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SURVEY PAPER
Approximation Methods in Multiobjective Programming

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Abstract. Approaches to approximate the efficient set and Pareto set of multiobjective programs are reviewed. Special attention is given to approximating structures, methods generating Pareto points, and approximation quality. The survey covers more than 50 articles published since 1975.

Key Words. Multiobjective programming, approximation, efficient sets, Pareto sets, nondominated sets.

1. Introduction

1.1. Motivation. In multiobjective programming, several conflicting and noncommensurate objective (criterion) functions have to be optimized over a feasible set determined by constraint functions. Due to the conflicting nature of the criteria, a unique feasible solution optimizing all the criteria does not exist. Based on the commonly used Pareto concept of optimality, one has to deal with a rather large number or infinite number of efficient solutions. Two different efficient solutions are characterized by the fact that each of them is better in one criterion but worse in another. The fact that improvement of one criterion results in a loss in another is known as the tradeoff between the solutions.

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The primary goal of multiobjective programming is to seek efficient solutions and/or Pareto outcomes of multiobjective programs (MOPs) and, if possible, support the decision maker (DM) in choosing a final preferred solution. Therefore, it is of interest to design methods for obtaining a complete or partial description of the Pareto set and efficient set, referred to as the solution sets.

An exact description of these sets might be available analytically as a closed-form formula, numerically as a set of points, or in mixed form as a parametrized set of points. Unfortunately, for the majority of MOPs, it is not easy to obtain an exact description of the solution set that includes typically a very large number or infinite number of points. Even if it is theoretically possible to find these points exactly, this is often computationally challenging and expensive and therefore is usually abandoned. For some problems, finding the elements of the solution set is even impossible due to the numerical complexity of the resulting optimization problems. On the other hand, if it is possible to obtain the complete solution set, one might not be interested in this task due to overflow of information.

Another reason for approximating the solution set, rather than finding the solution set exactly, is that many real-world problems (e.g. in engineering) cannot be completely and correctly formulated before a solution procedure starts. The formulating and solving have to be performed interactively with the DM as more details about the solution set become known. The information provided by an approximation can be sufficient to develop a formulation and an exact solution is not necessary.

Since the exact solution set is very often not attainable, an approximated description of the solution set becomes an appealing alternative. Approximating approaches have been developed for the following purposes: to represent the solution set when this set is numerically computable (linear or convex MOPs); to approximate the solution set when some but not all of the efficient or Pareto points are numerically computable (nonlinear MOPs); and to approximate the solution set when the efficient or Pareto points are not numerically computable (discrete MOPs).

For any MOP, the approximation requires less effort and often may be accurate enough to play the role of the solution set. Additionally, if the approximation represents this set in a simplified, structured, and understandable way, it may effectively support the DM. Therefore, approximation quality and a measure for evaluating it are important aspects of the approximating approaches.

In the literature, a variety of approaches to approximate the solution set of MOPs of different types have been proposed. A large majority of the approaches employ an iterative method to produce points or objects approximating this set. Some approaches are exact and based on algorithms equipped with theoretical proofs for correctness and optimality, while some other approaches are heuristic and often theoretically unsupported.

This work categorizes, summarizes, and compares methods presented in more than 50 articles published in the English language since 1975. We focus on exact approaches, since the other methods have been discussed in a number of recent publications [e.g. Deb (Ref. 1), Coello Coello et al. (Ref. 2), or Ehrgott and Gandibleux (Ref. 3)].

Besides the approximation, special attention is given to methods generating approximating points and the quality aspect of the approximation. All the approaches employ a solution method to obtain the points that either become the final approximation or are used to construct other approximating objects. The solution method is an integrated component of the resulting approximating algorithm. The reader is referred to Ehrgott and Wiecek (Ref. 4) for a recent review of solution methods in multiobjective programming.

In Subsection 1.2, we present basic concepts and definitions needed in the survey. In Subsection 1.3, we propose a classification scheme to review and compare the approaches. Sections 2, 3, 4 constitute the main body of the paper: they include a review of methods for biobjective programs (BOPs), MOPs, and a discussion of quality measures, respectively. The paper is concluded in Section 5.

In Sections 2 and 3, the articles are not reviewed in chronological order, but with respect to the increasing degree of complexity of the structure approximating the solution set. Given the same structure, the articles are discussed according to the complexity of the MOP they refer to. Given the same problem category, the articles are presented chronologically.

1.2. Notation and Definitions. In the following, basic definitions and notations used in this survey are given.

The multiobjective program (MOP) is formulated as

$$\begin{aligned} \min \quad & f(x) = (f_1(x), \dots, f_p(x))^T, \\ \text{s.t} \quad & x \in X, \end{aligned}$$

where $X \subseteq \mathbb{R}^n$ is the feasible set, $f: X \to \mathbb{R}^p$ is composed of p real-valued objective functions, and $\mathbb{R}^n$ and $\mathbb{R}^p$ are finite-dimensional Euclidean vector spaces. For the special case $p = 2$, we refer to this problem as the biobjective program (BOP). We define the set of attainable outcomes Y as follows:

$$Y := \{y \in \mathbb{R}^p : y = f(x), x \in X\} = f(X).$$

For $y^1, y^2 \in \mathbb{R}^p$, we use the following notation:

$$y^1 \le y^2 :\Leftrightarrow y^1 \leqq y^2 \text{ and } y^1 \neq y^2,$$
$$y^1 < y^2 :\Leftrightarrow y^1_k < y^2_k, \forall k = 1, \ldots, p.$$

The Pareto cone is defined as

$$\mathbb{R}^p_{\geqq} := \left\{ y \in \mathbb{R}^p : y_k \ge 0 \;\; k = 1, \ldots, p \right\}.$$

According to the Pareto concept of optimality, the efficient set X_E and the weakly-efficient set X_{wE} are defined as

$$X_E := \left\{ x \in X : \nexists \bar{x} \in X : f(\bar{x}) \le f(x) \right\},$$
$$X_{wE} := \left\{ x \in X : \nexists \bar{x} \in X : f(\bar{x}) < f(x) \right\}.$$

The images of these sets under the vector-valued mapping f,

$$Y_N := f(X_E), \quad Y_{wN} := f(X_{wE}),$$

are called the Pareto set and the weak Pareto set, respectively. If the optimality modeled by the Pareto cone is generalized to the optimality with respect to a pointed convex cone, then the solution set in the objective space is called the (weakly) nondominated set. Throughout the paper, we use the Pareto concept of optimality unless stated otherwise. A point $y^2 \in \mathbb{R}^p$ is called dominated by $y^1 \in \mathbb{R}^p$ if $y^1 \le y^2$.

