JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 127, No. 1, pp. 27–44, October 2005 (© 2005)
DOI: 10.1007/s10957-005-6391-6

Systems of Simultaneous Generalized Vector Quasiequilibrium Problems and their Applications[1]

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Communicated by H. P. Benson

Abstract. In this paper, we introduce systems of simultaneous generalized vector equilibrium problems and prove the existence of their solutions. As application of our results, we derive the existence theorems for solutions of systems of vector saddle–point problems. Consequently, we prove the existence of a solution of systems of generalized minimax inequalities. Further application of our results is also given to establish the existence of a solution of a Debreu-type equilibrium problem for vector-valued functions.

Key Words. Systems of simultaneous generalized vector equilibrium problems, systems of vector saddle point problems, systems of minimax inequalities, Debreu-type equilibrium problems, properly convex functions.

1. Introduction and Formulations

In 1994, Husain and Tarafdar (Ref. 1) introduced simultaneous variational inequalities and gave some applications to the minimization problems. These were further studied by Fu (Ref. 2) for the vector-valued case

[1]The first author thanks the Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia for providing excellent research facilities. The second and third authors were supported by the National Science Council of the Republic of China. The authors are grateful to the referees for valuable suggestions and comments.

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with applications to vector complementarity problems. Very recently, Lin (Ref. 3) considered and studied simultaneous vector equilibrium problems and proved existence results for their solution. By using these results, he derived existence results for the solution of a vector saddle-point problem.

In the recent past, systems of scalar (vector) equilibrium problems and scalar (vector) generalized equilibrium problems were used as tools to solve Nash equilibrium problems for vector-valued functions and Debreu-type equilibrium problems for vector-valued functions; see for example Refs. 4–10 and references therein.

Because of the applications to vector optimization, game theory, and economics, the saddle-point problem for vector-valued functions emerged as a new direction for the researchers; see for example Refs. 10–12 and references therein.

In this paper, we consider systems of simultaneous generalized vector equilibrium problems (SSGVQEP), which contains generalized vector equilibrium problems (Ref. 13), systems of vector equilibrium problems (Ref. 4), systems of generalized vector variational-like inequalities (Ref. 5), and simultaneous vector variational inequalities (Ref. 2) as special cases. By using the Kakutani fixed-point theorem, we establish an existence result for solutions of (SSGVQEP). In Section 3, we derive several existence results for solutions of the above mentioned problems. These existence results either improve or extend known results in the literature. We consider also systems of vector saddle-point problems (SVQSPP) and systems of minimax inequalities (SQMI). In Section 4, as applications of our existence result for solutions of (SSGVQEP), we prove existence of solutions of (SVQSPP) and (SQMI). In Section 5, we give another application of our results to establish the existence of a solution of a Debreu-type equilibrium problem, also known as constrained Nash equilibrium problem.

Let A be a nonempty subset of a topological vector space (t.v.s.) $\mathcal{X}$; we denote by in A, $\bar{A}$, and $\overline{\text{co}}$A, the interior of A in $\mathcal{X}$, the closure of A in $\mathcal{X}$, and the closed convex hull of A, respectively. The family of all subsets of A is denoted by 2^A.

Let $\mathcal{Z}$ be a t.v.s. and let P be a closed convex cone in $\mathcal{Z}$ with $P \neq \emptyset$. Then, P induces the vector ordering in $\mathcal{Z}$ setting, $\forall x, y \in P$,

$$x \leq_P y \Leftrightarrow y - x \in P,$$
$$x \nleq_P y \Leftrightarrow y - x \notin P.$$

Since int $P \neq \emptyset$, we also have the weak ordering in $\mathcal{Z}$ setting, $\forall x, y \in P$,

$$x \leq_P y \Leftrightarrow y - x \in \text{int } P,$$
$$x \nless_P y \Leftrightarrow y - x \notin \text{int} P.$$

The ordering $\geq_P$, $\ngeq_P$, $>_P$, $\ngtr_P$ are defined similarly. A cone P is called pointed if $P \cap (-P) = \{0\}$.

Throughout the paper, unless specified otherwise, I is any index set (finite or infinite). For each $i \in I$, let X_i and Y_i be two nonempty convex subsets of locally convex t.v.s. E_i and F_i, respectively; let Z_i be a real t.v.s. Let

$$X = \prod_{i \in I} X_i \quad \text{and} \quad Y = \prod_{i \in I} Y_i.$$

For each $i \in I$, let $C_i : X \to 2^{Z_i}$ be a multivalued map such that, for all $x \in X, C_i(x)$ is a closed convex cone with apex at the origin. For each $i \in I$, let

$$P_i = \bigcap_{x \in X} C_i(x)$$

such that P_i defines a vector ordering on Z_i. For each $i \in I$, let $S_i : X \to 2^{X_i}$ and $T_i : X \to 2^{Y_i}$ be multivalued maps with nonempty values; let $f_i : X \times Y \times X_i \to Z_i$ and $g_i : X \times Y \times Y_i \to Z_i$ be trifunctions. We consider the following problems of systems of simultaneous generalized vector equilibrium problems (SSGVQEP) in three forms:

(SSGVQEP)(I) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that, for each $i \in I, \bar{x}_i \in S_i(\bar{x}), \bar{y}_i \in T_i(\bar{x})$,

$$f_i(\bar{x}, \bar{y}, x_i) \in C_i(\bar{x}), \quad \forall x_i \in S_i(\bar{x}),$$
$$g_i(\bar{x}, \bar{y}, y_i) \in C_i(\bar{x}), \quad \forall y_i \in T_i(\bar{x}).$$

(SSGVQEP)(II) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that, for each $i \in I, \bar{x}_i \in S_i(\bar{x}), \bar{y}_i \in T_i(\bar{x})$,

$$f_i(\bar{x}, \bar{y}, x_i) \notin -C_i(\bar{x}) \backslash \{0\}, \quad \forall x_i \in S_i(\bar{x}),$$
$$g_i(\bar{x}, \bar{y}, y_i) \notin -C_i(\bar{x}) \backslash \{0\}, \quad \forall y_i \in T_i(\bar{x}).$$

(SSGVQEP)(III) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that, for each $i \in I, \bar{x}_i \in S_i(\bar{x}), \bar{y}_i \in T_i(\bar{x})$,

$$f_i(\bar{x}, \bar{y}, x_i) \notin -\text{int } C_i(\bar{x}), \quad \forall x_i \in S_i(\bar{x}),$$
$$g_i(\bar{x}, \bar{y}, y_i) \notin -\text{int } C_i(\bar{x}), \quad \forall y_i \in T_i(\bar{x});$$

in this case, we assume that int C_i is nonempty for each $i \in I$.

