the form $\mathrm{x}(u(t)), t \in [0, 1]$. Since all the geodesics joining two arbitrary points in this neighborhood can be extended to an open interval (t_1, t_2), the Γ-convexity of the single variable C^2 function $f(x(u(t))), t \in (t_1, t_2)$, is equivalent to the nonnegativeness of the second derivative at every point.

By differentiating twice the function $f(x(u(t))), t \in (t_1, t_2)$, we obtain that

$$(d/dt) f(x(u(t))) = \nabla_x f(x(u(t))) Jx(u(t)) u'(t), \tag{12a}$$

$$\begin{aligned}(d^2/dt^2) f(x(u(t))) =& u'(t)^T Jx(u(t))^T H_x f(x(u(t))) J_x(u(t)) u'(t) \\ &+ \nabla_x f(x(u(t))) \left(u'(t)^T H_u x(u(t)) u'(t)\right) \\ &+ \nabla_x f(x(u(t))) J_x(u(t)) u''(t).\end{aligned} \tag{12b}$$

As the curve $x(u(t)), t \in (t_1, t_2)$, is a geodesics, we can substitute the following system of differential equations for $u''(t)$:

$$u''(t) = -u'(t)^T \Gamma(u(t)) u'(t), \quad t \in (t_1, t_2), \tag{13}$$

where the $n \times n \times n$ matrix Γ contains the second Christoffel symbols and $u'(t), t \in (t_1, t_2)$, are the tangent vectors. Considering only geodesics at each point and in every direction, we obtain the geodesics Hessian matrix

$$\begin{aligned}H_u^g f(x(u)) =& Jx(u)^T H_x f(x(u)) Jx(u) + \nabla_x f(x(u)) Hx(u) \\ &- \nabla_x f(x(u)) Jx(u) \Gamma(u), \quad u \in U \subseteq R^n,\end{aligned} \tag{14}$$

where the matrix multiplication $Jx(u)\Gamma(u), u \in U \subseteq R^n$, is defined by the rule related to the multiplication of a row vector and a 3-dimensional matrix, applied consecutively for every row vector of $Jx(u)$; see formula (10). Note that the result does not depend on the order of the multiplication in the term

$$\nabla_x f(x(u)) Jx(u) \Gamma(u), \quad u \in U \subseteq R^n.$$

As the gradient $\nabla_u f(x(u))$ is equivalent to the expression

$$D_\Gamma f(x(u) = \nabla_x f(x(u)) Jx(u), \quad u \in U \subseteq R^n, \tag{15}$$

in any coordinate representation, where $D_\Gamma f$ denotes the first covariant derivative of f with respect to Γ, which is a covariant vector field; on a covariant vector field V, the covariant derivative with respect to Γ is equal to

$$D_\Gamma V = JV - V^T \Gamma,$$

where JV denotes the Jacobian matrix of the corresponding vector field at each point of an arbitrary coordinate neighborhood; the right-hand side of the expression (14) is exactly the second covariant derivative of $f(x(u)), u \in U \subseteq R^n$, with respect to Γ, i.e.,

$$\begin{aligned} D^2_\Gamma f(x(u)) = & Jx(u)^T H_x f(x(u)) Jx(u) + \nabla_x f(x(u)) Hx(u) \\ & - \nabla_x f(x(u)) Jx(u) \Gamma(u), \quad u \in U \subseteq R^n. \end{aligned} \tag{16}$$

From the smoothness property of the function and the manifold, as well as from the Whitehead theorem, it follows that f is locally (strictly) Γ-convex on A iff the second covariant derivatives $D^2_\Gamma f$ are positive semidefinite in every Γ-convex neighborhood. By considering the fact that $D^2_\Gamma f$ is a tensor at every point, we obtain the statement. □

Corollary 2.2. The Γ-convexity property of sets and functions defined on a differentiable manifold is invariant under regular nonlinear coordinate transformations.

We remark that Theorems 2.1 and 2.3–2.5 are derived from the corresponding statements in Rapcsák (Ref. 11); while the second covariant, derivative is a tensor field on M, Γ is not.