Several points in the outcome space serve as auxiliary points when constructing approximations. Given

$$y^I_k := \min\{f_k(x) : x \in X\}, \quad k = 1, \ldots, p,$$
$$y^{AI}_k := \max\{f_k(x) : x \in X\}, \quad k = 1, \ldots, p,$$

the ideal point y^I, utopia point y^U, anti-ideal point y^{AI}, and antiutopia point y^{AU} are defined as

$$y^I := (y^I_1, \ldots, y^I_p)^T,$$
$$y^U := y^I - \epsilon,$$
$$y^{AI} := (y^{AI}_1, \ldots, y^{AI}_p)^T,$$
$$y^{AU} := y^{AI} + \epsilon,$$

where $\epsilon \in \mathbb{R}^p$ is a vector with small positive components. An attainable Pareto outcome y^{IM_k} having y^I_k as its k^{th} component is called an individual minimum of the k^{th} objective function (note that y^{IM_k} is not unique).

The set of all individual minima is denoted by IM. Furthermore, given

$$y_k^N := \max\{f_k(x) : x \in X_E\}, \quad k = 1, \ldots, p,$$

the point $y^N := (y_1^N, \ldots, y_p^N)^T$ is called the nadir point. Since its calculation is not easy, the nadir point is often estimated.

The ideal and the (estimated) nadir points determine bounds $[y^I, y^N]$ of the Pareto set such that $y^I \leq y \leq y^N$ for all Pareto outcomes $y \in Y_N$. The projection of Y_N onto the axis of the k^{th} objective function, $k = 1, \ldots, p$, is contained in the interval given by

$$Y_N^k := \left[y_k^I, y_k^N\right], \quad k = 1, \ldots, p.$$

The finite family of all these intervals is referred to as the range of the Pareto set, whereas the range of the attainable set is given by the intervals

$$Y^k := \left[y_k^I, y_k^{AI}\right], \quad k = 1, \ldots, p.$$

An aspiration point is a desired point in the outcome space which helps defining parameters for solution methods such as weights, reference points, etc.

The convex Pareto hull of a set of point $\{v^1, \ldots, v^s\} \subset \mathbb{R}^p$ is defined as $\{y \in \mathbb{R}^p : y \geq \sum_{k=1}^s \lambda_k v^k, \sum_{k=1}^s \lambda_k = 1, \lambda_k \geq 0\}$. The convex hull of the individual minima of the objective functions, abbreviated CHIM, is defined as

$$\text{CHIM} := \left\{ y \in \mathbb{R}^p : y = \sum_{k=1}^{p} \lambda_k y^{IM_k}, \sum_{k=1}^{s} \lambda_k = 1, \ \lambda_k \geq 0 \right\}.$$

Different weight vectors are used in the reviewed approximation methods. One particular set of weights is denoted by W and is defined as

$$W := \left\{ w \in \mathbb{R}^p : \sum_{i=1}^{p} w_i = 1, w \geq 0 \right\}.$$

1.3. Classification of Approximation Approaches. The subject of the approximation of the solution set of MOPs has been of interest to scientists and engineers for a period of thirty years. To our knowledge, attention to this subject was given first in the 1970s [see for example Payne et al. (Ref. 5) or Polak (Ref. 6)]. A large majority of articles deal with the approximation of the Pareto set and only few articles study the approximation of the efficient set. In this paper, an approximation $\mathcal{A}$ is a set of points in $\mathbb{R}^p$ (or in $\mathbb{R}^n$) considered a surrogate of (a part of) the Pareto (efficient) set usually of a simpler structure than the approximated set.

We introduce a classification scheme that helps comparing the approximation approaches. The articles are categorized with the respect to the number of criterion functions of the MOPs they addres. We distinguish between approaches for BOPs and approaches for MOPs. Within each of the two groups, we further classify the approaches according to the structure of the approximating functions. Table 1 lists the approximation classes and the related approximating functions.

A subclass of 1[st] order approximations are the so-called sandwich approximations. They are composed of a piecewise linear inner approximation, $\mathcal{A}_I$ and a piecewise linear outer approximation $\mathcal{A}_O$.

Inner means that, for a fixed f_i-value, $i \in \{1, \dots, p\}$, every f_j-value, $j \neq i$, of the approximation is never smaller than the f_j-value of any Pareto point with the same f_i-value. Outer is defined accordingly. Given $\mathcal{A}_I$ and $\mathcal{A}_O$ the area in between the inner and the outer approximation includes all the theoretically possible Pareto sets having the generated Pareto points as elements; i.e., the Pareto set satisfies the following property:

$$Y_N \subset ((\mathcal{A}_I + (-\mathbb{R}^p_{\geqq})) \cap (\mathcal{A}_O + \mathbb{R}^p_{\geqq})).$$

2. Biobjective Approaches

2.1. Approximations of 0th Order. As indicated above, pointwise or discrete approximations are called approximations of the 0[th] order. Discrete approximations are among the simplest structures one can think of (which does not diminish their effectiveness). Pareto outcomes are generated by a solution method and serve directly as approximating points. No further structure is computed.

Jahn and Merkel (Ref. 7) and Helbig (Ref. 8) propose methods for continuous BOPs. Jahn and Merkel (Ref. 7) use the ε-constraint method solved with a tunneling technique to obtain Pareto solutions. The interval

Table 1. Approximation classes.

Approximation class	Approximating functions and sets
0th order	Points
1st order	(Piecewise) linear functions
2nd order	(Piecewise) quadratic functions
3rd order	(Piecewise) cubic functions
Other	Other functions and sets

on the f_1-axis defined by the individual minima is discretized. The resulting points are used as the right-hand-side parameters in the ε-constraint method. An interactive algorithm is proposed to find a preferred solution.

In Helbig (Ref. 8), the CHIM is discretized and, for each discretizing point, the max-ordering method with a feasible reference point is used to obtain Pareto outcomes. An interactive algorithm for finding a preferred solution is also proposed.

Schandl et al. (Ref. 9) propose an approach for discrete BOPs. A combination of the lexicographic direction method and a norm method yields the Pareto outcomes. Since the method applies to other problems as well, more details are given in Subsection 2.2.

2.2. Approximations of 1st Order. For piecewise linear approximations, Pareto points are obtained first with a solution method and then connected with line segments (the convex hull of two adjacent outcomes).

Inner Approximations. Piecewise linear inner approximations are proposed by Das (Ref. 10) and Schandl et al. (Ref. 9). Although each method yields a piecewise linear curve connecting adjacent Pareto points found as the approximating structure, the methods are based on different concepts.

