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Matrix Inequality Approach to a Novel Stability Criterion for Time-Delay Systems with Nonlinear Uncertainties

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Abstract. In this paper, a novel stability criterion is presented for time-delay systems which have nonlinear uncertainties. Based on the Lyapunov method, a stability criterion is derived in terms of matrix inequalities which can be solved easily by efficient convex optimization algorithms. Numerical examples are included to show the effectiveness of the proposed method.

Key Words. Time-delay systems, Lyapunov method, nonlinear uncertainties, convex optimization.

1. Introduction

Time delays occur in various industrial systems such as tandem mills, remote control systems, long transmission lines in pneumatic systems, and chemical systems. Frequently, the delays are a source of instability and poor performance (Ref. 1). For more characteristics of the system, see Refs. 2–4.

Practical systems have some type of uncertainties because it is almost impossible to obtain an exact mathematical models due to the difficulty of measuring various parameters, environmental noises, hysteresis or friction, poor plant knowledge, reduced-order models, uncertain or slowly varying parameters, and the complexity of the system. This leads the system to unexpectedly complicated situations. Therefore, many researchers have

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studied extensively the stability analysis of time-delay systems with uncertainties (Refs. 5–8).

The stability criteria developed in the literature are classified often into two categories according to their nature and the size of the delays, namely, delay-independent criteria and delay-dependent criteria. In general, a delay-dependent stability criterion is less conservative than a delay-independent one when the size of the time delay is small.

In this paper, we present a novel delay-dependent stability criterion for uncertain time-delay systems with nonlinear uncertainties. Based on the Lyapunov second method, two stability criteria are derived in terms of matrix inequalities. The proposed method employs free weighting matrices, which are easy to select, to obtain less conservative stability criteria. Furthermore, the matrix inequalities can be solved easily by using various convex optimization algorithms (Ref. 9). Three numerical examples are given to show the superiority of the present result to those available in the published literature.

Notations. $\mathcal{R}^n$ is the n-dimensional Euclidean space. $\mathcal{R}^{m\times n}$ denotes the set of $m\times n$ real matrices. A star denotes the symmetric part of a matrix. $X>0$ $[X\geq 0]$ means that X is a real symmetric positive-definitive matrix [positive-semidefinite matrix]. I denotes the identity matrix of appropriate dimensions. $||\cdot||$ refers to the induced matrix 2-norm. diag$\{\cdots\}$ denotes a block diagonal matrix. $\mathcal{C}_{n,h}=\mathcal{C}([-h,0],\mathcal{R}^n)$ denotes the Banach space of continuous functions mapping the interval $[-h,0]$ into $\mathcal{R}^n$, with the topology of uniform convergence.

2. Main Results

Consider the following system:

$$\dot{x}(t)=Ax(t)+A_1x(t-h)+f(t,x(t))+f(t,x(t-h)), \tag{1a}$$

$$x(s)=\phi(s), \quad s\in[-h,0], \tag{1b}$$

where $x(t)\in\mathcal{R}^n$ is the state vector, $A\in\mathcal{R}^{n\times n}$ and $A_1\in\mathcal{R}^{n\times n}$ are known real parameter matrices, $h>0$ is a constant delay, $\phi(s)\in\mathcal{C}_{n,h}$ is a given continuous vector-valued initial function, the functions f and f_1 are nonlinear uncertainties with respect to the current state $x(t)$ and the delayed state $x(t-h)$. In this paper, the nonlinear uncertainties are assumed to be bounded in magnitude,

$$\|f(t,x(t))\| \leq \alpha\|x(t)\|, \tag{2a}$$

$$\|f_1(t,x(t-h))\| \leq \alpha_1\|x(t-h)\|, \tag{2b}$$

where α and α_1 are positive scalars.

Now, define an operator $\mathcal{D}(x_t):\mathcal{C}_{n,h}\to\mathcal{R}^n$ as

$$\mathcal{D}(x_t)=x(t)+\int_{t-h}^{t} Gx(s)ds, \tag{3}$$

where

$$x_t=x(t+s), \quad s\in[-h,0],$$

and $G\in\mathcal{R}^{n\times n}$ is a constant matrix which will be chosen so as to make the system asymptotically stable. With the above operator, the transformed system is

$$\begin{aligned}\dot{\mathcal{D}}(x_t)&=\dot{x}(t)+Gx(t)-Gx(t-h)\\&=(A+G)x(t)+(A_1-G)x(t-h)+f(t,x(t))+f_1x(t,x(t-h)).\end{aligned} \tag{4}$$

We will need the following well-known facts and lemmas to obtain the main results.