Remark 1.1. For each $i \in I$ and for all $x \in X$, let $C_i(x)$ be a pointed cone and let $P_i = \cap_{x \in X} C_i(x)$; it is easy to see that P_i is also pointed.

Remark 1.2. For each $i \in I$ and for all $x \in X$, if $C_i(x)$ is also pointed, then every solution of (SSGVQEP)(I) is a solution of (SSGVQEP)(II) and every solution of (SSGVQEP)(II) is a solution of (SSGVQEP)(III). But the reverse implication does not hold.

Indeed, let $(\bar{x}, \bar{y}) \in X \times Y$ be a solution of (SSGVQEP) (I); then for each $i \in I, \bar{x}_i \in S_i(\bar{x}), \bar{y}_i \in T_i(\bar{x})$,

$$f_i(\bar{x}, \bar{y}, x_i) \in C_i(\bar{x}), \quad \forall x_i \in S_i(\bar{x}),$$
$$g_i(\bar{x}, \bar{y}, y_i) \in C_i(\bar{x}), \quad \forall y_i \in T_i(\bar{x}).$$

Since for each $i \in I$ and $\forall x \in X$, $C_i(x)$ is a pointed cone, we have

$$C_i(x) \cap (-C_i(x)) = \{0\};$$

therefore,

$$C_i(x) \cap (-C_i(x) \backslash \{0\}) = \emptyset.$$

Hence,

$$f_i(\bar{x}, \bar{y}, x_i) \notin -C_i(\bar{x}) \backslash \{0\}, \quad \forall x_i \in S_i(\bar{x}),$$
$$g_i(\bar{x}, \bar{y}, y_i) \notin -C_i(\bar{x}) \backslash \{0\}, \quad \forall y_i \in T_i(\bar{x}).$$

The second statement follows from the fact that

$$-\text{int } C_i(x) \subseteq -C_i(x) \backslash \{0\}, \quad \forall x \in X \text{ and for each } i \in I.$$

For each $i \in I$, we denote by $L(E_i, Z_i)$ the space of all continuous linear operators from E_i into Z_i and let Y_i be a nonempty subset of $L(E_i, Z_i)$. For each $i \in I$, let $g_i = 0$. Then, (SSGVQEP)(I) reduces to the following systems of generalized implicit vector variational inequalities problem:

(SGIVQVIP)(I) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that, for each $i \in I, \bar{x}_i \in S_i(\bar{x}), \bar{y}_i \in T_i(\bar{x})$ satisfying

$$f_i(\bar{x}, \bar{y}, x_i) \in C_i(\bar{x}), \quad \forall x_i \in S_i(\bar{x}).$$

Analogously, we can define (SGIVQVIP)(II) and (SGIVQVIP)(III) problems of systems of generalized implicit vector quasivariational inequalities corresponding to (SSGVQEP)(II) and (SSGVQEP)(III), respectively.

The (SGIVQVIP) contains the problem of systems of generalized vector variational like inequalities (SGVQVLIP) as a special case. Recently, the weak formulation of (SGVQVLIP) was introduced and studied by Ansari and Khan (Ref. 5). They used (SGVQVLIP) as a tool to prove the existence of a solution of the Debreu-type equilibrium problem for non-differentiable and nonconvex vector-valued functions.

When for each $i \in I$, $X_i = Y_i$, $S_i \equiv T_i$, and $f_i \equiv g_i$, then (SSGVQEP) is called system of vector equilibrium problems. In this case, (SSGVQEP)(III) was considered and studied by Ansari et al (Ref. 4) for $f_i(x, y, y_i) = h_i(x, y_i)$. They gave further applications to systems of generalized vector variational-like inequalities and Debreu-type equilibrium problems for vector-valued functions.

When I is a singleton set and $g_i \equiv 0$, then (SSGVQEP)(I) was considered and studied by Fu (Ref. 13).

When I is singleton set, $X = Y$, $S_i \equiv T_i$, $S_i(x) = X$, $f_i(x, y, x_i) = \varphi(x, y)$, $g_i(x, y, y_i) = \phi(x, y)$, then (SSGVQEP)(III) reduces to the problem of simultaneous vector variational inequalities. Fu (Ref. 2) considered and studied the problem of simultaneous vector variational inequalities for a fixed cone C. If $C = \mathbb{R}_+$, the problem of simultaneous variational inequalities was introduced and studied by Husain and Tarafdar (Ref. 1) with applications to optimization problems.

By making suitable choices of f_i and g_i, we can derive several systems of variational inequalities and systems of (quasi) equilibrium problems studied in the literature; see for example Refs. 4–8, 14, and references therein.

2. Preliminaries

Throughout the paper, all topological spaces are assumed to be Hausdorff. Let $\mathcal{Z}^*$ be the dual of a locally convex t.v.s. $\mathcal{Z}$; let $P^* \subseteq \mathcal{Z}^*$ be the polar cone of P, that is,

$$P^* = \{z^* \in \mathcal{Z}^* : \langle z^*, z\rangle \geq 0, \ \ \forall z \in P\}.$$

We assume that P^* has a weak* compact convex base B^*. This means that $B^* \subseteq P^*$ is a weak* compact convex set such that $0 \notin B^*$ and $P^* = \cup_{\lambda \geq 0} \lambda B^*$; see for example Ref. 15.