The initial approximation in Das (Ref. 10) is provided by y^{IM_1}, y^{IM_2}, and the Pareto point having the maximum l_2-distance from the CHIM. This point and subsequent Pareto points are found with a modified normal-boundary intersection approach. This approach is applied iteratively to the convex hull of each pair of adjacent points that do not meet a prescribed value of the l_2-distance from the Pareto set. In each iteration, under convexity assumptions, an additional Pareto point with maximum l_2-distance to the Pareto set is found.

The approximation proposed by Schandl et al. (Ref. 9) is part of the unit ball of an oblique norm. The initial approximation is given by two Pareto points, each of which is obtained with a modified direction approach. Consecutive points are found with the norm method in which the norm of the currently approximating unit ball is maximized. Similarly to Das (Ref. 10), under convexity assumptions, the point of worst approximation is added to the approximation in each step of the procedure. The quality is measured by the approximating norm itself. The approximation is scaling independent and allows us to explore regions of interest in more detail.

Sandwich Approximations. Two types of sandwich approximations have been proposed. One of them uses exclusively piecewise linear curves to construct the inner and outer approximations for convex problems. In the other type, piecewise linear curves result in rectangular approximating sets for general problems.

Cohon et al. (Ref. 11), Fruhwirth et al. (Ref. 12), and Yang and Goh (Ref. 13) propose similar sandwich approaches belonging to the first type. Adjacent Pareto points are used to construct an approximation. An inner approximation comes from the convex hull of these points, while an outer approximation is provided by the concatenation of line segments that support the Pareto set at the generated Pareto points.

In each of the three algorithms, the initial approximation is given by the CHIM. The initial sandwich approximation is given by a triangle spanned between y^{IM_1}, y^{IM_2} and the ideal point [Cohon et al. (Ref. 11)] or by y^{IM_1}, y^{IM_2} and the intersection point of the hyperplanes supporting the Pareto set at y^{IM_1} and y^{IM_2} [Fruhwirth et al. (Ref. 12)], or by a trapezoid determined by y^{IM_1}, y^{IM_2} and a generated Pareto point [Yang and Goh (Ref. 13)]. To generate Pareto points, Cohon et al. (Ref. 11) and Yang and Goh (Ref. 13) use the weighted-sum method with weights whose ratio is equal to the negative slope of the current inner approximation, while Fruhwirth et al. (Ref. 12) use the hybrid method and offer different choices for the weights. The process of calculating new Pareto outcomes and updating the approximation is repeated until a distance between the inner and outer approximation is satisfactory.

Solanki and Cohon (Ref. 14) refine the outer approximation of Cohon et al. (Ref. 11) using a multiparametric decomposition, while still using the same inner approximation and stopping criterion. The approach results in a more accurate outer approximation, thus reducing the computational effort for meeting a prescribed accuracy level.

Ruhe and Fruhwirth (Ref. 15) modify the algorithm proposed by Fruhwirth et al. (Ref. 12) in order to compute an ε-approximation of the Pareto set additionally to the sandwich approximation proposed in Fruhwirth et al. (Ref. 12). This ε-approximation is a piecewise linear inner approximation with the property that every Pareto point becomes dominated by some approximating point when contracting the approximation by the factor $1/(1+\varepsilon)$.

Payne and Polak (Ref. 16), Solanki (Ref. 17), and Payne (Ref. 18) propose sandwich approximations of the second type. In the first two papers, nested rectangles are constructed, while in the latter a chain of rectangles is built to approximate the Pareto set. The Polak-Payne method is used to find Pareto points in Payne and Polak (Ref. 16) and Payne (Ref. 18).

In the first step of the method in Payne and Polak (Ref. 16), a rectangle enclosing the Pareto set in computed. New Pareto points are then found to define new rectangles contained in the initial rectangle. The DM chooses interactively one of the rectangles as the rectangle of interest. The approximation of the part of the Pareto set that is contained in the rectangle of interest is refined. New Pareto points are determined and the rectangle of interest is cut down into smaller rectangles. This step of finding new points and decomposing big-rectangles into smaller ones is repeated until the approximation is accurate enough.

A similar scheme takes place in the approach by Solanki (Ref. 17) for mixed-integer linear BOPs. The initial rectangle is spanned between $y^{IM_k}, k = 1, 2$. The augmented weighted Chebyshev method yield a new Pareto outcome within the rectangle. This outcome is used to construct two new rectangles, each spanned between a point y^{IM_k} and the outcome itself. In every iteration, the solution method is employed within the rectangle with the biggest approximation error that is calculated as the maximum height or width of all the rectangles constructed so far. The algorithm produces an approximation of the Pareto set of a desired accuracy with as few Pareto outcomes as possible.

In Payne (Ref. 18), the rectangles are constructed sequentially starting at y^{IM_1} and working toward y^{IM_2}. Each rectangle in the chain has a prescribed area. An upper bound for the number of rectangles of a desired area can be computed.

2.3. Approximations of the 2nd Order. Wiecek et al. (Ref. 19) propose a piecewise quadratic approximation for nonconvex BOPs. They use the lexicographic weighted Chebyshev method and the ε-constraint method to find Pareto outcomes. Based on the Lagrangian dual problem associated with the weighted Chebyshev problem, a quadratic approximating function is derived and is available immediately as a closed-form description of the approximation. If the maximum distance between the known Pareto points and the approximation is not satisfactory, the interval of interest is shortened and the procedure is started again. Approximation of the entire Pareto set may be achieved by concatenation of local approximating curves.

2.4. Approximations of 3rd Order. Payne et al. (Ref. 5) and Polak (Ref. 6) use cubic functions to interpolate Pareto points that are found with the ε-constraint method. The quality of the approximation is measured by the interpolation error. If the desired quality is not yet obtained, additional Pareto points are computed.

2.5. Other Approximations. Some authors have proposed approximations of unspecified orders or orders higher than three.

The sandwich algorithms of Fruhwirth et al. (Ref. 12) and Yang and Goh (Ref. 13) are used by Liu et al. (Ref. 20) to approximate a nonconvex Pareto curve after it has been convexified by an r^{th} power transformation.

The southwest quarter of a hyperellipse is used in Li et al. (Ref. 21) and Fadel and Li (Ref. 22) to approximate the Pareto set for convex BOPs. Since three points can determine uniquely a hyperellipse, the IM is supplemented by one additional Pareto outcome found with the weighted-sum method. Knowing three different points, a root finding problem has to be solved to compute the power of the hyperellipse passing through these three points. While the closed-form formula of the approximating hyperellipse is found quickly and almost effortlessly, information about the approximation error or quality is not available.