Fact 2.1. For given matrices D, E, F, with $F^TF\le I$, and for a scalar $\epsilon>0$, the following inequality is always satisfied:

$$DFE+E^TF^TD^T\le\varepsilon DD^T+\varepsilon^{-1}E^TE.$$

Fact 2.2. Schur Complement. The linear matrix inequality

$$\begin{bmatrix} Z(x) & Y(x)\\ Y^T(x) & W(x)\end{bmatrix}>0 \tag{5}$$

is equivalent to

$$W(x)>0 \quad\text{and}\quad Z(x)-Y(x)W^{-1}(x)Y^T(x),$$

where

$$Z(x)=Z^T(x), \quad W(x)=W^T(x),$$

and $Y(x)$ depends affinely on x.

Lemma 2.1. See Ref. 10. For any constant matrix $M\in\mathcal{R}^{n\times n}$, $M=M^T>0$, scalar $\gamma>0$, vector function $\omega:[0,\gamma]\to\mathcal{R}^n$ such that the integrations are well defined, the following inequality holds:

$$\left(\int_0^{\gamma}\omega(s)ds\right)^T M\left(\int_0^{\gamma}\omega(s)ds\right)\le\gamma\int_0^{\gamma}\omega^T(s)M\omega(s)ds. \tag{6}$$

Lemma 2.2. See Ref. 6. Consider an operator $\mathcal{D}(\cdot): \mathcal{C}_{n,h} \to \mathcal{R}^n$, with $\mathcal{D}(x_t) = x(t) + \hat{B}\int_{t-h}^{t} x(s)ds$, where $x(t) \in \mathcal{R}^n$ and $\hat{B} \in \mathcal{R}^{n\times n}$. For a given scalar δ, with $0<\delta<1$, if there exists a positive-definite symmetric matrix M such that the following inequality holds:

$$\begin{bmatrix} -\delta M & h\hat{B}^T M \\ hM\hat{B} & -M \end{bmatrix} < 0, \tag{7}$$

then the operator $\mathcal{D}(x_t)$ is stable.

We have the following theorem.

Theorem 2.1. For given h, α, α_1, the system (1a) is asymptotically stable if there exist positive-definite matrices X, W, F_{11}, F_{33}, positive scalars ε_i, $i=0,\dots,3$, and matrices Y, F_{12}, F_{13}, F_{23} satisfying the following inequalities:

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 & 0 & \Xi_1 \\ \star & -h^{-1}X & Q_{23} & I & I & 0 & 0 & 0 \\ \star & \star & Q_{33} & 0 & 0 & \varepsilon_1\alpha_1 X & \varepsilon_3\alpha_1 X & 0 \\ \star & \star & \star & -\varepsilon_2 I & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\varepsilon_3 I & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\varepsilon_1 I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -\varepsilon_3 I & 0 \\ \star & \star & \star & \star & \star & \star & \star & \Xi_2 \end{bmatrix} < 0, \tag{8a}$$

with

$$Q_{11} = AX + XA^T + Y + Y^T + W + hF_{11}, \tag{8b}$$

$$Q_{12} = XA^T + Y^T + F_{12}, \tag{8c}$$

$$Q_{13} = A_1X - Y + hF_{13}, \tag{8d}$$

$$Q_{23} = A_1X - Y + F_{23}, \tag{8e}$$

$$Q_{33} = -W + hF_{33}, \tag{8f}$$

$$\begin{bmatrix} -X & hY^T \\ \star & -X \end{bmatrix} < 0, \tag{9}$$

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ \star & X & F_{23} \\ \star & \star & F_{33} \end{bmatrix} > 0, \tag{10}$$

where

$$\Xi_1 = [I\, I\, \varepsilon_0 \alpha X\, \varepsilon_2 \alpha X\, hY^T],$$
$$\Xi_2 = \mathrm{diag}\{-\varepsilon_0 I, -\varepsilon_1 I, -\varepsilon_0 I, -\varepsilon_2 I, -0.5hX\}.$$