Lemma 2.1. See Ref. 15. Let B^* be a weak* compact convex base of P^* and let $z \in \mathcal{Z}$. Then:

(i) $z \geq_P 0 \Leftrightarrow \langle z^*, z\rangle \geq 0, \quad \forall z^* \in P^*$;

(ii) $z \geq_P 0 \Leftrightarrow \langle z^*, z\rangle \geq 0, \quad \forall z^* \in B^*$.

Definition 2.1. See Ref. 16. Let $(\mathcal{Z}, P)$ be an ordered t.v.s. and let $\mathcal{K}$ be a nonempty convex subset of a vector space $\mathcal{X}$. A map $f:\mathcal{K}\to\mathcal{Z}$ is said to be:

(i) convex if, $\forall x, y\in\mathcal{K}$ and $t\in[0,1]$, we have

$$f(tx+(1-t)y)\le_P tf(x)+(1-t)f(y);$$

(ii) properly quasiconvex if, $\forall x, y\in\mathcal{K}$ and $t\in[0,1]$, we have

$$\begin{aligned}&\text{either}\quad f(tx+(1-t)y)\le_P f(x)\\ &\text{or}\quad f(tx+(1-t)y)\le_P f(y);\end{aligned}$$

(iii) properly quasiconcave if $-f$ is properly quasiconvex.

Definition 2.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be two topological spaces. A multivalued map $T:\mathcal{X}\to 2^{\mathcal{Y}}$ is said to be:

(i) compact if there exist a compact subset $\mathcal{K}\subseteq\mathcal{Y}$ such that $T(\mathcal{X})\subseteq\mathcal{K}$;
(ii) closed if its graph $Gr(T)=\{(x,y)\,|\,x\in\mathcal{X}, y\in T(x)\}$ is closed in $X\times Y$.

Lemma 2.2. See Ref. 17. Let $\mathcal{X}$ and $\mathcal{Y}$ be two topological spaces $T:\mathcal{X}\to 2^{\mathcal{Y}}$ be a continuous multivalued map such that, for each $x\in\mathcal{X}$, $T(x)$ is a nonempty compact set of $\mathcal{Y}$. Let $\varphi:\mathcal{X}\times\mathcal{Y}\to\mathbb{R}$ be a continuous function. Then, the function $M(x)=\min_{y\in T(x)}\varphi(x,y)$ is continuous.

Lemma 2.3. See Ref. 18. Let $\mathcal{X}$ and $\mathcal{Y}$ be nonempty subsets of the topological spaces E_1 and E_2, respectively, and let $S:\mathcal{X}\to 2^{\mathcal{X}}$ and $T:\mathcal{X}\times\mathcal{Y}\times\mathcal{X}\to 2^{\mathbb{R}}$ be multivalued maps. Let the multivalued map $m:\mathcal{X}\times\mathcal{Y}\to 2^{\mathbb{R}}$ be defined by

$$m(x,y)=\min T(x,y,S(x)),\quad \forall(x,y)\in\mathcal{X}\times\mathcal{Y},$$

and let the multivalued map $M:\mathcal{X}\times\mathcal{Y}\to 2^{\mathcal{X}}$ be defined by

$$M(x,y)=\{u\in S(x):m(x,y)\in T(x,y,u)\}.$$

If both S and T are compact continuous multivalued maps with closed values, then both m and M are closed compact upper semicontinuous (u.s.c.).

We close this section by mentioning the following Kakutani fixed-point theorem (Ref. 19), which is the main tool to prove the existence of a solution of (SSGVQEP).

Theorem 2.1. See Ref. 19. Let $\mathcal{X}$ be a nonempty compact convex subset of a locally convex t.v.s. and let $T:\mathcal{X}\to 2^{\mathcal{X}}$ be an u.s.c. multivalued map with nonempty compact convex values. Then, T has at least one fixed point in $\mathcal{X}$.

3. Existence Results for Solutions of (SSGVQEP)

In this section, we establish an existence result for solutions of (SSGVQEP) by using the Kakutani fixed-point theorem. We derive also existence results for solutions of (SGIVQVIP)(I), simultaneous generalized vector equilibrium problems, and systems of generalized vector variational-like inequalities.

Theorem 3.1. For each $i\in I$, let E_i, F_i, Z_i be real locally convex t.v.s. and let F_i be also complete. For each $i\in I$, let $X_i\subseteq E_i$ be a nonempty compact convex set and let $Y_i\subseteq F_i$ be a nonempty convex set. Let $X=\Pi_{i\in I}X_i$ and $Y=\Pi_{i\in I}Y_i$. For each $i\in I$, let $S_i:X\to 2^{X_i}$ be a continuous multivalued map with nonempty closed convex values and let $T_i:X\to 2^{Y_i}$ be a continuous multivalued map with nonempty compact convex values. For each $i\in I$, assume that the following conditions are satisfied.

(i) $C_i:X\to 2^{Z_i}$ is a closed multivalued map such that, $\forall x\in X, C_i(x)$ is a closed convex cone with apex at the origin and $P_i=\cap_{x\in X}C_i(x)$.

(ii) P_i^* has a weak* compact convex base B_i^* and Z_i is ordered by P_i.

(iii) $f_i:X\times Y\times X_i\to Z_i$ is a continuous function such that:

 (a) $\forall x\in X$ and $y\in Y, f_i(x,y,x_i)\geq_{P_i}0$,

 (b) $\forall(x,y)\in X\times Y$, the map $u_i\mapsto f_i(x,y,u_i)$ is properly quasi-convex.