Corollary 2.3. If $A \subseteq R^n$ is an open set, if $f \in C^2(A, R)$, and if $\Gamma(x), x \in A$, is a continuously differentiable linear connection, then a locally (strictly) Γ-convex function f is (strictly) Γ-convex on A iff the set A is Γ-convex.

In Corollary 2.3, the local-global property of the C^2 function f on A is stated, which can be proved directly for continuous functions following the proof of Theorem 6.1.2 in Rapcsák (Ref. 11). The local-global property for pseudoconvex functions was stated first by Komlósi (Ref. 19).

Pini (Ref. 18) investigated invexity on manifolds and the relationship, based on the integrability of the invexity map, between convexity along curves on a manifold and invexity. This corollary and Theorem 2.2 show that the local-global property, the invexity, and the integrability of the invexity map are equivalent in this framework; therefore, results based on the latter two notions might be considered the reformulation of the original problem without constructing at least one new family of curves satisfying the assumptions.

Corollary 2.4. If $A \subseteq R^n$ is an open set $f \in C^2(A, R)$, and $\Gamma(x), x \in A$, is a continuously differentiable linear connection, then f is locally (strictly) Γ-convex on A iff the matrices

$$D^2_\Gamma f(x) = Hf(x) - \nabla f(x)\Gamma(x), \quad x \in A, \tag{17}$$

are (strictly) positive semidefinite.

Proof. If $M = R^n$, then there exists a coordinate representation of R^n such that $J\mathrm{x}(u) = I_n$ (the identity matrix in R^n) and $Hx(u)$ is equal to the null matrix for all $u \in R^n$; thus, the formula of $D^2_\Gamma f$ derives from (16). □

Remark 2.1. Let $A \subseteq R^n$ be an open subset. The linear connection $\Gamma^{l_3}_{l_1,l_2}, l_1, l_2, l_3 = 1, \dots, n$ is locally Lipschitz if each point of A has a neighborhood such that a constant K exists satisfying

$$\left|\Gamma^{l_3}_{l_1,l_2}(x_1) - \Gamma^{l_3}_{l_1,l_2}(x_2)\right| \le K \|x_1 - x_2\|, \quad l_1, l_2, l_3 = 1, \dots, n,$$

for all pairs x_1, x_2 in this neighborhood, where the symbol $\|\cdot\|$ denotes the Euclidean norm.

By the theory of differential equations, the local Γ-convexity of the set A can be obtained by the local Lipschitz property of the linear connection which may substitute the continuous differentiability of the linear connection in Corollary 2.4.

Corollary 2.4 results directly in a condition for the local Γ-convexity of a smooth function.

Corollary 2.5. If $A \subseteq R^n$ is an open set, if $f \in C^2(A, R)$, and if $\partial f/\partial x_i \neq 0$, $i = 1, \dots k$, $\partial f/\partial x_i = 0, i = k+1, \dots, n$, for some $0 \le k \le n$ on A then f is locally strictly Γ-convex on A with respect to

$$\Gamma^i(x) = [1/(k\partial f(x)/\partial x_i)]\,(Hf(x) - I_n), \quad i = 1, \dots, k, \quad x \in A, \tag{18a}$$

$$\Gamma^i(x) = I_n, \quad i = k+1, \dots, n, \quad x \in A. \tag{18b}$$

3. Concluding Remarks

The local convexity of sets based on the space of paths belonging to smooth manifolds is investigated; then, the local convexity of functions defined on local convex sets is introduced and characterized by using the Whitehead theorem (Ref. 17). A characterization like this may be useful not only in optimization theory, but in the image analysis of optimization theory, the main principles of which were established by Giannessi in 1984 (Ref. 13). Theorems 2.3, 2.4, 2.5 provide some results to form a new subclass of locally convexlike mappings consisting of a finite number of locally Γ-concave functions with the same linear connection Γ. We remark that locally convexlike mappings like these can be constructed by given linear connections; see e.g. Corollary 2.4. By Corollary 2.3, a mapping belonging to this subclass is convexlike iff the set $A \subseteq M$ is Γ-convex with the same linear connection. It follows that the notion of convexlike mapping defined in the image space is in connection with the given space and the local-global property.

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