Proof. Consider a legitimate Lyapunov function candidate V as

$$V = V_1 + V_2 + V_3 + V_4, \tag{11}$$

where

$$V_1 = \mathcal{D}^T(x_t) P \mathcal{D}(x_t), \tag{12}$$

$$V_2 = 2\int_{t-h}^{t} \int_{s}^{t} x^T(u) G^T P G x(u) du\, ds, \tag{13}$$

$$V_3 = \int_{t-h}^{t} x^T(s) T x(s)\, ds, \tag{14}$$

$$\begin{aligned} V_4 = \int_0^t \int_{s-h}^{s} &\begin{bmatrix} x(s) \\ Gx(u) \\ x(s-h) \end{bmatrix}^T \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ \star & P^{-1} & F_{23} \\ \star & \star & F_{33} \end{bmatrix} \\ &\times \begin{bmatrix} P & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & P \end{bmatrix} \begin{bmatrix} x(s) \\ Gx(u) \\ x(s-h) \end{bmatrix} du\, ds. \end{aligned} \tag{15}$$

Taking the time-derivative of V leads to

$$\begin{aligned}
\dot{V}_1 &= 2\mathcal{D}^T(x_t)P\dot{\mathcal{D}}(x_t)\\
&= 2\left\{x(t)+\int_{t-h}^{t} Gx(s)ds\right\}^T P\{(A+G)x(t)+(A_1-G)x(t-h)\\
&\quad +f(t,x(t))+f_1(t,x(t-h))\}\\
&= x^T(t)\{P(A+G)+(A+G)^T P\}x(t)+2x^T(t)P(A_1-G)x(t-h)\\
&\quad +2x^T(t)Pf(t,x(t))+2x^T(t)Pf_1(t,x(t-h))\\
&\quad +2\left(\int_{t-h}^{t} Gx(s)ds\right)^T P(A+G)x(t)\\
&\quad +2\left(\int_{t-h}^{t} Gx(s)ds\right)^T P(A_1-G)x(t-h)\\
&\quad +2\left(\int_{t-h}^{t} Gx(s)ds\right)^T Pf(t,x(t))\\
&\quad +2\left(\int_{t-h}^{t} Gx(s)ds\right)^T Pf_1(t,x(t-h)),
\end{aligned} \tag{16}$$

$$\begin{aligned}
\dot{V}_2 &= 2hx^T(t)G^T PGx(t)-2\int_{t-h}^{t} x^T(s)G^T PGx(s)ds\\
&\le 2hx^T(t)G^T PGx(t)-\int_{t-h}^{t} x^T(s)G^T PGx(s)ds\\
&\quad -h^{-1}\left(\int_{t-h}^{t} Gx(s)ds\right)^T P\left(\int_{t-h}^{t} Gx(s)ds\right),
\end{aligned} \tag{17}$$

$$\dot{V}_3 = x^T(t)Tx(t)-x^T(t-h)Tx(t-h), \tag{18}$$

$$\begin{aligned}
\dot{V}_4 &= hx^T(t)PF_{11}Px(t)+2x^T(t)PF_{12}P\int_{t-h}^{t} Gx(s)ds\\
&\quad +2hx^T(t)PF_{13}Px(t-h)\\
&\quad +\int_{t-h}^{t} x^T(s)G^T PGx(s)ds+2\left(\int_{t-h}^{t} Gx(s)ds\right)^T PF_{23}Px(t-h)\\
&\quad +hx^T(t-h)PF_{33}Px(t-h),
\end{aligned} \tag{19}$$

where Lemma 2.1 is utilized in (17).

Using Fact 2.1, we obtain

$$2x^T(t)Pf(t,x(t)) \le \varepsilon_0^{-1}x^T(t)PPx(t) + \varepsilon_0\alpha^2 x^T(t)x(t), \tag{20}$$

$$2x^T(t)Pf_1(t,x(t-h)) \le \varepsilon_1^{-1}x^T(t)PPx(t) + \varepsilon_1\alpha_1^2 x^T(t-h)x(t-h), \tag{21}$$

$$\begin{aligned} &2\left(\int_{t-h}^{t} Gx(s)ds\right)^T Pf(t,x(t)) \\ &\le \varepsilon_2^{-1}\left(\int_{t-h}^{t} Gx(s)ds\right)^T PP\left(\int_{t-h}^{t} Gx(s)ds\right) + \varepsilon_2\alpha^2 x^T(t)x(t), \end{aligned} \tag{22}$$

$$\begin{aligned} &2\left(\int_{t-h}^{t} Gx(s)ds\right)^T Pf_1(t,x(t-h)) \\ &\le \varepsilon_3^{-1}\left(\int_{t-h}^{t} Gx(s)ds\right)^T PP\left(\int_{t-h}^{t} Gx(s)ds\right) + \varepsilon_3\alpha_1^2 x^T(t-h)x(t-h). \end{aligned} \tag{23}$$