(iv) $g_i:X\times Y\times Y_i\to Z_i$ is a continuous function such that

 (a) $\forall x\in X$ and $y\in Y, g_i(x,y,y_i)\geq_{P_i}0$,

 (b) $\forall(x,y)\in X\times Y$, the map $v_i\mapsto g_i(x,y,v_i)$ is properly quasi-convex.

Then, there exists a solution $(\bar{x},\bar{y})\in X\times Y$ of (SSGVQEP)(I).

Proof. For each $i\in I$ and for any fixed $(x,y,u_i)\in X\times Y\times X_i, (x,y,v_i)\in X\times Y\times Y_i$, $\langle z_i^*, f_i(x,y,u_i)\rangle$ and $\langle z_i^*, g_i(x,y,v_i)\rangle$ are weak* continuous on B_i^*. For each $i \in I$, let

$$h_i(x, y, u_i) = \min_{z_i^* \in B_i^*} \langle z_i^*, f_i(x, y, u_i) \rangle$$

and

$$p_i(x, y, v_i) = \min_{z_i^* \in B_i^*} \langle z_i^*, g_i(x, y, v_i) \rangle.$$

By Lemma 2.2, for each $i \in I$, h_i and p_i are continuous on $X \times Y \times X_i$ and $X \times Y \times Y_i$, respectively. For each $i \in I$ and $\forall (x, y) \in X \times Y$, define two multivalued maps $\Phi_i : X \times Y \to 2^{X_i}$ and $\Psi_i : X \times Y \to 2^{Y_i}$ by

$$\Phi_i(x, y) = \left\{ u_i \in S_i(x) : h_i(x, y, u_i) = \min_{u_i' \in S_i(x)} h_i(x, y, u_i') \right\},$$

$$\Psi_i(x, y) = \left\{ v_i \in T_i(x) : p_i(x, y, v_i) = \min_{v_i' \in T_i(x)} p_i(x, y, v_i') \right\}.$$

Following the approach adopted in the proof of Theorem 1 in Ref. 13, it is easy to show that, for each $i \in I$ and $\forall (x, y) \in X \times Y$, $\Phi_i(x, y)$ and $\Psi_i(x, y)$ are closed convex subsets of $S_i(x)$ and $T_i(x)$, respectively.

For each $i \in I$ and $\forall x \in X$, $S_i(x)$ is closed subset of a compact set X is compact, by assumptions on S_i and T_i and Proposition 3 in Ref. 20, pp. 42, $S_i(X)$ and $T_i(X)$ are compact.

For each $i \in I$, let $L_i = \overline{\mathrm{co}} T_i(X)$. Since F_i is quasicomplete, L_i is a compact convex subset of F_i (see, for example Ref. 21, pp. 241) and $L = \Pi_{i \in I} L_i$ is a compact convex set of $\Pi_{i \in I} F_i$. Since $X \times L \times X_i$ and $X \times L \times L_i$ are compact sets and h_i and p_i are continuous maps, h_i and p_i are compact continuous maps. By Lemma 2.3 (T is a single-valued map in Lemma 2.3), for each $i \in I, \Phi_i$ and Ψ_i are compact upper semicontinuous multivalued maps.

For each $i \in I$, define the multivalued map $G_i : X \times L \to 2^{X_i \times L_i}$ by

$$G_i(x, y) = (\Phi_i(x, y), \Psi(x, y)), \quad \forall (x, y) \in X \times L.$$

Then, by Lemma 3 in Ref. 22, for each $i \in I, G_i$ is u.s.c. with nonempty compact convex values. The multivalued map $G : X \times L \to 2^{X \times L}$ defined by $G(x, y) = \Pi_{i \in I} G_i(x_i, y_i)$ is u.s.c. with nonempty compact convex values. By Theorem 2.1, there exists a point $(\bar{x}, \bar{y}) \in X \times L$ such that $(\bar{x}, \bar{y}) \in G(\bar{x}, \bar{y})$. Therefore, for each $i \in I, \bar{x}_i \in S_i(\bar{x})$ and $\bar{y}_i \in T_i(\bar{x})$ such that

$$h_i(\bar{x}, \bar{y}, x_i) \geq h_i(\bar{x}, \bar{y}, \bar{x}_i), \quad \forall x_i \in S_i(\bar{x}),$$

$$p_i(\bar{x}, \bar{y}, y_i) \geq p_i(\bar{x}, \bar{y}, \bar{y}_i), \quad \forall y_i \in T_i(\bar{x}).$$

By (iii)(a) and (iv)(b), for each $i \in I$ we have

$$f_i(\bar{x}, \bar{y}, x_i) \geq_{P_i} 0,$$
$$g_i(\bar{x}, \bar{y}, y_i) \geq_{P_i} 0;$$

therefore, from Lemma 2.1, we have

$$\langle z_i^*, f_i(\bar{x}, \bar{y}, \bar{x}_i)\rangle \geq 0,$$
$$\langle z_i^*, g_i(\bar{x}, \bar{y}, \bar{y}_i)\rangle \geq 0, \quad \forall z_i^* \in B_i^*.$$

So,

$$\min_{z_i^* \in B_i^*} \langle z_i^*, f_i(\bar{x}, \bar{y}, \bar{x}_i)\rangle \geq 0,$$
$$\min_{z_i^* \in B_i^*} \langle z_i^*, g_i(\bar{x}, \bar{y}, \bar{y}_i)\rangle \geq 0.$$

Hence, for each $i \in I, \bar{x}_i \in S_i(\bar{x})$ and $\bar{y}_i \in T_i(\bar{x})$ such that

$$h_i(\bar{x}, \bar{y}, x_i) \geq 0, \quad \forall x_i \in S_i(\bar{x}),$$
$$p_i(\bar{x}, \bar{y}, y_i) \geq 0, \quad \forall y_i \in T_i(\bar{x}).$$

Again by using Lemma 2.1, we get

$$f_i(\bar{x}, \bar{y}, x_i) \geq_{P_i} 0, \quad \forall x_i \in S_i(\bar{x}),$$
$$g_i(\bar{x}, \bar{y}, y_i) \geq_{P_i} 0, \quad \forall y_i \in T_i(\bar{x}).$$

that is,

$$f_i(\bar{x}, \bar{y}, x_i) \in P_i \subseteq C_i(\bar{x}), \quad \forall x_i \in S_i(\bar{x}),$$
$$g_i(\bar{x}, \bar{y}, y_i) \in P_i \subseteq C_i(\bar{x}), \quad \forall y_i \in T_i(\bar{x}).$$

This completes the proof □

For each $i \in I$, if $g_i \equiv 0$, then we have the following result.