In Li et al. (Ref. 21), an interactive optimization step is added after an approximating hyperellipse has been found informing the DM about the overall shape of the Pareto set. The DM can look for specific Pareto points by specifying a preferred value for one or each objective function. For the former, the ε-constraint method is applied to find the Pareto point with the desired value in one objective. For the latter, goal programming delivers a Pareto point with minimum deviation from the selected approximating outcome.

3. Multiobjective Approaches

In contrast to methods for BOPs, only three types of approximating structures are used to approximate the Pareto set of MOPs. This is probably due to the fact that it is often difficult to deal with objects in more than two dimensions. Therefore, methods for MOPs are divided into only three subgroups: approximations of the 0^{th} order using discrete points, approximations of the 1^{st} order using polyhedral sets, and approximations using other sets.

3.1. Approximations of 0th Order. Among all the articles we have reviewed, only two deal with the pointwise approximations of the efficient set. Therefore, approximations of the 0^{th} order are divided into approaches dealing with the efficient set and the Pareto set.

Approximations of the Efficient Set. Approaches to find a pointwise approximation of the efficient set of a desired quality are given by Popov (Ref. 23) and Nefëdov (Ref. 24). Both authors assume that the data about the MOP are incomplete, i.e., the feasible set X and the objective functions $f_k, k=1,\dots,p$, are not available. Instead, the given data include the set X^τ and functions $f_k^\epsilon(x), k=1,\dots,p$, such that the Hausdorff distance between X and X^τ is bounded by τ and

$$|f_k^\epsilon(x) - f_k(x)| \leq \epsilon, \quad \text{for some } \tau, \epsilon \geq 0.$$

In Popov (Ref. 23), a two-stage weighted max-ordering method is used to find efficient points. Under regularity conditions and given the initial parameters τ and $\epsilon \geq 0$ as well as some other parameters, it is shown how to construct a grid W^δ on the set of weights W so that the efficient points obtained with the max-ordering method result in an approximation of the efficient set of a desired accuracy.

Nefëdov (Ref. 24) builds upon previous results of Nefëdov (Ref. 25). The approximation process goes through two stages. In the first stage, a set approximating the Pareto set is constructed from which in the second stage an inverse is separated in the set X^τ. Two general approaches for stage one are discussed: a set approximating the Pareto set can be found using points from the set X^τ or by means of methods approximating the Pareto set and available in the literature [e.g. Nefëdov (Ref. 25)]. The author proposes a measure of the approximation error that, under suitable conditions, implies the measure based on the Hausdorff distance between the efficient set and the approximating set. For both stage-one approaches, he derives results on how to choose approximation parameters in order to construct a pointwise approximation of the efficient set so that (i) the maximum distance between the image of the approximating set and the Pareto set tends to zero as the approximation parameters go to zero and (ii) the maximum distance between the image of the approximating set and the Pareto set is bounded by a desired quantity.

Approximations of the Pareto Set. A single-point approximation is proposed by Das (Ref. 26) who advocates that, due to the high computational costs of solving nonlinear MOPs, the Pareto point with the maximum l_2-distance to the CHIM is a good approximation of the Pareto set. That point is found with modified normal-boundary intersection method and is referred to as the maximum bulge of the Pareto set.

Approaches for linear MOPs have been proposed by Steuer and Harris (Ref. 27), Reuter (Ref. 28), and Sayın (Ref. 29). Steuer and Harris (Ref. 27) assume the Pareto set to be known prior to the procedure. For each Pareto face, approximating points in the interior of the face are generated using convex combinations of the extreme points defining this face. The approximating points having a small, unsatisfactory l_p-distance from each other are removed temporarily from consideration. The DM then chooses a first preferred outcome and approximating points being in the l_p-neighborhood of this points are presented to the DM. Among these, the DM chooses a final preferred outcome.

In Reuter (Ref. 28), a concept of ε-approximation is introduced which differs from the one used by Ruhe and Fruhwirth (Ref. 15). The range of the attainable set is determined first. Depending on a parameter $\varepsilon \geq 0$, the range is divided into several intervals of equal length. Pareto points are found using the hybrid method with the right-hand side parameters provided by the intervals bounds. The obtained Pareto points form an ε-approximation in the sense that the translation of the discrete approximation by the vector $-\varepsilon$ makes all the Pareto points dominated by the approximating points; in other words, the Pareto set is contained in the union of the Pareto cones attached to the translated approximating points.

Like in Steuer and Harris (Ref. 27), Sayın (Ref. 29) assumes that the Pareto set of the linear MOP is known already and the main goal of her approach is to compute a discrete approximation of a certain quality. While the former suggest to first generate and then delete approximating points to achieve a quality approximation, Sayın (Ref. 29) generates only points which improve the current level of quality. Given an initial Pareto outcome, other Pareto points in the same Pareto face are found iteratively by solving a single-objective program whose numerical complexity depends on the norm used in this program. For the case of the l_∞-norm, it is shown that the problem becomes a mixed linear-integer program. In each iteration, a point of worst approximation is added to the representation of the face.

Smirnov (Ref. 30) deals with convex and linear MOPs. A subset W^δ of the set W is chosen such that, for each $w^i \in W^\delta$, there is a $w^j \in W^\delta$ such that $\|w^i - w^j\|_\infty \leq \delta$, where δ is a chosen scalar. A weighted max-ordering method is applied for each weight vector in W^δ. The resulting approximation forms an ε_1-grid on Y_N, where ε_1 depends on δ and some other parameters. In the convex case, the approximation is also ε_2-optimal in the sense of Reuter (Ref. 28), where again ε_2 depends on δ and some other parameters. For the linear case, it is additionally shown how unnecessary computations can be avoided.

Two approximation approaches based on two different versions of the Chebyshev problem are investigated numerically in Buchanan and Gardiner (Ref. 31) and applied to approximating the Pareto set of an n-dimensional unit hypersphere and linear MOPs. The hyperrectangle defined by the ideal points and nadir points is discretized by evenly distributed points serving as aspiration points, that in turn are used to define weights with which both Chebyshev methods are solved. The Pareto points obtained with each formulation of the Chebyshev method form two approximations. The authors observe that one method produces less uniformly distributed approximating points having larger gaps near the coordinate axes than the other.

A 0^{th} order approximation is constructed in Fliege and Heseler (Ref. 32) for problems with convex quadratic objective functions and polyhedral feasible sets. Approximating points are generated by solving weighted-sum problems with an interior-point method. The weight vectors of these problems are changed sufficiently small to apply a warm-start strategy to the interior-point method: the weight vector is altered systematically so that a point found by solving a previous weighted-sum problem can be used as a starting point for solving the current weighted-sum problem. The approximating points are generated iteratively until all the components of the weight vectors have covered the range between 0 and 1.