From (16)–(23), the time-derivative of V has a new upper bound as follows:

$$\dot{V} \le \begin{bmatrix} x(t) \\ \int_{t-h}^{t} Gx(s)ds \\ x(t-h) \end{bmatrix}^T \Omega_1 \begin{bmatrix} x(t) \\ \int_{t-h}^{t} Gx(s)ds \\ x(t-h) \end{bmatrix}, \tag{24}$$

where

$$\Omega_1 = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ \star & Q_{22} & Q_{23} \\ \star & \star & Q_{33} \end{bmatrix} \tag{25a}$$

$$\begin{aligned} Q_{11} = {} & P(A+G) + (A+G)^T P + \varepsilon_0^{-1}PP + \varepsilon_1^{-1}PP + \varepsilon_0\alpha^2 I + \varepsilon_2\alpha^2 I \\ & + 2hG^T PG + T + hPF_{11}P, \end{aligned} \tag{25b}$$

$$Q_{12} = (A+G)^T P + PF_{12}P, \tag{25c}$$

$$Q_{13} = P(A_1 - G) + hPF_{13}P, \tag{25d}$$

$$Q_{22} = -h^{-1}P + \varepsilon_2^{-1}PP + \varepsilon_3^{-1}PP, \tag{25e}$$

$$Q_{23} = P(A_1 - G) + PF_{23}P, \tag{25f}$$

$$Q_{33} = -T + \varepsilon_1\alpha_1^2 I + \varepsilon_3\alpha_1^2 I + hPF_{33}P. \tag{25g}$$

Hence, if $\Omega_1 < 0$, then a positive scalar λ exists which satisfies

$$\dot{V} < -\lambda \parallel x(t) \parallel^2. \tag{26}$$

Let

$$X = P^{-1}, \quad W = XTX, \quad Y = GX. \tag{27}$$

By premultiplying and postmultiplying the inequality $\Omega_1 < 0$ by $\operatorname{diag}\{X, X, X\}$ and using Fact 2.2 (Schur complement), the resulting inequality is equivalent to (8).

Inequality (9) is equivalent to

$$\begin{bmatrix} -P & hG^T P \\ \star & -P \end{bmatrix} < 0 \tag{28}$$

as can be seen by premultiplying and postmultiplying the inequality (9) by $\operatorname{diag}\{X^{-1}, X^{-1}\}$. If inequality (28) holds, then we can prove that there exists a positive scalar δ, which is less than one, such that

$$\begin{bmatrix} -\delta P & hG^T P \\ \star & -P \end{bmatrix} < 0, \tag{29}$$

according to matrix theory. Therefore, from Lemma 2.2, if the inequality (9) holds, then the operator $\mathcal{D}(x_t)$ is stable. The inequality (10) means that V_4 is nonnegative. According to Theorem 9.8.1 in Ref. 1, we conclude that, if the matrix inequalities (8)–(10) holds, then the system (1) is asymptotically stable. This completes our proof. □

Suppose that the nonlinear uncertainties are norm-bounded and time-varying,

$$f(t, x(t)) = DF(t)Ex(t), \tag{30a}$$

$$f_1(t, x(t-h)) = D_1F_1(t)E_1x(t-h), \tag{30b}$$

$$\parallel F(t) \parallel \leq 1, \tag{30c}$$

$$\parallel F_1(t) \parallel \leq 1, \tag{30d}$$

where D, D_1, E, E_1 are known real constant matrices of appropriate dimensions. In this case, we have the following result.

Corollary 2.1. For given h, α, α_1, the system (1a) is asymptotically stable if there exist positive definite matrices X, W, F_{11}, F_{33}, positive scalars ε_i, $i=0,\dots,3$, and matrices Y, F_{12}, F_{13}, F_{23} satisfying the following inequalities:

$$\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0 & 0 & \Xi_1 \\ \star & -h^{-1}X & Q_{23} & D & D_1 & 0 & 0 & 0 \\ \star & \star & Q_{33} & 0 & 0 & \varepsilon_1 E_1^T X & \varepsilon_3 E_1^T X & 0 \\ \star & \star & \star & -\varepsilon_2 I & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\varepsilon_3 I & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & -\varepsilon_1 I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -\varepsilon_3 I & 0 \\ \star & \star & \star & \star & \star & \star & \star & \Xi_2 \end{bmatrix} < 0, \tag{31a}$$