Corollary 3.1. For each $i \in I$, let E_i, F_i, Z_i be real locally convex t.v.s and let F_i be also complete. For each $i \in I$, let $X_i \subseteq E_i$ be a nonempty compact convex set and let $Y_i \subseteq F_i$ be a nonempty convex set. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. For each $i \in I$, let $S_i : X \to 2^{X_i}$ be a continuous multivalued map with nonempty closed convex and values and let $T_i : X \to 2^{Y_i}$ be a continuous multivalued map with nonempty compact convex values. For each $i \in I$, assume that the following conditions are satisfied.

(i) $C_i : X \to 2^{Z_i}$ is a closed multivalued map such that, $\forall x \in X, C_i(x)$ is a closed convex cone with apex at the origin and $P_i = \bigcap_{x\in X} C_i(x)$.

(ii) P_i^* has a weak* compact convex base B_i^* and Z_i is ordered by P_i.

(iii) $f_i : X \times Y \times X_i \to Z_i$ is a continuous function such that
(a) $\forall x \in X$ and $y \in Y$, $f_i(x, y, x_i) \geq_{P_i} 0$;
(b) $\forall (x, y) \in X \times Y$, the map $u_i \mapsto f_i(x, y, u_i)$ is properly quasiconvex.

Then, there exists a solution $(\bar{x}, \bar{y}) \in X \times Y$ of (SGIVQVIP)(I).

Remark 3.1. Corollary 3.1 is an extension of Theorem 1 in Ref. 13 to systems of quasiequilibrium problems with a moving cone.

When I is a singleton set, then we have the following result.

Corollary 3.2. Let E, F, Z be real locally convex t.v.s. and let F be also complete. Let $X \subseteq E$ be a nonempty compact convex set and let $Y \subseteq F$ be a nonempty convex set. Let $S : X \to 2^X$ be a continuous multivalued map with nonempty closed convex values and let $T : X \to 2^Y$ be a continuous multivalued map with nonempty compact convex values. Assume that the following conditions are satisfied.

(i) $C : X \to 2^Z$ is a closed multivalued map such that, $\forall x \in X, C(x)$ is a closed convex cone with apex at the origin and $P = \bigcap_{x\in X} C(x)$.

(ii) P^* has a weak* compact convex base B^* and Z is ordered by P.

(iii) $f : X \times Y \times X \to Z$ is a continuous function such that:

   (a) $\forall x \in X$ and $y \in Y, f(x, y, x) \geq_P 0$;
   (b) $\forall (x, y) \in X \times Y$, the map $u \mapsto f(x, y, u)$ is properly quasiconvex.

(iv) $g : X \times Y \times Y \to Z$ is a continuous function such that, (a) $\forall x \in X$ and $y \in Y, g(x, y, y) \geq_P 0$,
(b) $\forall (x, y) \in X \times Y$, the map $v \mapsto g(x, y, v)$ is properly quasiconvex.

Then, there exists a solution $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in S(\bar{x}), \bar{y} \in T(\bar{x})$,

$$f(\bar{x}, \bar{y}, x) \in C(\bar{x}), \quad \forall x \in S(\bar{x}),$$
$$g(\bar{x}, \bar{y}, y) \in C(\bar{x}), \quad \forall y \in T(\bar{x}).$$

Let $\mathcal{E}$ and $\mathcal{Z}$ be Hausdorff topological vector spaces. Let σ be the family of bounded subsets of $\mathcal{E}$ whose union is total in $\mathcal{E}$; that is, the linear hull of $\cup\{U : U \in \sigma\}$ is dense in $\mathcal{E}$. Let $\mathcal{B}$ be a neighborhood base of 0 in $\mathcal{Z}$. When U runs through σ, V through $\mathcal{B}$, the family

$$M(U, V) = \{\xi \in L(\mathcal{E}, \mathcal{Z}) : \cup_{x \in U} \langle \xi, x \rangle \subseteq V\}$$

is a neighborhood base of 0 in $L(\mathcal{E}, \mathcal{Z})$ for a unique translation-invariant topology, called the topology of uniform convergence of the sets $U \in \sigma$, briefly the σ-topology (see Ref. 23, pp. 79–80).

In order to drive existence results for a solution of the problem of system of simultaneous generalized vector variational-like inequalities from Theorem 3.1, we need the following result due to Ding and Tarafdar (Ref. 23).

Lemma 3.1. Let $\mathcal{E}$ and $\mathcal{Z}$ be Hausdorff t.v.s. and $L(\mathcal{E}, \mathcal{Z})$ be the t.v.s. under the σ-topology. Then, the bilinear mapping $\langle \cdot, \cdot \rangle : L(\mathcal{E}, \mathcal{Z}) \times \mathcal{E} \to \mathcal{Z}$ is continuous on $L(\mathcal{E}, \mathcal{Z}) \times \mathcal{E}$.