Das and Dennis (Ref. 33) propose an approach for convex MOPs. A set of equidistant reference points on the CHIM is generated and, for each of them, a Pareto point is found using the normal-boundary intersection method. While the solution method may overlook a portion of the Pareto set for $p>2$, it is expected to produce a set of equidistant approximating points.

MOPs with convex objective functions are also addressed in Churkina (Ref. 34). First, it is algebraically shown that an infinite set of reference points, together with the Chebyshev method applied to this set, characterize the set of weak Pareto points. The Chebyshev method is then applied to a finite subset of this set of reference points to produce an approximation. Since the resulting outcomes may not be Pareto, the weighted-sum method is applied to discard the weak Pareto solutions from the approximating points found. While an approximation algorithm is not given, it is proved that there exists a finite number of reference points such that the Chebyshev distance from any Pareto point to some approximating point is satisfactory.

The approximation method in Helbig (Ref. 35) applies to MOPs whose outcome set is C-convex, where C is a convex cone subsuming the Pareto cone with respect to which nondominated points are sought. The use of a nonPareto cone yields two benefits: (i) a subset of the Pareto set can be potentially reached and (ii) perturbation of the original cone allows for the exploration of Pareto points located within a small region of interest. Each point of the approximation is generated with the Pascoletti-Serafini method, which is a direction method using a general convex cone. The algorithm requires a parameter for cone perturbation, a direction, and a set of reference points for each of which a Pareto point is found with the solution method. If the resulting set of Pareto points is satisfactory, they comprise the final approximation. Otherwise, the set of reference points and/or the perturbation parameter is changed and the process continues. Theoretical results on the relationships between weakly nondominated solutions with respect to a perturbed cone and on the relationships between nondominated solutions with respect to the original one are included.

Approximation methods for general MOPs have been developed by many authors. In Nefëdov (Ref. 25) and Abramova (Ref. 36), it is assumed that data about the MOP is incomplete in the sense of Popov (Ref. 23) and Nefëdov (Ref. 24). In Nefëdov (Ref. 25), two approximation approaches using a variation of the weighted max-ordering method are presented; in Abramova (Ref. 36), the weighted-norm scalarization is applied to find Pareto points. In these articles, the approximation results are similar but based on different solution methods. The quality of the approximation is understood in the sense of Popov (Ref. 23) and Reuter (Ref. 28). Given the initial parameters τ and $\epsilon \geq 0$ as well as some other parameters, it is shown how to construct a grid W^{δ} on the set of weights W so that the points obtained with the employed solution methods result in an approximation of the Pareto set of a desired accuracy measured with the Hausdorff metric. The approximating set has several properties. The distance between this set and the Pareto set is measured by the Hausdorff metric and is bounded. The bound is calculated based on the parameters $\tau, \epsilon \geq 0$ and other parameters used in the derivation of the bounds. It is shown that, if the parameters tend to zero, the measured Hausdorff distance also goes to zero. Additionally, for every Pareto outcome, there exists a point in the approximating set within a desired distance. In Nefëdov (Ref. 25), a method of filtering (contracting) the approximating set is also included.

The approach of Armann (Ref. 37) produces a discrete approximation of the Pareto set in which a desired number of Pareto points is contained. The payoff table is found first yielding p Pareto points and initial bounds on the objective functions. An auxiliary single-objective integer program is solved to find the number of Pareto points that need to be found for each objective function so that the total number of generated Pareto outcomes is equal to the desired number. Pareto points are generated with two variations of the hybrid method enabling us to find these points with respect to maximally dispersed bounds on each axis in the objective space.

Kostreva et al. (Ref. 38) address very general problems: the objective functions may be discontinuous and the feasible set may be disconnected. A suitable utopia point as well as a finite set of weight vectors are chosen prior to the procedure. For each weight vector, the weighted Chebyshev method is solved using the utopia point as a reference point. The resulting approximating points may be connected with simplices, which will serve as the final approximating structure.

A global shooting procedure is proposed by Benson and Sayın (Ref. 39). The anti-ideal point y^{AI} is used to construct a simplex S

containing the outcome set. The constructing of the simplex requires the solution of the weighted-sum problem with equal weights. Directions in the subsimplex $S_0 \subset S$ defined by the extreme points of $S \setminus \{y^{AI}\}$ are determined and used to run the direction method, which produces points on the boundary of the outcome set. The Benson method examines whether these boundary points are Pareto or not, in which case new Pareto points are obtained.

Karaskal and Köksalan (Ref. 40) propose an approach for general MOPs but emphasize its effectiveness for convex problems. They estimate first the nadir point and compute initial Pareto points using a constrained Chebyshev method. A weighted L_p-hypersurface is then fitted through the Pareto points around the nadir point so that the sum of the squared deviations between the hypersurface and the Pareto points is minimized. Once equidistant points on the hypersurface are determined, they are projected on the Pareto set using the achievement scalarizing function method. The resulting Pareto points form a discrete approximation.

The articles reviewed below originate from the engineering community. While the proposed approaches are not mathematically justified, in a rigorous way, they are motivated by real-life applications and offer insight in and valuable information about the state of the art of engineering perspective.

Wilson et al. (Ref. 41) propose a method of surrogate approximations for MOPs that are not available in a closed form, which is quite common in engineering applications. The feasible set is sampled first using an experimental design technique and a surrogate approximation of the outcome set is constructed based on a surrogate model of choice (e.g., second-order polynomial surfaces, kriging models). If the approximation is not satisfactory, additional sample points may be taken to improve the accuracy. Otherwise, points of the Pareto set of the approximation are generated with a Pareto fitness function.

Physical programming is used in Messac and Mattson (Ref. 42) as a basis for approximation. After the range of the Pareto set has been determined; a set of equidistant parameters covering that range is found. For each of these parameters, a Pareto outcome is generated with the scalarization method of physical programming.

Similarly to Das and Dennis (Ref. 33), Ismail-Yahaya and Messac (Ref. 43) determine equidistant reference points on the CHIM expecting that they will yield equidistant approximating points. The normal-constraint method is used for each of the reference points to get approximating points that are not guaranteed to be Pareto. Within the set of produced candidates, dominated points are filtered and deleted, while the remaining points, being Pareto with respect to one another, form an approximating set.

Mattson et al. (Ref. 44) build upon the ideas of Ismail-Yahaya and Messac (Ref. 43) and Steuer and Harris (Ref. 27) and refine the filtering process. Not only dominated points are deleted from the candidate set, but also points that are too close to each other. In contrast to Steuer and Harris (Ref. 27), Mattson et al. (Ref. 44) do not use an l_p-metric, but a cross-shaped region to measure the closeness of the approximating points.