with

$$Q_{11} = AX + XA^T + Y + Y^T + W + hF_{11}, \tag{31b}$$

$$Q_{12} = XA^T + Y^T + F_{12}, \tag{31c}$$

$$Q_{13} = A_1 X - Y + hF_{13}, \tag{31d}$$

$$Q_{23} = A_1 X - Y + F_{23}, \tag{31e}$$

$$Q_{33} = -W + hF_{33}. \tag{31f}$$

$$\begin{bmatrix} -X & hY^T \\ \star & -X \end{bmatrix} < 0, \tag{32}$$

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} \\ \star & X & F_{23} \\ \star & \star & F_{33} \end{bmatrix} > 0, \tag{33}$$

where

$$\Xi_1 = [D D_1 \varepsilon_0 E^T X \varepsilon_2 E^T X h Y],$$
$$\Xi_2 = \mathrm{diag}\{-\varepsilon_0 I, -\varepsilon_1 I, -\varepsilon_0 I, -\varepsilon_2 I, -0.5hX\}.$$

Remark 2.1. In this paper, we use the operator

$$\mathcal{D}(x_t) = x(t) + \int_{t-h}^{t} Gx(s)ds$$

to transform the original system. Note that, if G is A_1, then the transformation is the neutral model transformation (Ref. 1). Since the operator $\mathcal{D}(x_t)$ has a free weighting matrix, it yields results less conservative than the results obtained by using the neutral model transformation.

Remark 2.2. The solutions of Theorem 2.1 can be obtained by solving the generalized eigenvalue problem in X, W, F_{11}, F_{33}, Y, F_{12}, F_{13}, F_{23}, ε_i, $i = 0, \ldots, 3$, which is a quasiconvex optimization problem. Note that a locally optimal point of a quasiconvex optimization problem with strictly quasiconvex objective is globally optimal (Ref. 9). In this paper, we utilize the Matlab LMI Control Toolbox (Ref. 13), which implements interior-point algorithms. These algorithms are significantly faster than classical convex optimization algorithms (Ref. 9).

3. Numerical Examples

Example 3.1. Consider the following time-delay systems with nonlinear uncertainties (Ref. 11):

$$\dot{x}(t) = Ax(t) + A_1x(t-h) + f(t, x(t)) + f_1(t, x(t-h)),$$

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

$$f(t, x(t)) = [\delta_1 \cos t \,|\, x_1(t) \,|, \delta_2 \sin t \,|\, x_1(t) \,|]^T,$$

$$f_1(t, x(t-h)) = [\gamma_1 \cos t \,|\, x_1(t-h) \,|, \gamma_2 \sin t \,|\, x_1(t-h) \,|]^T,$$

where

$$|\, \delta_i \,| \le \alpha = 0.05, \quad |\, \gamma_i \,| \le \alpha_1 = 0.1, \quad i = 1, 2.$$

Thus, we have

$$\| f(t, x(t)) \| \le \alpha \| x(t) \|, \ \| f_1(t, x(t-h)) \| \le \alpha_1 \| x(t-h) \|.$$

If we apply Theorem 2.1 to the above system, the result is that the system is asymptotically stable for any h which is less than 1.3334. Note that the stability bound on h is 0.7062 by the Cao and Wang criterion (Ref. 11). The solution via Theorem 2.1 when $h = 1.3333$ is as follows:

$$X = \begin{bmatrix} 2.3668 \times 10^2 & 1.4176 \times 10^2 \\ 1.4176 \times 10^2 & 1.5654 \times 10^6 \end{bmatrix},$$

$$W = \begin{bmatrix} 2.3670 \times 10^2 & 2.1816 \times 10^2 \\ 2.1816 \times 10^2 & 1.4390 \times 10^6 \end{bmatrix},$$

$$Y = \begin{bmatrix} 2.7828 \times 10^{-3} & 2.3039 \times 10^1 \\ -5.9988 \times 10^1 & -4.2754 \times 10^{-5} \end{bmatrix},$$

$$G = YX^{-1} = \begin{bmatrix} 2.9424 \times 10^{-6} & 1.4717 \times 10^{-5} \\ -8.9872 \times 10^{-2} & -2.7310 \times 10^{-1} \end{bmatrix},$$