In addition to the assumptions on $C_i : K \to 2^{Z_i}$, in the following corollary, we assume further that $C_i(x)$ is pointed, for each $i \in I$ and for all $x \in K$. Then, the following result can be derived easily from Corollary 3.1 by setting

$$f_i(x, y, u_i) = \langle \theta_i(x, y), \eta_i(u_i, x_i) \rangle.$$

Corollary 3.3. Let E_i, X_i, Z_i, S_i, X be the same as in Theorem 3.1. For each $i \in I$, let $L(E_i, Z_i)$ be quasicomplete, let $Y_i \subseteq L(E_i, Z_i)$ be a nonempty convex set, and let $Y = \prod_{i \in I} Y_i$. For each $i \in I$, let $T_i : X \to 2^{Y_i}$ be a continuous multivalued map with nonempty compact convex values. For each $i \in I$, assume that the following conditions are satisfied:

(i) $C_i : X \to 2^{Z_i}$ is a closed multivalued map such that, $\forall x \in X$, $C_i(x)$ is a nonempty closed convex pointed cone and $P_i = \bigcap_{x \in X} C_i(x)$.

(ii) P_i^* has a weak* compact convex base B_i and Z_i is ordered by P_i.

(iii) $\theta_i : X \times Y \to Y_i$ and $\eta_i : X_i \times X_i \to X_i$ are continuous bifunctions such that, for each $i \in I$:

  (a) $\forall x_i \in X_i, \eta_i(x_i, x_i)\rangle \geq_{P_i} 0$;

  (b) $\forall (x, y) \in X \times Y$, the map $u_i \mapsto \langle \theta_i(x, y), \eta_i(u_i, x_i) \rangle$ is properly quasiconvex.

Then, there exists a solution $(\bar{x}, \bar{y}) \in X \times Y$ of (SGVQVLIP) (I): find $(\bar{x}, \bar{y}) = ((\bar{x}_i)_{i\in I}, (\bar{y}_i)_{i\in I}) \in X \times Y$ such that, for each, $i \in I, \bar{x}_i \in S_i(\bar{x}), \bar{y}_i \in T_i(\bar{x})$ and

$$\langle \theta_i(\bar{x}, \bar{y}), \eta_i(x_i, \bar{x}_i)\rangle \in C_i(\bar{x}), \quad \forall x_i \in S_i(\bar{x}).$$

Remark 3.2. It is worth to mention that the weak formulation of (SGVQVLIP)(III) was considered and studied by Ansari and Khan (Ref. 5). Corollary 3.3 provides the existence of a solution of a more general problem than (SGVQVLIP)(III).

4. Systems of Vector Saddle-Point Problems

In this section, we define systems of quasisaddle point problems and system of quasiminimax inequalities. As application of the results of the last section, we derive existence result for the solutions of these problems.

Let X, Y, X_i, Y_i, Z_i, and C_i be the same as defined in the formulations of (SSGVQEP). Let $\ell_i : X_i \times Y_i \to Z_i$ be a bifunction. We consider the following system of saddle-point problems:

(SVQSPP)(I) Find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that, for each, $i \in I, \bar{x}_i \in S_i(\bar{x}), \bar{y}_i \in T_i(\bar{x})$,

$$\ell_i(x_i, \bar{y}_i) - \ell_i(\bar{x}_i, \bar{y}_i) \in C_i(\bar{x}), \quad \forall x_i \in S_i(\bar{x}),$$
$$\ell_i(\bar{x}_i, \bar{y}_i) - \ell_i(\bar{x}_i, y_i) \in C_i(\bar{x}), \quad \forall y_i \in T_i(\bar{x}).$$

(SVQSPP)(II) Find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that, for each, $i \in I, \bar{x}_i \in S_i(\bar{x}), \bar{y}_i \in T_i(\bar{x})$,

$$\ell_i(x_i, \bar{y}_i) - \ell_i(\bar{x}_i, \bar{y}_i) \notin -C_i(\bar{x})\backslash\{0\}, \quad \forall x_i \in S_i(\bar{x}),$$
$$\ell_i(\bar{x}_i, \bar{y}_i) - \ell_i(\bar{x}_i, y_i) \notin -C_i(\bar{x})\backslash\{0\}, \quad \forall y_i \in T_i(\bar{x}).$$

(SVQSPP)(III) Find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that, for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}), \bar{y}_i \in T_i(\bar{x})$,

$$\ell_i(x_i, \bar{y}_i) - \ell_i(\bar{x}_i, \bar{y}_i) \notin -\text{int } C_i(\bar{x}), \quad \forall x_i \in S_i(\bar{x}),$$
$$\ell_i(\bar{x}_i, \bar{y}_i) - \ell_i(\bar{x}_i, y_i) \notin -\text{int } C_i(\bar{x}), \quad \forall y_i \in T_i(\bar{x}).$$

Remark 4.1. For each $i \in I$ and $\forall x \in X$, if $C_i(x)$ is a convex pointed cone, then every solution of (SVQSPP)(I) is a solution of (SVQSPP)(II) and every solution of (SVQSPP)(II) is a solution of (SVQSPP)(III). But the converse implication is not true.

If I is a singleton set and $Z=\mathbb{R}$, then (SVQSPP)(I), (SVQSPP)(II), and (SVQSPP)(III) are called quasisaddle-point problem [for short, (QSPP)].

Of course, if I is a singleton set, $S_i(x)=X_i$, and $T_i(x)=Y_i, \forall x \in X$. and $Z_i=\mathbb{R}$, then the above mentioned (SVQSPP)s reduce to the classical saddle-point problem.