3.2. Approximations of 1st Order. Similar to the biobjective case, approximations of the 1st order are classified according to their structure and location with respect to the Pareto set.

Inner Approximations. Piecewise linear inner approximations are proposed by Chernykh (Ref. 45) and Schandl et al. (Ref. 46). Chernykh (Ref. 45) deals with a special class of convex MOPs whose outcome set is convex. He approximates the Pareto hull of the outcome set from inside by the intersection of linear halfspaces. First, a weak Pareto outcome is found by minimizing one of the objective functions. The Pareto hull of this outcome is constructed and given in the form of a system of inequalities. For each newly created inequality, the weighted-sum method with weights coming from the coefficients of these inequalities is applied to determine other weak Pareto outcomes. Among the weak Pareto outcomes found, the one with the biggest orthogonal projective distance from the hyperplane defined by the inequality producing this outcome is identified as a candidate to be added to the set of approximating points. If this projective distance is satisfactory, the approximation is considered good enough and the algorithm stops. Otherwise, the approximation is updated by calculating the convex hull of the current approximation and the candidate point.

In Schandl et al. (Ref. 46), a polyhedral inner approximation is presented for convex, nonconvex, and discrete MOPs. Gauges [see Minkowski (Ref. 47)] measuring the distance between a feasible reference point and the Pareto outcomes are used and the polyhedral structure of their unit ball is exploited heavily in the proposed algorithm. For convex problems, the approximating structure is the unit ball of a gauge, while for nonconvex problems the constructed polyhedral set is nonconvex and may be viewed as the level curve of another function. The gauge method delivers a set of candidate approximating points and, at the same time, the distances between the candidate points and the current approximation. If the maximum of these distances is not satisfactory, the candidate point of worst approximation is added to the set of approximating points and the next iteration is started (for the convex case, the added point is of globally worst approximation). Otherwise, the iterative step ends and the final approximation is obtained. For nonconvex and discrete MOPs, the lexicographic weighted Chebyshev method is used to generate candidate points in a cone. The final set of approximating points may include non-Pareto points that should be removed.

Outer Approximations. In this category, there are two approaches designed for linear MOPs. Similar to Wiecek et al. (Ref. 19), Voinalovich (Ref. 48) uses the notion of duality to derive an approximating structure. A dual linear program to a single objective formulation of the Chebyshev problem is found and solved for optimal dual variables with which a system of linear inequalities is set up. The resulting outer polyhedral approximation is algebraically given by the inequalities that hold with equality at the weak Pareto outcomes.

The other approach is proposed by Benson (Ref. 49). A polyhedron, referred to as an efficiency-equivalent polyhedron (EEP) and having the same Pareto set as the original problem, is approximated by means of an approximating polyhedron (AP). Upon the initialization, a reference point in the interior of the EEP is found and a polyhedral set, being the first AP and containing the EEP, is constructed. The algorithm stops if the AP equals the EEP. If not, a point of the AP is found that is not contained in the EEP. This point and the reference point determine the direction used in the direction method to find a Pareto point. A Pareto face that contains this Pareto point is then computed. The AP is updated by intersecting the halfspace defined by this face with the current AP. The algorithm terminates after a finite number of steps and constructs the Pareto set exactly.

The only approach for nonconvex MOPs is drafted by Kaliszewski (Ref. 50). Using the augmented or modified weighted-Chebyshev method, a finite set of Pareto outcomes is generated. At every outcome, the cone determined by the level set associated with the Chebyshev norm level curve generating and passing through that point is constructed. The intersection of the complements of all the cones provides an approximation of the set of outcomes and implicitly an approximation of the Pareto set.

Sandwich Approximations. Solanki et al. (Ref. 51) and Klamroth et al. (Ref. 52) propose sandwich approximations that combine inner and outer approximations into an approximating structure.

The work of Solanki et al. (Ref. 51) is an extension of the bicriteria approach for linear MOPs. As in the bicriteria case, a polyhedral inner and a polyhedral outer approximations are constructed, which is a more complex task due to multiple dimensions. Initially, the hyperplanes supporting Y_N at y_k^{IM}, for all $k = 1, \ldots, p$, and the CHIM define the outer and inner approximation, respectively. For each face of the inner approximation, the weighted-sum problem is solved with the weights defined by the hyperplane including this face. If the projective distance from the newly-generated points to the corresponding faces is satisfactory, the algorithm stops. Otherwise, a point with the biggest distance is added to the set of approximating points and the inner approximation is updated by computing the convex hull of all the approximating points that have been found so far. The outer approximation is updated by computing the intersection of the previous outer approximation and the halfspace defined by the hyperplane passing through the newly found approximating point. Unlike the biobjective case, several difficulties can arise such as generation of dominated approximating points or nonconvexity of the approximation. Methods for tasking care of these difficulties are proposed.

We observe that the inner approximation in Chernykh (Ref. 45) and the inner approximation in the sandwich algorithm of Solanki et al. (Ref. 51) both require the computation of the convex hull of points found by the weighted-sum method. We note also that the outer approximation of Benson (Ref. 49) and the sandwich algorithm of Solanki et al. (Ref. 51) are updated in the same fashion: First, a hyperplane is generated using the weighted-sum method and the direction method, respectively. Then, the approximation is updated by intersecting it with a suitable halfspace defined by the hyperplane.

Klamroth et al. (Ref. 52) build upon and improve significantly the approach of Schandl et al. (Ref. 46) in the sense that all the generated points are weak Pareto and, for all types of MOPs and in each iteration, the point of (globally) worst approximation is added. Inner and outer approximation algorithms for convex and nonconvex MOPs are developed. All the four algorithms produce an approximating set based on gauges and their unit balls. Similar to Schandl et al. (Ref. 46), the algorithms use an initial attainable reference point becoming the origin of the unit ball of a gauge. In the inner approach, facets of the cones with the apex at an initial attainable reference point determine the approximation. In each cone, a modified gauge problem is solved to generate (globally) weak Pareto candidate outcomes. The outer approach can be considered the dual to the inner approach. The approximation is determined by the intersection of hyperplanes that generate halfspaces containing an attainable reference point and are identified by solving the direction problem with the directions provided by the extreme points of the fundamental vectors of the currently constructed unit ball of a gauge.

3.3. Other Approximations. The Pareto set of general MOPs is approximated in Mateos and Rios-Insua (Ref. 53) and in Mateos et al. (Ref. 54) by means of a vector-valued utility function specified by the DM.