$$F_{11} = \begin{bmatrix} 2.6627 \times 10^2 & 9.6091 \times 10^1 \\ 9.6091 \times 10^1 & 8.0718 \times 10^5 \end{bmatrix},$$

$$F_{12} = \begin{bmatrix} 2.3668 \times 10^2 & 1.0165 \times 10^2 \\ 1.0165 \times 10^2 & 1.0077 \times 10^6 \end{bmatrix},$$

$$F_{13} = \begin{bmatrix} 8.8758 \times 10^1 & 6.4204 \times 10^2 \\ 1.2220 \times 10^2 & 104.5271 \times 10^5 \end{bmatrix},$$

$$F_{23} = \begin{bmatrix} 1.1834 \times 10^2 & 8.5509 \times 10^1 \\ 1.6112 \times 10^2 & 5.7724 \times 10^5 \end{bmatrix},$$

$$F_{33} = \begin{bmatrix} 8.8759 \times 10^1 & 6.7875 \times 10^1 \\ 6.7875 \times 10^1 & 3.7524 \times 10^5 \end{bmatrix},$$

$$\varepsilon_0 = 2.1317 \times 10^6, \quad \varepsilon_1 = 3.3218 \times 10^6,$$
$$\varepsilon_2 = 2.5304 \times 10^6, \quad \varepsilon_3 = 4.2574 \times 10^6.$$

Example 3.2. Consider the following system present in Ref. 11 $[f(t,x(t))=0,\ f_1(t,x(t-h))=0]$:

$$\dot{x}(t)=Ax(t)+A_1x(t-h),$$

where

$$A=\begin{bmatrix}-3 & -2\\ 1 & 0\end{bmatrix},\quad A_1=\begin{bmatrix}-0.5 & 0.1\\ 0.3 & 0\end{bmatrix}.$$

Applying Theorem 2.1 to the above system, the upper bound of the time delay h which assures the asymptotic stability of the above system is 1.5875. Table 1 shows the comparison of our result with those of others. From Table 1, one can see that our method provides a larger stability bound.

Example 3.3. Consider the following uncertain system with constant time delay (Ref. 12):

$$\begin{aligned}\dot{x}(t)=&\begin{bmatrix}-2+\delta_1\cos(t) & 0\\ 0 & -1+\delta_2\sin(t)\end{bmatrix}x(t)\\ &+\begin{bmatrix}-1+\gamma_1\cos(t) & 0\\ -1 & -1+\gamma_2\sin(t)\end{bmatrix}x(t-h),\end{aligned}$$

where $\delta_1,\delta_2,\gamma_1,\gamma_2$ satisfy

$$|\delta_1|\le 1.6,\quad |\delta_2|\le 0.05,\quad |\gamma_1|\le 0.1,\quad |\gamma_2|\le 0.3.$$

Note that the above system has norm-bounded time-varying uncertainties. The uncertain matrices are chosen as

$$D=E=\mathrm{diag}\left\{\sqrt{1.6},\sqrt{0.05}\right\},\quad D_1=E_1=\mathrm{diag}\left\{\sqrt{0.1},\sqrt{0.3}\right\}.$$

By applying Corollary 2.1 to the above system, we obtained the stability bound h as 1.3334.

In Table 2, we compare our result with those of others. From Table 2, we see that the size of h which guarantees the asymptotic stability of the above system is larger than that provides by the results in other papers.

Table 1. Stability bounds of the time delay for Example 3.2.

Xu (Ref. 5)	Su (Ref. 7)	Yan (Ref. 8)	Cao and Wang (Ref. 11)	This paper
$h=0.0667$	$h=0.1298$	$h=0.4991$	$h=0.7602$	$h=1.5875$

Table 2. Stability bounds of the time delay for Example 3.3.

Li (Ref. 15)	Kim (Ref. 12)	Kharitomov (Ref. 16)	This paper
$h = 0.2013$	$h = 0.2412$	$h = 0.6096$	$h = 1.3334$

4. Conclusions

In this paper, we present a novel stability criterion for the asymptotic stability of time-delay systems with nonlinear uncertainties. Utilizing an operator with a free weighting matrix, we transform the original system to an equivalent time-delay system. Then, the delay-dependent stability criterion is derived in terms of matrix inequalities by establishing a Lyapunov functional with free weighting matrices. Through numerical examples, we showed that the derived criteria are less conservative than those in other published papers.

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