For each $i \in I$, let ℓ_i be a real-valued bifunction. We consider also the following problem of systems of minimax inequalities: find $\bar{x}=(\bar{x}_i)_{i\in I} \in X$ and $\bar{y}=(\bar{y}_i)_{i\in I} \in Y$ such that, for each $i \in I$, $\bar{x}_i \in S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$, and

$$\min_{u_i \in S_i(\bar{x}_i)} \max_{v_i \in T_i(\bar{x}_i)} \ell_i(u_i, v_i) = \ell_i(\bar{x}_i, \bar{y}_i) = \max_{v_i \in T_i(\bar{x}_i)} \min_{u_i \in S_i(\bar{x}_i)} \ell_i(u_i, v_i).$$

As application of Theorem 3.1, we derive the following existence result for the solution of (SGVQSPP)(I).

Theorem 4.1. In Theorem 3.1, if conditions (iii) and (iv) are replaced by the (iii′) then (SVQSPP)(I) has a solution, where

(iii′) $\ell_i : X_i \times Y_i \to Z_i$ is a continuous function such that:
- (a) for each fixed $y_i \in Y_i, x_i \mapsto \ell_i(x_i, y_i)$ is properly quasiconvex;
- (b) for each fixed $x_i \in X_i, y_i \mapsto \ell_i(x_i, y_i)$ is properly quasiconcave.

Proof. For each $i \in I$, let

$$f_i(x, y, u_i) = \ell_i(u_i, y_i) - \ell_i(x_i, y_i)$$
$$g_i(x, y, v_i) = \ell_i(x_i, y_i) - \ell_i(x_i, v_i),$$

$\forall x=(x_i)_{i\in I}, u=(u_i)_{i\in I} \in X$, and $y=(y_i)_{i\in I}, v=(v_i)_{i\in I} \in Y$.

For each $i \in I$ and $\forall x=(x_i)_{i\in I} \in X$ and $y=(y_i)_{i\in I} \in Y$, we have

$$f_i(x, y, x_i) = 0 \in P_i \quad \text{and} \quad g_i(x, y, y_i) = 0 \in P_i.$$

For each $i \in I$, since ℓ_i is continuous on $X_i \times Y_i$, f_i and g_i are continuous on $X \times Y \times Y_i$ and $X \times Y \times Y_i$, respectively. By condition (iii), $\forall (x, y) \in X \times Y$, $u_i \mapsto \ell_i(u_i, y_i) - \ell_i(x_i, y_i)$ and $v_i \mapsto \ell_i(x_i, y_i) - \ell_i(x_i, v_i)$ are properly quasiconvex. The conclusion follows from Theorem 3.1. □

If I is a singleton set and $Z=\mathbb{R}$, then we have following existence result for a solution of the quasisaddle-point problem.

Corollary 4.1. Let E and F be real locally convex t.v.s. and also let F be quasicomplete. Let $X \subseteq E$ be a nonempty compact convex set and let $Y \subseteq F$ be a nonempty convex set. Let $S: X \to 2^X$ be a continuous multivaluated map with nonempty closed convex values and let $T: X \to 2^Y$ be a continuous multivalued map with nonempty compact convex values. Assume that $\ell: X \times Y \to Z$ is a continuous function such that

(a) for each fixed $y \in Y, x \mapsto l(x, y)$ is quasiconvex;
(b) for each fixed $x \in X, x \mapsto l(x, y)$ is quasiconcave.

Then, (QSPP) has a solution.

As a consequence of Theorem 4.1, we have the following existence result for the solution of a system of quasiminimax inequalities.

Theorem 4.2. Let $E_i, F_i, X_i, Y_i, X, Y, S_i,$ and T_i be the same as in Theorem 3.1. For each $i \in I$, assume that $\ell_i : X_i \times Y_i \to \mathbb{R}$ is a continuous function satisfying the following conditions:

(i) For each fixed $y_i \in Y_i, x_i \mapsto \ell_i(x_i, y_i)$ is quasiconvex.
(ii) For each fixed $x_i \in X_i, y_i \mapsto \ell_i(x_i, y_i)$ is quasiconvex.

Then, (SQMIP) has a solution.

Proof. For each $i \in I, Z_i = \mathbb{R}$, which has a weak* compact base. For each $i \in I$ let $C_i(x) = [0, \infty), \forall x \in X$. By Theorem 4.1, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that, for each $i \in I, \bar{x}_i \in S_i(\bar{x})$ and $\bar{y}_i \in T_i(\bar{x})$ satisfy

$$\ell_i(x_i, \bar{y}_i) - \ell_i(\bar{x}_i, \bar{y}_i) \geq 0, \quad \forall x_i \in S_i(\bar{x}),$$
$$\ell_i(\bar{x}_i, \bar{y}_i) - \ell_i(\bar{x}_i, y_i) \geq 0, \quad \forall y_i \in T_i(\bar{x}),$$

and so,

$$\ell_i(\bar{x}_i, \bar{y}_i) = \min_{u_i \in S_i(\bar{x})} \ell_i(u_i, \bar{y}_i) \leq \max_{v_i \in T_i(\bar{x})} \min_{u_i \in S_i(\bar{x})} \ell_i(u_i, v_i),$$
$$\ell_i(\bar{x}_i, \bar{y}_i) = \max_{v_i \in T_i(\bar{x})} \ell_i(\bar{x}_i, v_i) \geq \min_{u_i \in S_i(\bar{x})} \max_{v_i \in T_i(\bar{x})} \ell_i(u_i, v_i).$$

Therefore,

$$\min_{u_i \in S_i(\bar{x})} \max_{v_i \in T_i(\bar{x})} \ell_i(u_i, v_i) \leq \ell_i(\bar{x}_i, \bar{y}_i) \leq \max_{v_i \in T_i(\bar{x})} \min_{u_i \in S_i(\bar{x})} \ell_i(u_i, v_i),$$
$$\max_{u_i \in T_i(\bar{x})} \min_{v_i \in S_i(\bar{x})} \ell_i(u_i, v_i) \leq \ell_i(\bar{x}_i, \bar{y}_i) \leq \min_{u_i \in S_i(\bar{x})} \max_{v_i \in T_i(\bar{x})} \ell_i(u_i, v_i).$$