The utility function can be modified with a cone (subsuming the Pareto cone) also specified by the DM, which reduces the subset of the Pareto set being approximated. The Pareto points needed for an initial approximation are found with the lexicographic method. The initial approximation is constructed as the intersection of the outcome set with the level sets of a utility function (of unspecified order) passing through these points. The convergence property of the approximation is proved; i.e., a sequence of nested cones converging to a halfspace (while the sequence of the dual cones converges to a ray) results in a sequence of nested approximating sets each of which includes the same Pareto point.

4. Quality Aspects and Measures

Measures of an approximation error or quality have not gained much attention although almost every approach with an iterative step uses implicitly an error or quality measure. Papers authored by Popov (Ref. 23), Nefëdov (Ref. 25), Nefëdov (Ref. 24), and Abramova (Ref. 36) seem to be the first ones to focus closely on these issues in the context of their own approximation approaches. They are the only authors to consider the Hausdorff distance between the solution set and the approximating set and they obtained results on how to choose approximation parameters to control this distance.

Independently of the literature mentioned above and more recently, Benson and Sayın (Ref. 39) have recognized the need for comprehensive measures of approximation quality. Sayın (Refs. 55 and 29) continues on the earlier ideas and proposes three measures of approximation quality: the cardinality of the approximating set, the maximum distance between two approximating points (uniformity), and the distance of the worst represented element to its closest approximating point (coverage). The work of Benson and Sayın is universal, since it has established tools for measuring the quality independently of a specific approximation approach. Those tools are employed by Karaskal and Köksalan in their computational study.

We classify quality measures into two groups: the first group deals only with the approximating structure, whereas the second group uses the approximating structure and the solution set.

4.1. Measures of Approximation. (i) The cardinality (C) is defined as the number of approximating points generated for the approximation and used to control the size of the approximation. The approaches by Armann (Ref. 37), Solanki (Ref. 17), Jahn and Merkel (Ref. 7), Helbig

(Ref. 8), Das (Ref. 26), Schandl et al. (Ref. 9), Schandl et al. (Ref. 46), and Klamroth et al. (Ref. 52) are capable of either controlling directly the number or at least an upper bound on the number of generated approximating points.

(ii) The dispersion (Di) measures the scattering or distribution of the approximating points. The concept is used to avoid clusters of approximating points, i.e., overrepresented regions of the solution set, and to achieve an even, possibly equidistant spread of approximating points over this set. Dispersion is achieved and measured by different means. Sayın (Ref. 29) maximizes an l_p-distance between the approximating points. Mattson et al. (Ref. 44) and Steuer and Harris (Ref. 27) delete points that are too close to each other. While the former uses a cross-shaped region for measuring distance, thus reducing the cardinality of the approximating set, the latter employs l_p-metrics for the same purpose. Helbig (Ref. 8), Das and Dennis (Ref. 33), and Ismail-Yahaya and Messac (Ref. 43) determine equidistant points on the CHIM, whereas Karaskal and Köksalan (Ref. 40) determine equidistant points on an l_p-hypersurface. Various projections of these points deliver approximating points. Armann (Ref. 37) and Reuter (Ref. 28) compute maximally dispersed reference points on the axes in the outcome space, which are used as parameters for the hybrid method. Messac and Mattson (Ref. 42) use equidistant points in the range of the Pareto set as bounding parameters for a solution method. Buchanan and Gardiner (Ref. 31) discretize the bounds of the Pareto set with equidistant aspiration points. The algorithms in Steuer and Harris (Ref. 27), Mattson et al. (Ref. 44), and Sayın (Ref. 29) control dispersion directly, while in all other approaches the resulting approximating points are expected to be dispersed similarly to the set of auxiliary points.

(iii) The spread (S) informs about the bounds of the Pareto set. In all sandwich approaches, an initial area subsuming the Pareto set or a part of the Pareto set to be approximated is determined. In each iteration, this area is made smaller in order to obtain sharper bounds. The approaches by Fruhwirth et al. (Ref. 12) and Yang and Goh (Ref. 13) measure the Hausdorff distance between $\mathcal{A}_I$ and $\mathcal{A}_O$, while Payne and Polak (Ref. 16), Solanki (Ref. 17), and Payne (Ref. 18) use the size of the approximating rectangles to evaluate the quality. However, in some other sandwich approaches [e.g. Solanki et al. (Ref. 51)], information about Y_N is gathered to update the approximating structure.

4.2. Measures of Approximation and Solution Set. (i) The error (E) of the approximation is commonly measured by some notion of distance between the solution set and the approximation. The length of the maximum orthogonal line segment from the approximation to Y_N is used by Cohon et al. (Ref. 11), Solanki and Cohon (Ref. 14), Solanki et al. (Ref. 51), Chernykh (Ref. 45), and Das (Ref. 10). Additionally generated points and their distance to the approximation, like in Wiecek et al. (Ref. 19), or upper bounds on the maximum distance between the approximation and Y_N, like in Payne et al. (Ref. 5), are other means to serve the same purpose. Schandl et al. (Ref. 9), Schandl et al. (Ref. 46), and Klamroth et al. (Ref. 52) use their approximations property of being a gauge to measure the distance from the approximation to Y_N and change their distance measure when updating the approximation. In their work, the approximation itself is used to measure the quality and there is no need to introduce an external, user-dependent measure of quality. Sayın (Ref. 29) assumes Y_N to be known and formulates an algorithm that, in each step, reduces the error and at the same time informs the DM about the cardinality and the dispersion of the approximating points.

(ii) The dominance (Do) is used in two different concepts, both being called ε-approximations. Reuter (Ref. 28) calls an approximation $\mathcal{A}$ an ε-approximation if $Y_N + \varepsilon, \varepsilon \in \mathbb{R}^p$ and $\varepsilon_1 = \cdots = \varepsilon_p$, is dominated by $\mathcal{A}$. An approximation is called ε-approximation in the sense of Ruhe and Fruhwirth (Ref. 15) if $(1 + \varepsilon)Y_N$ is dominated by $\mathcal{A}$, with $\varepsilon \in \mathbb{R}$. The concept of ε-approximation in the sense of Reuter (Ref. 28) was used implicitly by Nefëdov (Ref. 25), Abramova (Ref. 36), and Smirnov (Ref. 30).

5. Summary

This survey contains an overview of the research that has been done since 1975 in the area of the solution set approximation for continuous multiobjective programs.

Due to intended brevity, conciseness, and legibility, the articles are presented in a facilitated version with (important) technical details hidden. Certain aspects of the approaches, such as the solution methods and quality measures, have been selected subjectively by the authors as significant components of the approximation methodology.