Therefore, there exist $\bar{x} = (\bar{x}_i)_{i \in I} \in X, \bar{y} = (\bar{y}_i)_{i \in I} \in Y$ such that, for each $i \in I, \bar{x}_i \in S_i(\bar{x}), \bar{y}_i \in T_i(\bar{x}_i)$, and

$$\min_{u_i \in S_i(\bar{x})} \max_{v_i \in T_i(\bar{x})} \ell_i(u_i, v_i) = \ell_i(\bar{x}_i, \bar{y}_i) = \max_{v_i \in T_i(\bar{x})} \min_{u_i \in S_i(\bar{x})} \ell_i(u_i, v_i). \qquad \square$$

For each $i \in I$, if X_i and Y_i are nonempty compact convex sets, and $S_i(x) = X_i$ and $T_i(x) = Y_i, \forall x \in X$, then from Theorem 4.2 we derived the following corollary.

Corollary 4.2. For each $i \in I$, let X_i and Y_i be nonempty compact convex subsets of E_i and F_i, respectively. For each $i \in I$, assume that $\ell : X_i \times Y_i \to \mathbb{R}$ is a continuous function satisfying the following conditions:

(i) For each fixed $y_i \in Y_i, x_i \mapsto \ell_i(x_i, y_i)$ is quasiconvex.
(ii) For each fixed $x_i \in X_i, y_i \mapsto \ell_i(x_i, y_i)$ is quasiconcave.

Then, there exist $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i \in I} \in Y$ such that

$$\min_{u_i \in X_i} \max_{v_i \in Y_i} \ell_i(u_i, v_i) = \ell_i(\bar{x}_i, \bar{y}_i) = \max_{v_i \in Y_i} \min_{u_i \in X_i} \ell_i(u_i, v_i).$$

If I is a singleton, then Theorem 4.2 reduces to the following Corollary 3.2 in Ref. 11.

Corollary 4.3. See Ref. 11. Let E, F, X, Y, S, and T be the same as in Corollary 4.1. Assume that $\ell : X \times Y \to \mathbb{R}$ is a continuous function satisfying the following conditions:

(i) For each fixed $y \in Y, x \mapsto \ell(x, y)$ is quasiconvex.
(ii) For each fixed $x \in X, y \mapsto \ell(x, y)$ is quasiconcave.

Then, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in S(\bar{x}), \bar{y} \in T(\bar{y})$, and

$$\min_{u \in S(\bar{x})} \max_{v \in T(\bar{x})} \ell(u, v) = \ell(\bar{x}, \bar{y}) = \max_{v \in T(\bar{x})} \min_{u \in S(\bar{x})} \ell(u, v).$$

5. Debreu-Type Equilibrium Problem

In this section, we give another application of Corollary 3.1 to prove the existence of a solution of the Debreu-type equilibrium problem for vector-valued functions.

Let X, X_i, Z_i, and C_i be the same as defined in the formulations of (SSGVQEP). For each $i \in I$, let $\varphi_i : X \to Z_i$ be a vector-valued function and let $X^i = \prod_{j \in I, j \neq i} X_j$. We write $X = X^i \times X_i$. For $x \in X, x^i$ denotes the projection of x onto X^i; hence, we write $x = (x^i, x_i)$. We consider the following Debreu-type equilibrium problem for vector-valued functions (Debreu VEP).

(Debreu VEP)(I) Find $\bar{x} \in X$ such that, for each $i \in I, \bar{x}_i \in S_i(\bar{x})$, and $\varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \in C_i(\bar{x}), \quad \forall y_i \in S_i(\bar{x})$.

(Debreu VEP)(II) Find $\bar{x} \in X$ such that, for each $i \in I, \bar{x}_i \in S_i(\bar{x})$ and $\varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -C_i(\bar{x}) \setminus \{0\}, \quad \forall \bar{y}_i \in S_i(\bar{x})$.

(Debreu VEP)(III) Find $\bar{x} \in X$ such that, for each $i \in I, \bar{x}_i \in S_i(\bar{x})$, and $\varphi_i(\bar{x}^i, y_i) - \varphi_i(\bar{x}) \notin -\text{int } C_i(\bar{x}), \quad \forall \bar{y}_i \in S_i(\bar{x})$;

in the case, we assume that int C_i is nonempty for each $i \in I$.

Of course, for each $i \in I$, if φ_i is a scalar-valued function, then the (Debreu VEP)s are the same as the one introduced and studied by Debreu in Ref. 24. In this case, a large number of papers have appeared already in the literature; see for example Refs. 10, 14, and references therein. Ansari et al (Ref. 4) introduced and studied (Debreu VEP) (III) and established several existence results for its solution with or without involving Φ-condensing maps. This is the first paper in the literature in which the (Debreu-type equilibrium problem for vector-valued function is considered.

As in the case of (SSGVQEP)s, for each $i \in I$ and $\forall x \in X$, if $C_i(x)$ is also pointed, then every solution of (Debreu VEP) (I) is a solution of (Debreu VEP) (II) and every solution of (Debreu VEP) (II) is a solution of (Debreu VEP) (III). But the reverse implication does not hold.

Theorem 5.1. Let E_i, X_i, S_i, Z_i be the same as Theorem 3.1. For each $i \in I$ assume conditions (i) and (ii) of Theorem 3.1 and in addition:

(iii) $\varphi_i : X \to Z_i$ is continuous and properly quasiconvex in each argument.

Then, there exists a solution $\bar{x} \in X$ of (Debreu VEP)(I).

Proof. For each $i \in I$, set $X_i = Y_i$; let $f_i(x, y, u_i) = \varphi(x) - \varphi(x^i, u_i)$, $\forall x, y \in X$, let $u_i \in X_i$, and let $T_i(x) = X_i \forall x \in X$ in Corollary 3.1; we get the conclusion. □

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