The survey is not complete, since the authors are aware of additional publications that, due to publication language other than English or nonavailability of a paper, are not covered in this review. However, they shall be listed in alphabetical order: Beauzamy (Ref. 56), Karyakin (Ref. 57), Lotov et al. (Ref. 58), Polishchuk (Ref. 59), Popov (Ref. 60), Popovici (Ref. 61), Postolica and Scarelli (Ref. 62), Smirnov (Ref. 63), and Yannakakis (Ref. 64).

The approaches covered in this review are listed in alphabetical order in Table 2 together with a brief description of the MOPs which they address (problem description), the solution methods used to obtain approximating points (solution methods), the geometric structure forming the final approximation (structure), and the quality measure employed (quality).

Table 2. Approximation approaches.

Reference	Problem description	Solution methods	Structure	Quality
Abramova (Ref. 36)	M|C^0, nonnegative, bounded|.	Weighted-norm method	0th order	Do, E
Armann (Ref. 37)	M|nonconvex|nonconvex	Hybrid method	0th order	C, Di
Benson (Ref. 49)	M|linear|polyhedral	Direction method	1st order	
Benson and Sayin (Ref. 39)	M|nonconvex|.	Weighted-sum method Direction method Benson method	0th order	
Buchanan and Gardiner (Ref. 31)	M|convex|.	Chebyshev method	0th order	Di
Chernykh (Ref. 45)	M|.|.	Weighted-sum method	1st order	E
Churkina (Ref. 34)	M|C^0, convex|.	Chebyshev method Weighted-sum method	0th order	
Cohon et al. (Ref. 11)	2|linear|convex	Weighted-sum method	1st order	E, S
Das and Dennis (Ref. 33)	M|.|.	NBI method	0th order	Di
Das (Ref. 10)	2(M)|convex|convex	NBI method	1st order	E
Das (Ref. 26)	M|.|.	NBI method	0th order	C
Fadel and Li (Ref. 22)	2|(non)convex|.	ε-constraint method Lexicographic method Weighted-sum method	other	
Fliege and Heseler (Ref. 32)	M|convex, quadratic|polyhedral	Weighted-sum method	0th order	
Fruhwirth et al. (Ref. 12)	2|linear|convex	Hybrid method	1st order	E, S
Helbig (Ref. 35)	2|convex|convex	Direction method	0th order	
Helbig (Ref. 8)	2|quasiconvex|convex	Max-ordering method	0th order	C, Di

Table 2. (Continued).

Reference	Problem description	Solution methods	Structure	Quality
Ismail–Yahaya and Messac (Ref. 43)	M|.|.	Normal constraint method	0th order	Di
Jahn and Merkel (Ref. 7)	$2\|C^0\|.$	ε-constraint method	0th order	C
Kaliszewski (Ref. 50)	M|nonconvex|nonconvex	Modified Chebyshev method Augmented Chebyshev method	1st order	
Karaskal and Köksalan (Ref. 40):	M|.|.	Constrained Chebyshev method Achievement scalarizing fcts.	0th order	Di
Klamroth et al. (Ref. 52)	M|convex|convex	Modified direction method Gauge method Multidirection method	1st order	C, E
Kostreva et al. (Ref. 38)	M|.|.	Chebyshev method	0th order	
Li et al. (Ref. 21)	$2\|C^1\|$convex	ε-constraint method Lexicographic method Weighted-sum method	other	
Liu et al. (Ref. 20)	2|nonconvex|.	Weighted-sum method	other	S
Mateos and Rios-Insua (Ref. 53)	M|.|.	Lexicographic method	other	
Mateos et al. (Ref. 54)	M|.|.	Lexicographic method	other	
Mattson et al. (Ref. 44)	M|.|.	Normal-constraint method	0th order	Di
Messac and Mattson (Ref. 42)	M|nonconvex|nonconvex	Physical programming	0th order	Di
Nefëdov (Ref. 25)	$M\|C^0$, nonnegative, bounded|.	Max-ordering method	0th order	Di, Do, E
Nefëdov (Ref. 24)	$M\|C^0$, nonnegative, bounded|.	Max-ordering method	0th order	C, E
Payne (Ref. 5)	$2\|C^0\|C^1$	ε-constraint method	3rd order	E
Payne and Polak (Ref. 16)	$2\|C^0\|.$	Polak-Payne method	1st order	S, C
Payne (Ref. 18)	$2\|C^0\|.$	Polak-Payne method	1st order	S
Polak (Ref. 6)	$2\|C^5\|C^5$	ε-constraint method	3rd order	E

Table 2. (Continued).

Reference	Problem description	Solution methods	Structure	Quality
Popov (Ref. 23)	M|C^0|.	Max-ordering method	0th order	E
Reuter (Ref. 28)	M|linear|polyhedral	Hybrid method	0th order	Di, Do
Ruhe and Fruhwirth (Ref. 15)	2|linear|convex	Hybrid method	1st order	Do, S
Sayin (Ref. 55)	M|linear|polyhedral	–	0th order	C, Di, E
Sayin (Ref. 29)	M|linear|polyhedral	–	0th order	C, Di, E
Schandl et al. (Ref. 9)	2|·|several	Direction method Chebyshev method	1st order	C, E
Schandl et al. (Ref. 46)	M|convex|convex	Direction method Gauge method	1st order	C, E
Smirnov (Ref. 30)	M|linear|polyhedral/ convex	Max-ordering method	0th order	Do, E
Steuer and Harris (Ref. 27)	M|linear|polyhedral	–	0th order	Di
Solanki and Cohon (Ref. 14)	2|linear|polyhedral	Weighted-sum method	1st order	E
Solanki (Ref. 17)	2|linear|polyhedral (integer)	Chebyshev method	1st order	C, S
Solanki et al. (Ref. 51)	M|linear|polyhedral	Weighted-sum method	1st order	E, S
Voinalovich (Ref. 48)	M|linear|polyhedral	Chebyshev method	1st order	
Wiecek et al. (Ref. 19)	2|C^1|.	Chebyshev method ε-constraint method	2nd order	E
Wilson et al. (Ref. 41)	M|.|.	Pareto fitness functions	0th order	
Yang and Goh (Ref. 13)	2|linear|convex	Weighted-sum method	1st order	E, S

A scheme of the form $A|B|C$ is used for the problem description. The first position A shows whether the approach addresses BOPs or MOPs. The second position B specifies requirements on the objective functions, while the third position C states requirements on the constraint functions. For example, 2|linear|linear means that the article deals with BOPs with linear criteria and the feasible set is a polyhedron; $M|\text{convex}|C^1$ stands for an MOP with more than two convex criteria and continuously differentiable constraint functions. A dot indicates that no restrictions are made.

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